

THE APPLICATION OF TOPOLOGICAL METHODS IN THE STUDY AND SOLUTION OF PROBLEMS WITH PARAMETERS

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Abstract. This article investigates the possibilities of applying topological methods in the study and resolution of parameter problems. The conditions under which the topological properties of spaces and applications allow for the determination of the existence, number, or structure of solutions depending on the values of the parameters are analyzed. Examples of methods based on continuity, connectivity, the fixed-point principle, and compactness are provided. The application of these methods to solving parameter problems significantly simplifies the resolution of certain classes of parameter problems.

Keywords: study and resolution of parameter problems, Topology, topological methods, parameter problems, continuity, compactness, connectivity, fixed point.

APLICAREA METODELOR TOPOLOGICE ÎN STUDIUL ȘI REZOLVAREA PROBLEMELOR CU PARAMETRI

Rezumat. Acest articol investighează posibilitățile aplicării metodelor topologice în studiul și rezolvarea problemelor cu parametri. Sunt analizate condițiile în care proprietățile topologice ale spațiilor și aplicațiile permit determinarea existenței, numărului sau structurii soluțiilor în funcție de valorile parametrilor. Sunt furnizate exemple de metode bazate pe continuitate, conectivitate, principiul punctului fix și compactitate. Aplicarea acestor metode la rezolvarea problemelor cu parametri simplifică semnificativ rezolvarea anumitor clase de probleme cu parametri.

Cuvinte cheie: studiul și rezolvarea problemelor cu parametri, topologie, metode topologice, probleme cu parametri, continuitate, compactitate, conectivitate, punct fix.

The relevance of this topic is determined by the importance of topological methods in the research and resolution of mathematical problems that depend on one or more parameters. In various situations, determining the solutions of a nonlinear equation or problem can be too difficult, but topological methods allow for the study of the existence, uniqueness, and behavior of solutions without the need to obtain an explicit form of them. Topological methods based on properties such as continuity, compactness, and connectivity are important tools in nonlinear analysis. It should be noted that changing the parameters can lead to the emergence or disappearance of solutions, changes in the number of solutions, or alterations in their properties. In this context, topological methods allow for the highlighting of general properties and the formulation of conditions for the existence and stability of solutions. Therefore, the application of topological methods to solving parameter problems presents not only theoretical interest but also practical relevance.

An important role in the development of topological methods in solving mathematical problems has been played by Henri Poincaré, Luitzen E. J. Brouwer, Jean Leray, and Juliusz Schanuel. Their research has contributed to the emergence of methods based on continuity, compactness, fixed points, connectivity, and topological degree. Henri Poincaré's research has had a fundamental role in applying topological ideas to differential equations and problems dependent on one or more parameters.

The aim of the research is to study and highlight the possibilities of applying topological methods to solve mathematical problems with parameters to establish the existence, uniqueness, and number of solutions depending on the values of the parameters.

Research objectives:

- Analysis of fundamental concepts and results from topology for studying parameter problems;
- Study of the main topological methods used to solve parameter problems;
- Classification of problems based on the topological methods used;
- Selecting and analyzing model problems that illustrate the effectiveness of the method;
- Highlighting the advantages of topological methods compared to classical methods for solving parameter problems.

To introduce some notions and formulate some theorems applicable in solving parameter problems that are to be resolved.

The family $\mathcal{T}$ of subsets of the set X is called a topological structure on the set X if $\mathcal{T}$ satisfies the following conditions:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
2. The intersection of a finite number of subsets from $\mathcal{T}$ belongs to $\mathcal{T}$;
3. The union of any number of subsets from $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space. The subsets of the family $\mathcal{T}$ are called open sets. If $U \in \mathcal{T}$, then the complement of $X \setminus U$ is called a *closed set*.

The topological space X is called compact if from any open cover $\{U_i\}_{i \in I}$ a finite subcover can be extracted.

The topological space X is called connected if it cannot be represented as the union of two non-empty, disjoint open subsets.

The application $f: X \rightarrow Y$ is called continuous at the point $x_0 \in X$ if for any neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that $f(U) \subseteq V$.

Bolzano's Theorem. If the function $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) \cdot f(b) < 0$, then there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.

Brouwer's Theorem in the case of the interval. If $f: [a, b] \rightarrow [a, b]$ is continuous, then there exists at least one point $x_0 \in [a, b]$ for which $f(x_0) = x_0$.

Some of the most important topological methods used in solving parameter problems are presented in Table 1.

Table 1. Topological methods used in solving parameter problems

No.	Method	This demonstrates the method
1	Bolzano	Existence of a root
2	Intermediate values	Passing of a function through a value
3	Darboux	Intermediate value property
4	Connectedness	Images and intervals
5	Compactness	The existence of extremes/limits
6	Brouwer	The existence of fixed points

To apply topological methods in solving problems with parameters, we propose that you try the following resolution strategy:

1. Transform the equation into $f(x) = 0$ or in the fixed-point fix $x = f(x)$.
2. *Check continuity.*
3. *Look for:*
 - values at the ends of an interval;
 - limits at $\pm\infty$;
 - critical points;
 - the interval on which the function is monotonic;
 - a compact set K for which $f(K) \subseteq K$.
4. *For existence, use:*
 - Bolzano / intermediate value;
 - Brouwer / fixed point.
5. *For the number of solutions:*
 - track how the intersections change when the parameter varies.
 - «special» values of the parameter usually occur when $f(x) = 0$, $f'(x) = 0$ simultaneously.
6. For advanced problems, use:

Continuity + connectedness + compactness + fixed point.

Example 1. Demonstrate that for any $t \in \mathbb{R}$, the equation $x^3 + x + t = 0$ has at least one real solution.

Solution. To solve this problem, we will use the topological idea: the intermediate value theorem and the monotonicity of the function.

Let $f_t(x) = x^3 + x + t$. The given function is continuous on the set $\mathbb{R}$. We observe that: $\lim_{t \rightarrow -\infty} f_t(x) = -\infty$ and $\lim_{t \rightarrow +\infty} f_t(x) = +\infty$. From the intermediate value theorem, for any $t \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f_t(x) = 0$.

Therefore, $t \in \mathbb{R}$.

In the given example, the topology allows us to demonstrate the existence of a solution without solving the equation.

Example 2. Let $t > 0$. It is to be demonstrated that the equation

$$x^{2025} + tx = t + 1$$

has exactly one real solution.

Solution. We define the function $f(x) = x^{2025} + tx - t - 1$. The function $f(x)$ is continuous on the set $\mathbb{R}$. We calculate:

$$f(0) = -t - 1 < 0, f(1) = 1 + t - t - 1 = 0.$$

Therefore, $x = 1$ is a solution. It remains to demonstrate that this solution is unique.

We calculate the derivative

$$f'(x) = 2025 \cdot x^{2024} + t > 0.$$

Therefore, the function $f(x)$ is strictly increasing on $\mathbb{R}$. Any strictly increasing function can take the value 0 at most once. Thus, $x = 1$ is the unique solution.

In this example, continuity guarantees the existence of a root through a sign change, and the order structure on $\mathbb{R} +$ monotonicity guarantees uniqueness.

Example 3. Determine the values of the parameter t for which the equation

$$x^3 - 3x + t = 0$$

has at least one solution in the interval $[0,2]$.

Solution. The given equation is equivalent to the equation $t = -x^3 + 3x$. We define the function:

$$g(x) = -x^3 + 3x, x \in [0, 2].$$

Since the function $g(x)$ is continuous and $[0,2]$ is connected, the image $g([0,2])$ is an interval. Calculating the derivative of the function $g(x)$:

$$g'(x) = -3 \cdot x^2 + 3 = 3(1 - x^2)$$

we observe that the function $g(x)$ is increasing on $[0,1]$ and decreasing on $[1,2]$.

$$\text{We calculate: } g(0) = 0, g(1) = 2, g(2) = -2.$$

Therefore,

$$g([0, 2]) = [-2, 2].$$

So, the equation has at least one solution in $[0,2]$ for $-2 \leq t \leq 2$.

Answer: $-2 \leq t \leq 2$.

Continuity tells us that if the function takes the values 0 and -2, then it must take all values between 0 and -2. It is not necessary to check each value of t *separately*.

Example 4. Determine all values of the parameter t for which the equation:

$$x^2 + tx + 1 = 0$$

has at least one solution in the interval $[1,3]$.

Solution. The given equation is equivalent to the equation

$$t = -\frac{x^2+1}{x}.$$

For $x \in [1,3]$ we define the function

$$g(x) = -x - \frac{1}{x}.$$

The function $g(x)$ is continuous on $[1,3]$. Since the image of an interval through a continuous function is interval, the set of values of the function g is also an interval.

We calculate the values of the function $g(x)$ at points 1 and 3:

$$g(1) = -2, g(3) = -3 - \frac{1}{3} = -\frac{10}{3}.$$

We calculate the derivative:

$$g'(x) = -1 + \frac{1}{x^2} < 0, \text{ for } x > 1.$$

Therefore, $g(x)$ is strictly decreasing and

$$g[1,3] = [-\frac{10}{3}, -2].$$

Thus,

$$t \in [-\frac{10}{3}, -2].$$

Example 5. Determine, based on the values of the parameter t , the number of solutions: $\sin x = tx, x \in [0, \pi]$.

Solution. We consider the function $f(x) = \frac{\sin x}{x}, x > 0$.

Since $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, we can continuously extend f by $f(0) = 1$.

On the compact $[0, \pi]$, $f(x)$ reaches its maximum and minimum. Furthermore,

$$f(0) = 1, f(\pi) = 0$$

and $f(x)$ is strictly decreasing on $(0, \pi]$.

Therefore,

$$f([0, \pi]) = [0, 1].$$

The initial equation is equivalent to $f(x) = t$.

So:

- $t < 0$ or $t > 1$: 0 solutions;
- $t = 1$: the solution $x = 0$, if the interpretation by extension is included;
- $0 < t < 1$: exactly one solution;
- $t = 0$: $x = \pi$.

Example 5 shows in which the image of a compact set through a continuous function is compact, and the image of an interval through a continuous function is connected.

Example 6. Demonstrate that for any $t \in [0,1]$, the equation $x = t(1 - x^2)$ has at least one solution in $[0,1]$.

Solution. We consider the function

$$f(x) = t(1 - x^2).$$

For $x \in [0,1], t \in [0,1]$ we have: $0 \leq t(1 - x^2) \leq 1$.

Therefore,

$$f([0,1]) \subseteq [0,1].$$

The function $f(x)$ is continuous. According to Brouwer's fixed-point theorem, there exists $x_0 \in [0,1]$ such that

$$f(x_0) = x_0.$$

Therefore

$$x_0 = t(1 - x_0^2).$$

Thus, the initial equation has at least one solution in $[0,1]$ for any $t \in [0,1]$.

Example 7. Given the equation $x^3 - 3x = t$, determine the number of real solutions depending on the values of the perimeter t .

Solution. We consider the function

$$f(x) = x^3 - 3x$$

We calculate the derivative of the function:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1).$$

The critical points of the function f are $x = -1$ and $x = 1$.

We observe that:

f is increasing on $[-\infty; -1]$

f is increasing on $[-1; 1]$;

f is increasing on $[1; +\infty]$.

The values of the function f at the critical points are:

$$f(-1) = 2 \text{ and } f(1) = -2.$$

Therefore, f has a local maximum equal to 2 and a local minimum equal to -2.

The equation $f(x) = t$ has:

- One solution for $t < -2$ and $t > 2$;
- Two solutions, for $t = -2$ and $t = 2$;
- Three different solutions, for $-2 < t < 2$.

This problem illustrates the role of topology and continuity on each interval of monotonicity.

Example 8. Determine the number of solutions of the equation $x^2 e^{-x} = t, x \geq 0$, depending on the values of the parameter t .

Solution. We define the function

$$f(x) = x^2 e^{-x}.$$

We observe:

$$f(0) = 0, \lim_{x \rightarrow \infty} f(x) = 0.$$

We calculate the derivative:

$$f'(x) = x \cdot e^{-x} (2 - x).$$

We observe that $f(x)$ is increasing on $[0,2]$ and decreasing on $[2, +\infty]$.

The maximum of the function is

$$f(2) = \frac{4}{e^2}.$$

Therefore:

- If $t < 0$ or $t > 4/e^2$: one solution;
- If $t = 0$ or $4/e^2$: one solution;
- $0 < t < 4/e^2$: two solutions.

In this example, compactness can be applied separately on $[0, 2]$, where the function reaches its maximum, and connectedness explains why all values between 0 and $4/e^2$ are taken.

Example 9. Determine the number of solutions of the equation $x(1 - x) = t$, $x \in [0, 1]$ in depending on the values of the perimeter t .

Solution. We define the function $f(x) = x(1 - x)$.

Since the interval $[0, 1]$ is compact, the function $f(x)$ reaches its maximum and minimum (Weierstrass):

$$\min f(x) = 0, \max f(x) = \frac{1}{4}.$$

As the interval $[0, 1]$ is connected, the image $f([0, 1])$ is an interval, so:

$$f([0, 1]) = [0, \frac{1}{4}].$$

Therefore:

- If $t < 0$ or $t > \frac{1}{4}$: 0 solutions;
- If $t = 0$: two solutions: $x = 0, x = 1$;
- If $0 < t < \frac{1}{4}$: two solutions;
- If $t = \frac{1}{4}$: one solution: $x = \frac{1}{2}$.

In this example, compactness demonstrates the existence of the maximum, and connectedness allows us to state that all values between the minimum and maximum are attained.

Example 10. Depending on the values of the parameter t , determine the number of solutions of the equation

$$x^2 - t = 0.$$

Solution. We consider the function

$$f(x, t) = x^2 - t.$$

For $t > 0$, the given equation has two solutions:

$$x_1(t) = \sqrt{t}, x_2(t) = -\sqrt{t}.$$

For the critical value $t = 0$ the two solutions coincide.

$$x_1(0) = x_2(0) = 0.$$

For $t < 0$, the real solutions disappear.

This example illustrates a fundamental idea: changing a parameter can lead to the emergence, disappearance, or change in the number of solutions.

Example 11. Let $f: [0, 1] \rightarrow [0, 1]$ be a continuous function. It must be demonstrated that for any $t \in [0, 1]$ the equation

$$f(x) = tx + (1 - t)x^2$$

has at least one solution in the interval $[0, 1]$.

Solution. We define the function

$$g_t(x) = f(x) - tx - (1 - t)x^2.$$

The function $g_t(x)$ is continuous.

For $x = 0$,

$$g_t(0) = f(0) \geq 0.$$

For $x = 1$,

$$g_t(1) = f(1) - t - (1 - t) = f(1) - 1 \leq 0.$$

Therefore, we have

$$g_t(0) \geq 0, g_t(1) \leq 0.$$

From the intermediate value theorem, there exists $c \in [0, 1]$ such that

$$g_t(c) = 0.$$

So,

$$f(c) = tc + (1 - t)c^2.$$

for any $t \in [0, 1]$.

In this significant example, the fact is that nothing is known about the concrete form of the function other than continuity and the fact that $f([0, 1]) = [0, 1]$. This is a situation where topological methods are extremely powerful in solving it.

Topological methods allow:

- Determining the parameter values for which the problem has solutions;
- Establishing the number of solutions;
- Analyzing the existence and uniqueness of solutions;
- Identifying changes in the behavior of solutions with variations in the parameter;
- Avoiding complicated algebraic calculations in certain classes of problems.

The advantages of applying topological methods in solving parameterized problems lie in highlighting the structure of the problem; the possibility of addressing entire families of problems with parameters; reducing calculations.

Table 2. Application of topological methods in solving parameterized problems

No.	Mathematical structure	Topological method	Parameter condition	Nr. of solutions	Role of the parameter
P1	Algebraic equation	Continuity of the function	$t \in \mathbb{R}$	≥ 1	Shifts the function vertically and influences the solution
P2	Algebraic equation	Continuity+monotonicity	$t > 0$	1	Influences both the linear term and the free term at the same time
P3	Equation with a solution in an interval	The image of an interval through a continuous function	$t \in [-2, 2]$	≥ 1	Determines the translation and the possibility of intersection with the O axis

P4	Equation with interval restriction	The image of a continuous function	$t \in [-\frac{10}{3}, -2]$	≤ 1	This is the value of $g(x) = -x - \frac{1}{x}$. And selects the admissible values of the solution
P5	Transcendental equation	Analysis of a continuous image	$t \in R$	1 or 2	Represents a value from the image of the function $\sin x/x$ and determines the number of intersections
P6	Existence problem	Fixed-point method	$t \in [0,1]$	≥ 1	Controls the application of $f(x)$ and guarantees, for $t \in [0,1]$.
P7	Determine the number of solutions	Analyze the image of the continuous function	$t \in R$	1, 2 or 3	Select the horizontal line $y = t$, and its position determines the number of intersections
P8	Transcendental equation	Analyze the image of the continuous function	$t \in R$	0, 1 or 2	Represents the horizontal level that intersects the graph - shows n solutions
P9	Equation with parameters	Analyze the image of the continuous function	$t \in [0,1]$	0, 1 or 2	Is the level value, the position relative to 0 and $\frac{1}{4}$ determines the number of solutions
P10	Algebraic equation	Image analysis of the function	$t \in R$	0, 1 or 2	Is the sought value in the image of the function x^2 .
P11	Equations dependent on the parameter	Bolzano's theorem	$t \in [0,1]$	at least one solution	Determine the convex combination between x and x^2 .

The limits of applying these problems lie in the fact that not every problem with parameters can be solved more simply through topological methods, and sometimes it is necessary to combine several topological methods with algebraic or geometric ones.

For further research, it is proposed:

- To extend the results obtained to problems with multiple parameters:
- To develop algorithms for the automatic identification of conditions on the parameters.

Conclusions

The study conducted demonstrates that topological methods represent a relevant and effective tool for solving and analyzing problems with parameters. The examined applications highlight the ability of these methods to determine the conditions for the existence of solutions, to analyze the variation in the number of solutions depending on the parameter, and to highlight the global structure of the studied problems. The results obtained confirm the opportunity to use topological methods in complementarity with algebraic, analytical, and geometric methods. At the same time, the topological approach offers perspectives for generalizing the results and for developing solving methods applicable to broader classes of problems with parameters.

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