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DIDACTIC ASPECTS OF IMPLEMENTING GEOGEBRA IN TEACHING GAME THEORY WITH A FOCUS ON THE ANALYSIS OF GAMES OF DIMENSION m×2 – O

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Abstract. The paper presents an interdisciplinary approach to Game Theory within the STEAM education framework. Using GeoGebra, $m \times 2$ games are analyzed visually and interactively, enabling a better understanding of the Nash equilibrium and decision-making processes.

Keywords: Game theory; matrix game; zero-sum game; optimal strategy; upper value of the game; lower value of the game, STEAM education, GeoGebra.

ASPECTE DIDACTICE DE IMPLEMENTARE A GEOGEBRA ÎN PREDAREA TEORIEI JOCURILOR CU ACCENT PE ANALIZA JOCURILOR DE DIMENSIUNEA $m \times 2$ – O

Rezumat. Prezenta lucrare propune o abordare interdisciplinară a teoriei jocurilor, integrată în cadrul educației STEAM. Utilizarea aplicației GeoGebra permite analiza vizuală și dinamică a jocurilor de tip m×2 facilitând astfel o înțelegere mai profundă a echilibrului Nash și a proceselor decizionale.

Cuvinte-cheie: teoria jocurilor; joc matricial; joc cu sumă nulă; strategie optimă; valoarea superioară a jocului; valoarea inferioară a jocului; educație STEAM; GeoGebra.

Introduction

Contemporary education places increasing emphasis on interdisciplinary approaches designed to simultaneously develop students' cognitive, digital, and analytical competencies. In this context, the STEAM paradigm-which integrates Science, Technology, Engineering, Art, and Mathematics-offers a comprehensive framework for active and applied learning.

Digital tools such as GeoGebra, Desmos, and SMath Solver facilitate the interactive exploration of Game Theory concepts and transform theoretical problems into easily analyzable visual representations. Games of dimension $m \times 2$, due to their simplified structure, constitute efficient models for students, supporting the development of logical thinking, analytical capacity, and strategic problem-solving skills.

The fundamental elements of the STEAM concept are also reflected in *Game Theory* analysis, providing an integrated framework for evaluating strategies, optimizing decisions, and creating visual-creative representations of results:

Science – application of scientific methods for modeling decision-making situations and interpreting obtained data.

Technology – use of GeoGebra for simulation and interactive exploration of games, facilitating visual understanding of concepts.

Engineering – development and optimization of efficient strategies under multiple constraints.

Art – creation of creative visual representations of strategic relationships and payoff functions, supporting complex understanding of phenomena.

Mathematics – identification of optimal strategies and analysis of payoff functions through logical-mathematical methods.

Thus, the analysis of games of dimension $m \times 2$ through GeoGebra enables students to transition from abstraction to application, transforming theoretical content into a process of strategic thinking based on analysis, experimentation, and decision-making, presented in an interactive and engaging manner.

Theoretical Aspects of Games of Dimension $m \times 2$

Let there be a strategic game with two players A and B, zero-sum, defined by a payoff matrix $A = a[i, j]$, where each element $a[i, j]$ represents the payoff of player A in relation to the strategies chosen by both players. The objective of the first player is to maximize their own payoff, while the second player's task is to minimize the payoff of player A.

The following describes games of dimension $m \times 2$, games of dimension $2 \times n$ have been described in detail in [2].

The graphical representation of a game of dimension $m \times 2$ can be constructed as a function of the probability of the first strategy of player B, denoted, for example, as y_1, while the probability of the second strategy is calculated according to the relation $y_2 = 1 - y_1$. In this case, the first strategy of player B is represented on a line parallel to the ordinate axis, situated at a distance of one unit from it, a distance which reflects the fact that the sum of the strategy probabilities must equal 1, while the second strategy is represented on the ordinate axis itself. The points corresponding to the payoff values are then connected by straight lines, which express the payoff functions with respect to the probability of the first strategy of player B and each pure strategy of the first player.

Alternatively, one can work with the probability of the second strategy, i.e., y_2, in which case $y_1 = 1 - y_2$, and in this situation, the first strategy is represented on the ordinate axis, while the second is on a line parallel to the ordinate axis at a distance of one unit. In this case, the payoff functions are constructed with respect to the probability of the second strategy of player B and each pure strategy of the first player.

In a matrix game of dimension $m \times 2$, player A has m pure strategies, while player B has two pure strategies. In the absence of an equilibrium point, i.e., α ≠ β, the solution of the game is determined using mixed strategies.

The probabilities of the first player's strategies can be denoted as follows:

$$\bar{x} = (x_1, x_2, \ldots x_m),\ \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0$$

and a mixed strategy of player A can be represented in the form: $S_A = \begin{pmatrix} A_1, A_2, \ldots, A_m \\ x_1, x_2, \ldots, x_m \end{pmatrix}$.

The probabilities of the second player's strategies can be denoted as follows:

$$\bar{y} = (y_1, y_2),\ \sum_{j=1}^{2} y_j = 1,\ y_j \geq 0$$

and a mixed strategy of player B can be represented in the form $S_B = \begin{pmatrix} B_1, B_2 \\ y_1, y_2 \end{pmatrix}$ [3].

The payoff matrix of a game of dimension $m \times 2$ is represented in Table 1.

Table 1. Payoff matrix of a game of dimension $m \times 2$

		Player B's Strategies		
		$\boldsymbol{y}$	y_1	$y_2 = 1 - y_1$
Player A's Strategies	$\boldsymbol{x}$	A_i/B_j	B_1	B_2
	x_1	A_1	a_{11}	a_{12}
	x_2	A_2	a_{21}	a_{22}
	⋮	⋮	⋮	⋮
	x_m	A_m	a_{m1}	a_{m2}

Identification of Active Strategies of Player *A*

y_1 – the probability of player B applying strategy B_1;

y_2 – the probability of player B applying strategy B_2;

Assume that player A will act with pure strategy A_1, then the expected value of the loss for the second player will be:

$$a_{11} \cdot y_1 + a_{12} \cdot y_2 = a_{11} \cdot y_1 + a_{12} \cdot (1 - y_1) = (a_{11} - a_{12}) \cdot y_1 + a_{12}.$$

However, player A can act with each of their pure strategies. In this case, the value of the game can be represented through m linear dependencies that illustrate the expected loss of player B (the expected payoff of player A) using the probability of the first strategy y_1: $v = k \cdot y_1 + b;\ 0 \leq y_1 \leq 1$.

Table 2. Player *B*'s Loss

Player A 's Strategies	Payoff Functions with respect to y_1	
A_1	$v_1 = a_{11} \cdot y_1 + a_{12} \cdot y_2 = a_{11} \cdot y_1 + a_{12} \cdot (1 - y_1)$	$v_1 = (a_{11} - a_{12}) \cdot y_1 + a_{12}$
A_2	$v_2 = a_{21} \cdot y_1 + a_{22} \cdot y_2 = a_{21} \cdot y_1 + a_{22} \cdot (1 - y_1)$	$v_2 = (a_{21} - a_{22}) \cdot y_1 + a_{22}$
…	…	…
A_m	$v_m = a_{m1} \cdot y_1 + a_{m2} \cdot y_2 = a_{m1} \cdot y_1 + a_{m2} \cdot (1 - y_1)$	$v_m = (a_{m1} - a_{m2}) \cdot y_1 + a_{m2}$

To graphically illustrate the solution of the game in this case, the following steps should be followed:

- ✓ On the coordinate system y_1Ov, construct the lines corresponding to the m linear functions, taking into account that $0 \leq y_1 \leq 1$.
- ✓ These lines intersect with each other and form an open polygonal line. Above this broken curve, no other segment is located, thus highlighting the favorable strategies of player A with respect to the probability of the strategy selected by player B.
- ✓ Player B aims to minimize the payoff of player A. To this end, player B applies the *minimax* principle.
- ✓ Initially, they select the maximum values depending on the pure strategies applied by player A, after which, among these, they select the smallest one, which corresponds precisely to the intersection point of a pair of favorable strategies.
- ✓ Subsequently, the analysis is restricted exclusively to these two strategies, and they are used to determine the optimal mixed strategy of player A.
- ✓ Thus, the game is reduced to a game of dimension 2×2, which can be solved either by the graphical method or by the analytical method [1].

This graphical representation facilitates the visual analysis of the game, allowing identification of the intersection point of the lines, which corresponds to the equilibrium point. Thus, the relationships between possible strategies can be observed directly, without requiring an exclusively algebraic solution, which makes the learning process more intuitive and interactive.

Practical Application

Let there be given a zero-sum matrix game of dimension $m \times 2$, with $m = 4$. Thus, player A has 4 pure strategies and player B has 2 pure strategies. The payoff matrix A, which specifies the payments from player B to player A, is $A = \begin{pmatrix} 5 & 1 \\ 1 & 8 \\ 2 & 4 \\ 4 & 5 \end{pmatrix}$. The payoff matrix corresponding to the mixed strategies of player B and player A with respect to the probability of the first strategy y_1 of player B takes the form:

Table 3. Game Matrix

			Player B's Strategies	
		$\boldsymbol{y}$	y_1	$y_2 = 1 - y_1$
Player A's Strategies	$\boldsymbol{x}$	A_i/B_j	B_1	B_2
	x_1	A_1	5	1
	x_2	A_2	1	8
	x_3	A_3	2	4
	x_4	A_4	4	5

It is required to determine the complete solution of the game, including:

1. Analysis of the solution in pure strategies
   - ✓ Lower value of the game α;
   - ✓ Upper value of the game β;
   - ✓ Equilibrium point, if it exists in pure strategies.
2. If the game has no saddle point $\alpha \neq \beta$, it is required to find the optimal solution in mixed strategies:
   - ✓ Optimal mixed strategy of player A in the form $\bar{x} = (x_1, x_2, x_3, x_3)$, $\sum_{i=1}^{4} x_i = 1$, $x_i \geq 0$, which maximizes the minimum guaranteed payoff (*maximin*).
   - ✓ Optimal mixed strategy of player B in the form $\bar{y} = (y_1, y_2)$, $\sum_{j=1}^{2} y_j = 1, y_j \geq 0$, which minimizes the maximum guaranteed loss (*minimax*).
3. Solution of the game in mixed strategies and the value of the game v.

Solution:

1. Analysis of the solution in pure strategies

- ✓ Lower value of the game α: $\alpha = \max_i \left(\min_j \left(a_{ij}\right)\right), i = 1..4, j = 1..2;$

$$\alpha = \max_i \left(\min_j \left(a_{ij}\right)\right) = \max_i(\min(5,1)\,;\ \min(1,8)\,;\ \min(2,4);\ \min(4,5)) =$$
$$\max_i\,(1;\ 1;\ 2; 4) = 4.$$

- ✓ Upper value of the β: $\beta = \min_j \left(\max_i \left(a_{ij}\right)\right), i = 1..4, j = 1..2;$

$$\beta = \min_j(\max(5,1,2,4); \max\,(1,8,4,5)) = \min_j(5; 8) = 5.$$

- ✓ Equilibrium point, if it exists in pure strategies – since $\alpha \neq \beta$, the game has no saddle point in pure strategies.

2. Analysis of the solution in mixed strategies

Identification of active strategies of player A and determination of optimal strategy for player B

Since player A can act with each of their pure strategies, the game matrix takes the form:

Table 4. Game matrix with respect to mixed strategies of player *B*

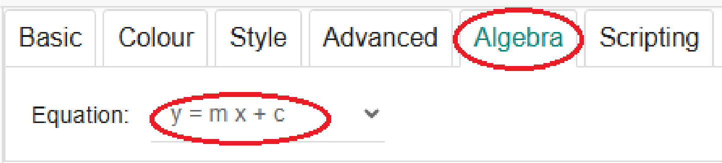
	Player B's mixed Strategies		
	$\boldsymbol{y}$	y_1	$y_2 = 1 - y_1$
Player A's pure Strategies	A_i/B_j	B_1	B_2
	A_1	5	1
	A_2	1	8
	A_3	2	4
	A_4	4	5

The expected payoff can be represented by four linear dependencies of the form $v = k \cdot y_1 + b;\ 0 \le y_1 \le 1$ with respect to the probability of the first strategy y_1.

$$A_1: 5y_1 + y_2 = 5y_1 + (1 - y_1):\ v_1 = 4y_1 + 1;$$
$$A_2: y_1 + 8y_2 = y_1 + 8(1 - y_1):\ v_2 = -7y_1 + 8;$$
$$A_3: 2y_1 + 4y_2 = 2y_1 + 4(1 - y_1):\ v_3 = -2y_1 + 4;$$
$$A_4: 4y_1 + 5y_2 = 4y_1 + 5(1 - y_1):\ v_4 = -y_1 + 5.$$

In GeoGebra, the lines corresponding to these analytically defined functions can be constructed, or the points corresponding to player B 's strategies can be graphically represented.

The analytical expressions of the payoff functions can be determined by accessing the context menu of the selected object and the *Settings* option, which provides access to the Algebra section, followed by the *Equation* option.



The first strategy of player B, $B_1 = (5, 1, 2, 4)$ is represented on a line parallel to the ordinate axis at a distance of one unit, while the second strategy $B_2 = (1, 8, 4, 5)$ is represented on the ordinate axis itself. The points corresponding to $a[1,1] - a[1,2]$; $a[2,1] - a[2,2]$; $a[3,1] - a[3,2]$ și $a[4,1] - a[4,2]$ represent player A 's strategies and are connected by straight lines to graphically illustrate the payoff functions.

The payoff functions defined by v_1, v_2, v_3, v_4 will be graphically illustrated using GeoGebra.

In Figure 1, the point corresponding to the *minimax*, $P[y_1, v]$ strategy is identified at the intersection of lines $v_1 = 4y_1 + 1$ and $v_4 = -y_1 + 5$, which determine the smallest of player B 's maximum payoffs.

The value of probability y_1 is obtained by solving the equation that results from the condition of equality of expected payoffs generated by the active pure strategies involved in determining the optimal mixed solution: $4y_1 + 1 = -y_1 + 5;\ 4y_1 + y_1 = 5 - 1;\ 5y_1 = 4;\ y_1 = \frac{4}{5}$. In this case, $y_2 = 1 - y_1 = 1 - \frac{4}{5} = \frac{1}{5}$.

The price of the game v can be determined from one of the equalities $v_1 = 4y_1 + 1$ or $v_4 = -y_1 + 5$ by substituting $y_1 = \frac{4}{5}$. Thus, $v_1 = 4y_1 + 1 = 4 \cdot \frac{4}{5} + 1 = \frac{21}{5} = 4.2$ and $v_4 = -y_1 + 5 = -\frac{4}{5} + 5 = \frac{21}{5} = 4.2$.

The optimal strategy of player B will be in this case $S_B = \left(\frac{4}{5}; \frac{1}{5}\right)$, and the price of the game $v = \frac{21}{5}$. The same values were also obtained by applying the graphical method.

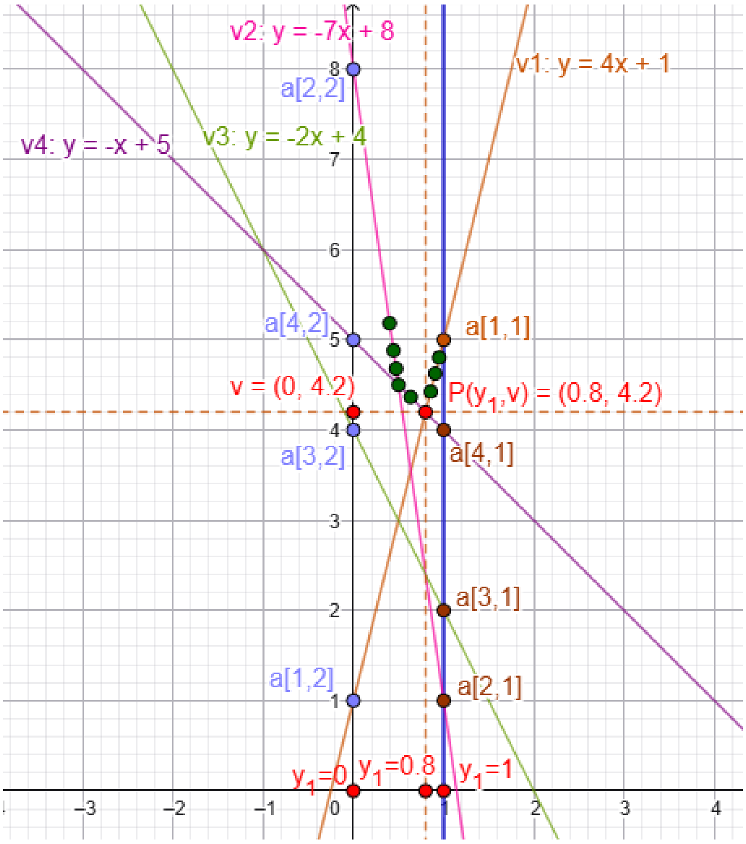


Figure 1. Graphical representation of player B's strategies

In GeoGebra, the *Solutions* function can be used to analytically calculate the corresponding point, thus verifying the correctness of the result obtained graphically:

$$y_1 = \text{Solutions}(4x + 1 = -x + 5)$$
$$= \{4 / 5\}$$

Next, the price of the game will be determined for the other lines $v_2 = -7y_1 + 8$ and $v_3 = -2y_1 + 4$. Thus, $v_2 = -7y_1 + 8 = -7 \cdot \frac{4}{5} + 8 = \frac{12}{5} = 2.4$ and $v_3 = -2 \cdot \frac{4}{5} + 4 = \frac{12}{5} = 2.4$. In both situations, the obtained price of the game is lower than the value 4.2, namely the value corresponding to the optimal mixed strategy. In this case, player *A* will not opt for strategies that bring a lower payoff, as these are unfavorable.

Determination of optimal strategy for player A

In what follows, the optimal mixed strategy for player *A* will be determined for the case when they work with strategies A_1 and A_4. In this case, the payoff matrix takes the form:

Table 5. Game matrix with respect to mixed strategies of player *A*

		Player B's pure Strategies	
		B_1	B_2
Player A's mixed Strategies	$A_1: x_1$	5	1
	$A_4: x_4 = 1 - x_1$	4	5

Since player A will not use the other strategies A_2 and A_3 (being unfavorable, their probabilities are equal to zero), they will work with strategies A_1 și A_4, which will be applied with certain non-zero probabilities.

Player B can act with each of their pure strategies, and the expected payoff of player A can be represented by two linear dependencies of the form $v = k \cdot x_1 + b; 0 \leq x_1 \leq 1$ with respect to the probability of the first strategy x_1. A_1: $v_1 = 5x_1 + 4(1 - x_1) = x_1 + 4$; A_2: $v_4 = x_1 + 5(1 - x_1) = -4x_1 + 5$.

The value x_1 can be calculated analytically by solving the equation resulting from equalizing the payoff lines corresponding to the active pure strategies involved: $x_1 + 4 = -4x_1 + 5;\ 5x_1 = 1;\ x_1 = \frac{1}{5}$. In this case, $x_4 = 1 - x_1 = 1 - \frac{1}{5} = \frac{4}{5}$. The price of the game v can be determined from one of the expressions $v_1 = x_1 + 4$ or $v_4 = -4x_1 + 5$ by substituting $x_1 = \frac{1}{5}$. Thus, $v_1 = -4x_1 + 5 = -4 \cdot \frac{1}{5} + 5 = \frac{21}{5} = 4.2$ and $v_4 = x_1 + 4 = \frac{1}{5} + 4 = \frac{21}{5} = 4.2$.

Next, the payoff functions v_1, v_4. will be graphically illustrated. The first strategy of player A, $A_1 = (5, 1)$ is represented on a line parallel to the ordinate axis, situated at a distance of one unit from it, while the second strategy $A_4 = (4, 5)$ is represented on the ordinate axis itself. The corresponding points $a[1,1] - a[2,1]$ and $a[1,2] - a[2,2]$ are connected by straight lines to graphically illustrate the payoff functions.

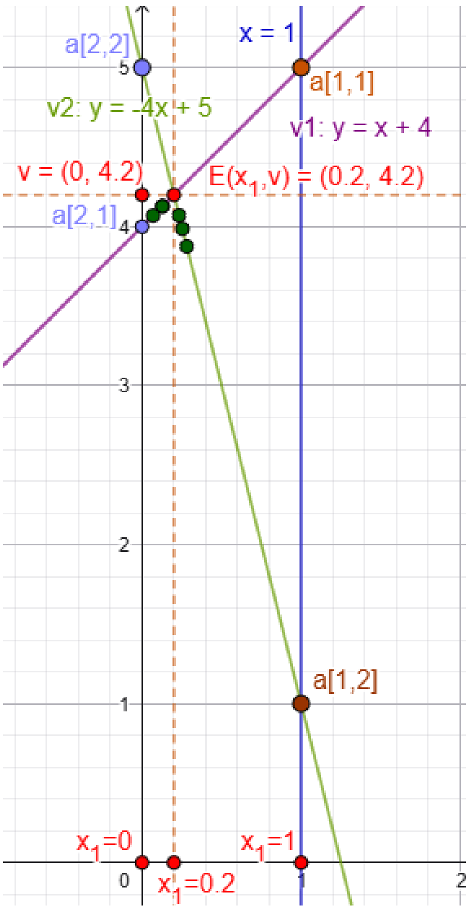


Figure 2. Graphical representation of player *B* 's strategies

In Figure 2, the point corresponding to the *maximin* strategy, $E[x_1, v]$, is identified at the intersection of lines $-4x_1 + 5$ and $x_1 + 4$, which determine the largest of player A 's minimum payoffs. The optimal strategy of player A in this case is $S_A = \left(\frac{1}{5}; 0; 0; \frac{4}{5}\right)$, and the price of the game is $v = 4.2$.

3. The solution of the game in mixed strategies and the value of the game v will take the form: $S = (S_A, S_B, v) = \left(\left(\frac{1}{5}; 0; 0; \frac{4}{5}\right), \left(\frac{4}{5}; \frac{1}{5}\right), \frac{21}{5}\right)$.

To validate the graphically obtained result, an initial verification was performed from the perspective of the first player, who acts with pure strategies, while the second player adopts mixed strategies, with the analysis subsequently repeated symmetrically from the perspective of the second player.

A – pure strategies; B – mixed strategies

$$v_A(A_1, y_j) = (a_{11}y_1 + a_{12}y_2)$$
$$v_A(A_2, y_j) = (a_{21}y_1 + a_{22}y_2)$$
$$v_A(A_3, y_j) = (a_{31}y_1 + a_{32}y_2)$$
$$v_A(A_4, y_j) = (a_{41}y_1 + a_{42}y_2)$$

A - mixed strategies; B - pure strategies

$$v_B(x_i, B_1) = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4$$
$$v_B(x_i, B_2) = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4$$

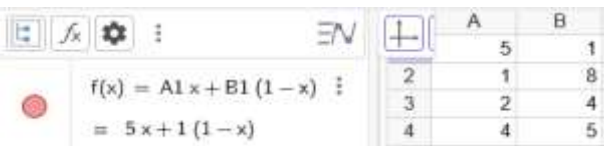
The correctness of the solution was thus verified from the perspective of both players, by calculating the values corresponding to the optimal mixed strategies, using the MS Excel application.

	A	B	C	D	E	F
1	Verification		B_1	B_2	A - pure strategies; B - mixed strategies	
2			4/5	1/5		
3	A_1	1/5	5	1	4 1/5	C3*C2+D3*D2
4	A_2	0	1	8	2 2/5	C4*C2+D4*D2
5	A_3	0	2	4	2 2/5	C5*C2+D5*D2
6	A_4	4/5	4	5	4 1/5	C6*C2+D6*D2
7	A - mixed strategies; B - pure strategies		4 1/5	4 1/5	A2, A3 - unfavourable strategies	
8						
9			C3*B3+C4*B4+C5*B5+C6*B6	D3*B3+D4*B4+D5*B5+D6*B6		

In the case of games of dimension $m \times 2$, deviation from these optimal strategies can yield a lower payoff for player A, as player B has only two strategies.

An alternative approach for defining the payoff matrix elements consists of entering them into a GeoGebra spreadsheet 

. Using the corresponding cell addresses, the payoff lines can be defined analytically, which offers a higher degree of flexibility in modifying the initial data.



This enables automatic recalculation of the payoff functions and corresponding updating of the graphical representations, facilitating the analysis of mixed strategies and rapid comparison of different strategic configurations.

Conclusions

The analysis of games of dimension $m \times 2$ using the GeoGebra application highlights the potential of digital technologies to transform the learning process into an active, exploratory, and reflective endeavor. Within a STEAM educational framework, problem-solving is not limited to the mechanical application of algorithms but involves modeling, experimentation, and interpretation of mathematical situations, contributing to the development of complex cognitive and digital competencies. In this context, GeoGebra emerges as an integrative educational tool that facilitates the connection between mathematics and visualization, between theory and practical application, and between formal analysis and graphical representation.

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