Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 18(2), 2024, Pages 18–37 E-ISSN: 2587-3644
CZU: 517.925 DOI: 10.36120/2587-3644.v18i2.18-37

The problem of the center for cubic differential systems with two affine non-parallel invariant straight lines of total multiplicity three

Alexandru Şubă

Abstract. In this paper, we show that a center-focus critical point of cubic differential systems with two affine non-parallel invariant straight lines of total multiplicity three is a center type if and only if the first five Lyapunov quantities vanish.
2020 Mathematics Subject Classification: 34C05.
Keywords: cubic differential system, multiple invariant line, the problem of the center.

Problema centrului pentru sistemele diferenţiale cubice cu două drepte afine invariante şi concurente de multiplicitate totală trei

Rezumat. În această lucrare se arată că punctul critic de tip centru-focar al sistemelor diferenţiale cubice cu două drepte afine invariante şi concurente de multiplicitate totală trei este centru, dacă şi numai dacă primele cinci mărimi Liapunov se anulează.
Cuvinte-cheie: sistem diferenţial cubic, dreaptă invariantă multiplă, problema centrului.

1. Introduction and statement of main results

We consider the real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad \gcd(P, Q) = 1 \tag{1}$$

and the vector fields $\mathbb{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$ associated to systems (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ (respectively, $n = 3$, $n = 4$), then the system (1) is called *quadratic* (respectively, *cubic, quartic*).

An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ (a function $f = \exp[\frac{g}{h}]$, $g, h \in \mathbb{C}[x, y]$) is called *invariant algebraic curve* (*exponential factor*) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K_f) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x, y)K_f(x, y)$ holds. In particular, *a straight line* $\mathcal{L} \equiv \alpha x + \beta y + \gamma = 0$, $\alpha, \beta, \gamma \in \mathbb{C}$ is called *invariant* for the system (1) if there exists a polynomial $K_{\mathcal{L}} \in \mathbb{C}[x, y]$ such that the identity $\alpha P(x, y) + \beta Q(x, y) \equiv (\alpha x + \beta y + \gamma)K_{\mathcal{L}}(x, y)$, $(x, y) \in \mathbb{R}^2$, i.e. $\mathbb{X}(\mathcal{L}) \equiv \mathcal{L}(x, y)K_{\mathcal{L}}(x, y)$, $(x, y) \in \mathbb{R}^2$, holds. If a straight line $\mathcal{L}$ is described by the equation

$y = \alpha x + \beta$, $\beta \neq 0$ (respectively, $x = \alpha$), then $\mathcal{L}$ is invariant for (1) if the following identity in x:

$$(\alpha P(x,y) - Q(x,y))|_{y=\alpha x+\beta} \equiv 0 \qquad (2)$$

(respectively, in y: $P(\alpha, y) \equiv 0$) holds.

If $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$) is the greatest natural number such that $\mathcal{L}^{m_p(\mathcal{L})}$ (respectively, $\mathcal{L}^{m_a(\mathcal{L})}$) divides $\mathbb{X}(\mathcal{L})$ (respectively, $E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$), then we say that the invariant straight line $\mathcal{L}$ has *parallel multiplicity* (*algebraic multiplicity*, or in brief, *multiplicity*) $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$).

Remark 1.1. $1 \leq m_p(\mathcal{L}) \leq n$ *and* $m_p(\mathcal{L}) \leq m_a(\mathcal{L})$.

The number $m_t(\mathcal{L}) = m_a(\mathcal{L}) - m_p(\mathcal{L}) + 1$ is called *transversally multiplicity* of the line $\mathcal{L}$.

Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

The cubic differential systems with multiple invariant straight lines (including the line at infinity) were studied in [5], [6], [10], [11], [12], [14].

Let $f_1, \dots, f_r$ ($f_{r+1} = \exp(g_{r+1}/h_{r+1}), \dots, f_s = \exp(g_s/h_s)$) be invariant algebraic curves (exponential factors) of (1) and let K_{f_j}, $j = \overline{1,s}$, be its cofactors [2]. The system (1) is called *Darboux integrable* if (1) has a first integral (an integrating factor) of the form $F(x,y) = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ ($\mu(x,y) = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$), $\alpha_j \in \mathbb{C}$, $j = \overline{1,s}$. Note that the constants $\alpha_1, \dots, \alpha_s$ are not all equal to zero.

It is easy to show that $F(x,y)$ ($\mu(x,y)$) is a Darboux first integral (a Darboux integrating factor) if and only if the following identity

$$\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \cdots + \alpha_s K_{f_s} \equiv 0$$
$$\left(\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \cdots + \alpha_s K_{f_s} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0\right)$$

holds in x and y.

In this work we consider the cubic systems of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \qquad \gcd(P,Q) = 1. \end{cases} \qquad (3)$$

The critical point $(0,0)$ of the system (3) is of a center-focus type, i.e. is either a focus or a center. The problem of distinguishing between a center and a focus is called *the problem of the center* or *the center-focus problem*.

It is known that $(0,0)$ is a center for (3) if and only if the system has a nonconstant analytic first integral $F(x,y)$ (an analytic integrating factor $\mu(x,y)$) in a neighborhood of $(0,0)$. Also, it is known that there exists a formal power series $F(x,y) = x^2 + y^2 + \sum_{j\geq 3} F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (3) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$, i.e. $\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. The quantities $L_j,\ j=\overline{1,\infty}$, are polynomials with respect to the coefficients of the system (3), called to be *the Lyapunov quantities*. For example, the first Lyapunov quantity looks as

$$L_1 = (bd - ac + 2bf - 2ag + dg - cf + 3k - 3l + p - q)/4.$$

The origin $(0,0)$ is a center for (3) if and only if $L_j = 0,\ j=\overline{1,\infty}$.

The problem of the center is completely solved for quadratic systems ($k = l = m = n = p = q = r = s = 0$) [4] and for symmetric cubic systems ($a = b = c = d = f = g = 0$) [8]. For other polynomial differential systems the necessary and sufficient conditions for the center-focus critical point to be a center were obtained in some particular cases (see, for example, [2], [7]).

The problem of coexistence in cubic systems of the distinct invariant straight lines and critical points of center type was studied in [2], [3], [9]. In [3] (see also [2]) it was proved that if the cubic system (3) has four distinct invariant straight lines of the form $1+\alpha_j x+\beta_j y = 0,\ j = 1,2,3,4$ $(y \pm ix = 0, 1+\alpha_j x+\beta_j y = 0,\ j=1,2)$ and the Lyapunov quantity vanishes: $L_1 = 0$ ($L_1 = L_2 = 0$), then the origin is a center. In the cases when (3) has three distinct invariant straight lines then $(0,0)$ is a center if the first seven Lyapunov quantities vanish $L_j = 0,\ j = 1,\ldots 7$.

In this article we investigate the problem of the center for (3) with two invariant affine straight lines of total multiplicity three. Our main result is the following one:

Main Theorem. *Let the cubic system have two distinct affine non-parallel invariant straight lines $\mathcal{L}_1$, $\mathcal{L}_2$ and a critical point $M_0(x_0,y_0)$ with pure imaginary eigenvalues. If $m(\mathcal{L}_1) = 2$, then M_0 is a center if and only if the first five Lyapunov quantity vanishe ($L_1 = L_2 = L_3 = L_4 = L_5 = 0$).*

2. Conditions of the existence of invariant straight lines

Let the system (3) have an invariant straight line $\mathcal{L}_1$. Using a transformation of the form $x \to \omega(x\cos\alpha - y\sin\alpha),\ y \to \omega(x\sin\alpha + y\cos\alpha)$, we do $\mathcal{L}_1$ to be described by the equation $\mathcal{L}_1 \equiv x - 1 = 0$. The identity $P(1,y) \equiv 0$ gives us

$$k = -a,\ m = -c-1,\ p = -f,\ r = 0. \tag{4}$$

For system $\{(3), (4)\}$ the identity (2) has the form

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 \equiv 0,$$

where

$$\begin{aligned}
A_0 &= -\beta(\alpha + b\beta + f\alpha\beta + l\beta^2),\\
A_1 &= -1 - \alpha^2 - (d + 2b\alpha + c\alpha + 2f\alpha^2)\beta - (n - f\alpha + 3l\alpha)\beta^2,\\
A_2 &= -g - q\beta - \alpha(a + d - \beta - c\beta + 2n\beta) - \alpha^2(b + c - 2f\beta + 3l\beta) - f\alpha^3,\\
A_3 &= -s + (a - q)\alpha + (1 + c - n)\alpha^2 + (f - l)\alpha^3.
\end{aligned}$$

The system $\{A_0 \equiv 0, A_1 \equiv 0, A_2 \equiv 0, A_3 \equiv 0, \beta \neq 0\}$ has the solution

$$\begin{aligned}
g &= \alpha(d + c\alpha - a) - (q - \alpha - c\alpha)\beta + \alpha(2 - \alpha^2)/\beta,\\
l &= -(\alpha + b\beta + f\alpha\beta)/\beta^2,\\
n &= f\alpha + (2\alpha^2 - 1 - \beta(d - b\alpha + c\alpha - f\alpha^2))/\beta^2,\\
s &= \alpha(a - q + \alpha + c\alpha) + \alpha^2(d + c\alpha)/\beta + \alpha^2(1 - \alpha^2)/\beta^2.
\end{aligned} \tag{5}$$

Therefore, the system $\{(3),(4),(5)\}$ has the invariant straight lines $\mathcal{L}_1 \equiv x - 1 = 0$, $\mathcal{L}_2 \equiv \alpha x - y + \beta = 0, \beta \neq 0$.

The invariant straight line $\mathcal{L}_1$ has parallel multiplicity two if $P(1, y)|_{\{(4),(5)\}} \equiv 0$, i.e. if

$$a = c + 2 = f = 0. \tag{6}$$

The equalities $\{(4), (5), (6)\}$ give us the first set of conditions

$$\begin{aligned}
& a = c + 2 = f = k = m - 1 = p = r = 0,\ l = -(\alpha + b\beta)/\beta^2,\\
& g = (2\alpha - \alpha(\alpha + \beta)^2 + \beta(d\alpha - q\beta))/\beta, n = (2\alpha^2 - d\beta + 2\alpha\beta + b\alpha\beta - 1)/\beta^2,\\
& s = \alpha(\alpha - \alpha(\alpha + \beta)^2 + \beta(d\alpha - q\beta))/\beta^2,
\end{aligned} \tag{7}$$

so that, the straight lines $\mathcal{L}_1$, $\mathcal{L}_2$ are invariant for (3) and $m_p(\mathcal{L}_1) = 2$.

If $m_a(\mathcal{L}_1) \geq 2 > m_p(\mathcal{L}_1) = 1$, then it is necessary that x^2 divide for $\{(3),(4),(5)\}$ the polynomial

$$\begin{aligned}
\kappa(x, y) = x(\alpha x - y)(&y^2(\alpha + b\beta + f\alpha\beta + f\beta^2) - xy(\alpha^2 - d\beta - c\alpha\beta - \beta^2 - c\beta^2 - 1)\\
&+x^2(\alpha - \alpha^3 + d\alpha\beta + c\alpha^2\beta + a\beta^2 - q\beta^2 + \alpha\beta^2 + c\alpha\beta^2))/\beta^2.
\end{aligned}$$

This implies that

$$b = -f(\alpha + \beta) - \alpha/\beta. \tag{8}$$

Taking into account (4) and (5), we obtain that

$$\beta^4(E(\mathbb{X})/((-1 + x)(-y + x\alpha + \beta)))|_{x=1} = f_1(y) f_2(y),$$

where

$$f_1(y) = \alpha^3 - \alpha - \beta(1 + d\alpha + c\alpha^2 - q\beta + \alpha\beta + c\alpha\beta) \\ +(\alpha^2 - 1 - \beta(d + \alpha + c\alpha))y + f\beta^2 y^2,$$

$$f_2(y) = a\beta^2(d + q - a) + (c + 2)(\alpha + \beta)(\alpha^3 - \alpha - \beta - d\alpha\beta + 2\alpha^2\beta + q\beta^2 + \alpha\beta^2) \\ -\alpha\beta(c + 2)^2(\alpha + \beta)^2 + 2((\alpha^2 - 1)(a + f\alpha^2) - \beta(ad - a\alpha + 2f\alpha + df\alpha^2 \\ -3f\alpha^3) - f\beta^2(1 + a\alpha + d\alpha - q\alpha - 3\alpha^2) - f\beta^3(a - q - \alpha) \\ -\beta(c + 2)(\alpha + \beta)(a + f\alpha^2 + f\alpha\beta))y(f\beta^2(a - d - q) \\ +(c + 2)(\alpha^2 - 1 - d\beta + \alpha\beta - f\alpha\beta^2 - f\beta^3) - \beta(\alpha + \beta)(c + 2)^2)y^2.$$

If $\{f_1(y) \equiv 0,\ \beta \neq 0\}$, then $\gcd(P, Q) = x - 1$, i.e. the system (3) is degenerate.

In the case $\{f_2(y) \equiv 0,\ \beta \neq 0\}$ we obtain the equalities (6) and the following three set of solutions

$$a = -(c + 2)(\alpha + \beta),\ d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta,\ f = 0; \tag{9}$$

$$d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta,\ q = (1 - \alpha^2 + a\beta - \alpha\beta)/\beta; \tag{10}$$

$$a = -(\alpha + \beta)(2 + c + f\alpha + f\beta), q = -(df\beta^2 + 2\beta(d + \alpha + 2\beta) \\ +(c + 2)(1 - \alpha^2) + c\beta(d + 3\alpha + 4\beta) \\ +\beta(\alpha + \beta)(c^2 + 4f\beta + 2cf\beta) + f^2\beta^2(\alpha + \beta)^2)/(f\beta^2). \tag{11}$$

Equalities (6) lead us to a particular case of the set (7) and each of the equalities (9), (10), (11), together with (4), (5) and (8) give us, respectively, the following three series of conditions

$$a = -(2 + c)(\alpha + \beta), b = -\alpha/\beta, d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta, f = 0, \\ g = (\alpha + \alpha\beta(c + 1)(\alpha + \beta) - q\beta^2)/\beta, k = -a, l = 0, m = -c - 1, r = 0, \\ n = (\alpha + 2\beta + c\beta)/\beta, p = -f, s = -(\alpha(q\beta + (\alpha + \beta)(\alpha + 2\beta + c\beta)))/\beta; \tag{12}$$

$$b = -(\alpha + f\alpha\beta + f\beta^2)/\beta, d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta, m = -c - 1, \\ g = (\alpha - \beta - a\alpha\beta - a\beta^2)/\beta, k = -a, l = f, n = (\alpha + 2\beta + c\beta)/\beta, \\ p = -f, q = (1 - \alpha^2 + a\beta - \alpha\beta)/\beta, r = 0, s = -\alpha/\beta; \tag{13}$$

$$a = -(\alpha + \beta)(2 + c + f\alpha + f\beta), b = -(\alpha + f\alpha\beta + f\beta^2)/\beta, r = 0, \\ g = (2f\alpha + (c + 2)(1 + d\beta) + (\alpha + \beta)(df\beta - (c + 2)(\alpha - 2\beta - c\beta)) \\ -f(\alpha + \beta)^2(\alpha - 4\beta - 2c\beta) + f^2\beta(\alpha + \beta)^3)/(f\beta), k = -a, l = f, \\ n = -(1 - \alpha^2 + d\beta + c\alpha\beta)/\beta^2, p = -f, q = -(df\beta^2 + 2\beta(d + \alpha + 2\beta) \\ +(c + 2)(1 - \alpha^2) + c\beta(d + 3\alpha + 4\beta) + \beta(\alpha + \beta)(c^2 + 4f\beta + 2cf\beta) \\ +f^2\beta^2(\alpha + \beta)^2)/(f\beta^2), s = \alpha(f\alpha + 2d\beta + 2\beta(\alpha + 2\beta) + (c + 2)(1 - \alpha^2) \\ +c\beta(d + 3\alpha + 4\beta) - (\alpha + \beta)(f(\alpha - 2\beta)(\alpha + \beta) - \beta(c^2 + df)) \\ +cf\beta(\alpha + \beta)^2)/(f\beta^2), m = -c - 1 \tag{14}$$

such that the straight lines $\mathcal{L}_1$, $\mathcal{L}_2$ are invariant for (3) and $m_a(\mathcal{L}_1) \geq 2$.

3. Sufficient conditions of the center

Lemma 3.1. *The following set of conditions is sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&a=\gamma(\gamma-\beta)/\beta, b=(\beta-\gamma)/\beta, c=-(\beta+\gamma)/\beta, d=(2\gamma^2-2\beta\gamma-1)/\beta,\\
&f=0, g=((\beta-\gamma)(\gamma^2-1)-q\beta^2)/\beta, k=\gamma(\beta-\gamma)/\beta, l=0, m=\gamma/\beta,\\
&n=0, p=0, r=0, s=q(\beta-\gamma);
\end{aligned}
\tag{15}
$$

Proof. In conditions (15) the system (3) has the integrating factor of the form

$$\mu(x,y)=\mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2},$$

where $\mathcal{L}_1=x-1,\ \mathcal{L}_2=(\beta-\gamma)x+y-\beta,\ \alpha_1=-2, \alpha_2=-1.$ □

Lemma 3.2. *The following six sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&b=f=g=l=p=q=r=s=0,\ d=(\beta^2-1)/\beta,\\
&a-\beta=c+3=k+\beta=m-2=n+1=0;
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&b=1, c=0, d=-1/\beta, f=0, g=-2, k=-a, l=0,\\
&m=-1, n=1, p=0, q=(1+a\beta)/\beta, r=0, s=1;
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&a=-1/(\beta(1+\beta^2)), b=1, c=-2\beta^2/(1+\beta^2), d=-1/\beta, f=-\beta/(1+\beta^2),\\
&g=-2, k=1/(\beta(1+\beta^2)), l=-\beta/(1+\beta^2), m=(\beta^2-1)/(1+\beta^2),\\
&n=(1-\beta^2)/(1+\beta^2), p=\beta/(1+\beta^2), q=\beta/(1+\beta^2), r=0, s=1;
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&a=\gamma^2(1+\beta\gamma-\gamma^2)/(2\beta(1-\gamma^2)), c=\gamma(\beta\gamma+\ \gamma^2-1)/(\beta(1-\gamma^2)),\\
&b=((\beta-\gamma)(\gamma^2-1)(\gamma^2-2)-\beta\gamma^2)/(2\beta(1-\gamma^2)),\\
&d=(\gamma(\beta-\gamma)(2\gamma^2-3)-1)/(\beta(1-\gamma^2)),\\
&f=\gamma((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(\gamma^2-1)),\\
&g=(\gamma^2-2)((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(\gamma^2-1)),\\
&k=\gamma^2(1+\beta\gamma-\gamma^2)/(2\beta(\gamma^2-1)), r=0, s=(\beta-\gamma)/\beta,\\
&l=\gamma((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(\gamma^2-1)),\\
&m=(\beta-\gamma+\gamma^3)/(\beta(\gamma^2-1)), n=1/(1-\gamma^2),\\
&p=\gamma((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(1-\gamma^2)),\\
&q=(1+\beta\gamma-\gamma^2)(\gamma^2-2)/(2\beta(\gamma^2-1));
\end{aligned}
\tag{19}
$$

$$\begin{aligned}
& b=(\beta-\gamma-f\beta\gamma)/\beta, \\
& c=(2\beta(f-\gamma)-\gamma(a+f)(2a\beta-2f\beta+3\beta\gamma-\gamma^2-1))/(\beta(\gamma \\
& \quad +(a+f)(\gamma^2-1))),\ d=((3a-\gamma)(\beta-\gamma)\gamma-\gamma(1-f\beta+3f\gamma) \\
& \quad +(a+f)(1+2a\beta\gamma^2-2f\beta\gamma^2))/(\beta(\gamma+(a+f)(\gamma^2-1))), \\
& g=(\gamma-2\beta-a\beta\gamma)/\beta,\ k=-a,\ l=f,\ m=(\beta(a-f+\gamma)+ \\
& \quad \gamma(a+f)(-1+2a\beta-2f\beta+2\beta\gamma-\gamma^2))/(\beta(\gamma+(a+f)(\gamma^2-1))), \\
& n=((1+2\gamma(a+f))(\gamma^2+f\beta-a\beta-\beta\gamma))/(\beta(\gamma+(a+f)(\gamma^2-1))), \\
& p=-f,\ q=(1+a\beta+\beta\gamma-\gamma^2)/\beta,\ r=0,\ s=(\beta-\gamma)/\beta, \\
& ((\beta-\gamma)\gamma^3-2\beta(f-\gamma-a\gamma^2)-\gamma^2)(a\beta\gamma+(a+f)(a\beta-\gamma^2-f\beta\gamma^2))=0;
\end{aligned} \tag{20}$$

$$\begin{aligned}
& a=-\gamma(c+2+f\gamma),\ b=(\beta-\gamma-f\beta\gamma)/\beta,\ k=\gamma(c+2+f\gamma),\ l=f, \\
& g=(\beta\gamma(c+2)^2+(c+2)(1+d\beta+\beta\gamma-\gamma^2+2f\beta\gamma^2)+f((\beta-\gamma)(\gamma^2-2) \\
& \quad +d\beta\gamma+f\beta\gamma^3))/(f\beta),\ n=((\beta-\gamma)(\beta+c\beta-\gamma)-d\beta-1)/\beta^2, \\
& m=-c-1,\ p=-f,\ q=-((c+2)^2\beta\gamma+(c+2)(d\beta+\beta\gamma+2f\beta^2\gamma-\gamma^2 \\
& \quad +1)+f\beta^2(d+f\gamma^2))/(f\beta^2),\ r=0,\ s=(\gamma-\beta)((c+2)^2\beta\gamma \\
& \quad +(c+2)(1+d\beta+\beta\gamma-\gamma^2+f\beta\gamma^2)+df\beta\gamma f(\beta-\gamma)(\gamma^2-1))/(f\beta^2), \\
& \quad 2(c+2)^2\beta\gamma+(c+2)(1+d\beta-f\beta+4f\beta\gamma^2)+f\beta\gamma(d-2f+2f\gamma^2).
\end{aligned} \tag{21}$$

Proof. In each of the sets of conditions (16)−(21), the system (3) has the integrating factor of the form

$$\mu(x,y)=\mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\mathcal{L}_3^{\alpha_3} \tag{22}$$

and therefore, in all cases the origin $(0,0)$ is a center for (3). Indeed, in *Case* (16):

$$\mathcal{L}_1=x-1,\ \mathcal{L}_2=y-\beta,\ \mathcal{L}_3=\beta y+1,\ \alpha_1=-3, \alpha_2=-1, \alpha_3=1;$$

in *Case* (17):

$$\begin{aligned}
& \mathcal{L}_1=x-1,\ \mathcal{L}_2=\beta x+y-\beta,\ \mathcal{L}_3=\exp[y/(x-1)], \\
& \alpha_1=2a\beta-2\beta^2-1,\ \alpha_2=2\beta^2-2a\beta-1,\ \alpha_3=-2\beta;
\end{aligned}$$

in *Case* (18):

$$\begin{aligned}
& \mathcal{L}_1=x-1,\ \mathcal{L}_2=\beta x+y-\beta,\ \mathcal{L}_3=\beta^2x+\beta y-\beta^2-1, \\
& \alpha_1=-3,\ \alpha_2=1,\ \alpha_3=-2;
\end{aligned}$$

in *Case* (19):

$$\begin{aligned}
& \mathcal{L}_1=x-1,\ \mathcal{L}_3=\exp[\beta\gamma(\gamma-y)(2\beta-\gamma-\beta\gamma^2+\gamma^3)/(2(\gamma^2-1)(x-1))], \\
& \mathcal{L}_2=(\beta-\gamma)x+y-\beta,\ \alpha_1=(1-2\beta(2\gamma-\beta)-\gamma^2(\gamma-\beta)^2)/(\gamma^2-1), \\
& \alpha_2=(1+(\beta-\gamma)^2(\gamma^2-2))/(\gamma^2-1),\ \alpha_3=-2/(\beta\gamma);
\end{aligned}$$

in *Case* (20):

$$\begin{aligned}
&\mathcal{L}_2 = (\beta-\gamma)x + y - \beta,\ \mathcal{L}_3 = \exp[(f\beta^2(y-\gamma))/(x-1)],\\
&\alpha_1 = (2\beta^2(\gamma-\beta)\gamma^2 f^2 + \beta(2a\beta^2 + 3\gamma - 2a\beta\gamma - 2\beta\gamma^2 + \gamma^3 + 2a\beta\gamma^3\\
&\qquad +\beta^2\gamma^3 - \beta\gamma^4)f + \gamma(a\beta - 2a^2\beta^2 - 3\beta\gamma - a\beta^2\gamma + \gamma^2 - a\beta\gamma^2\\
&\qquad +\beta^2\gamma^2 - 2\beta\gamma^3 + \gamma^4))/(\beta\gamma(\gamma + (a+f)(\gamma^2-1))),\\
&\alpha_2 = (2\beta(\beta-\gamma)\gamma^2 f^2 + (\gamma - 2a\beta^2 + 2a\beta\gamma + 2\beta\gamma^2 - 3\gamma^3 - 2a\beta\gamma^3\\
&\qquad -\beta^2\gamma^3 + \beta\gamma^4)f + \gamma(a + 2a^2\beta - \gamma + a\beta\gamma - 3a\gamma^2 - \beta\gamma^2\\
&\qquad +\gamma^3))/(\gamma(\gamma + (a+f)(\gamma^2-1))),\\
&\alpha_3 = (2a\beta - \gamma^2 - 2f\beta\gamma^2 + \beta\gamma^3 - \gamma^4)/(\beta^2\gamma(\gamma + (a+f)(\gamma^2-1)));
\end{aligned}$$

in *Case* (21):

$$\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = (\beta-\gamma)x + y - \beta,\\
&\mathcal{L}_3 = \exp[(f\beta^2 y + \gamma^2 - d\beta - 3\beta\gamma - c\beta\gamma - f\beta^2\gamma - 1)/(x-1)],\\
&\alpha_1 = -((c+2)^2\beta^2\gamma(1+2\gamma^2) + \beta(c+2)(2\beta^2\gamma^2 - 2\gamma^4 + (1+d\beta)(1+3\gamma^2)\\
&\qquad +f\beta(\beta^2 + 2\gamma^2(1+\beta^2+\gamma^2))) + \beta^2\gamma d^2 + \beta\gamma d(3 + 2f\beta + \beta^2 + f\beta^3 - \gamma^2\\
&\qquad +2f\beta\gamma^2) - 2\gamma(-(1+f\beta)(1+f\beta^3+f\beta^3\gamma^2) - 2\beta^2 + \gamma^2 + f\beta\gamma^4))/(\beta^2\gamma),\\
&\alpha_2 = \beta((c+2)(f\beta + 2\gamma^2 + 2f\beta\gamma^2) + \gamma(1+f\beta)(d + 2f + 2f\gamma^2))/\gamma,\\
&\alpha_3 = -\beta(c+2)(1+2\gamma^2) + 2\gamma + d\beta\gamma + 2f\beta\gamma(1+\gamma^2)/(\beta^2\gamma).
\end{aligned}$$

□

Lemma 3.3. *The following three sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$\begin{aligned}
&a = 0, b = 1, d = -1/\beta, f = 0, g = -1 - q\beta, k = 0,\\
&l = 0, m = -1 - c, n = 1 + c, p = 0, r = 0, s = q\beta;
\end{aligned} \tag{23}$$

$$\begin{aligned}
&a = 0, b = 1, c = -2, d = -1/\beta, g = -2, k = 0, l = f,\\
&m = 1, n = -1, p = -f, q = 1/\beta, r = 0, s = 1;
\end{aligned} \tag{24}$$

$$\begin{aligned}
&a = \gamma^2(1+\beta\gamma-\gamma^2)/(2\beta(1-\gamma^2)),\\
&b = (2\beta+\beta\gamma^2(\gamma^2-4)-\gamma(\gamma^2-1)(\gamma^2-2))/(2\beta(1-\gamma^2)),\\
&c = (\beta\gamma^2+2\gamma^3+\beta\gamma^4-\gamma^5-4\beta-\gamma)/(2\beta(1-\gamma^2)),\\
&d = (\gamma^4(\gamma^2-\beta\gamma-4)+(1+\beta\gamma)(5\gamma^2-2))/(2\beta(1-\gamma^2)),\\
&f = \gamma(\beta\gamma^2-\gamma^3-2\beta+\gamma)/(2\beta(\gamma^2-1)),\\
&g = (\gamma^2-2)(\beta\gamma^2-\gamma^3-2\beta+\gamma)/(2\beta(\gamma^2-1)),\\
&k = \gamma^2(1+\beta\gamma-\gamma^2)/(2\beta(\gamma^2-1)),\\
&l = \gamma(\beta\gamma^2-\gamma^3-2\beta+\gamma)/(2\beta(\gamma^2-1)),\\
&m = (2\beta+\gamma(\gamma^2-1)(\gamma^2-\beta\gamma-1))/(2\beta(1-\gamma^2)),\\
&n = (1+\gamma^2)((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(1-\gamma^2)),\\
&p = \gamma((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(1-\gamma^2)),\\
&q = (1+\beta\gamma-\gamma^2)(\gamma^2-2)/(2\beta(\gamma^2-1)), r=0, s=(\beta-\gamma)/\beta.
\end{aligned} \tag{25}$$

Proof. If one of the conditions (23) - (25) holds, then the cubic system (3) has a first integral $F(x, y)$ of the form

$$\mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\mathcal{L}_3^{\alpha_3}\mathcal{L}_4^{\alpha_4}. \tag{26}$$

In *Case* (23):

$$\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = \beta x+y-\beta,\ \mathcal{L}_3 = \exp[(\beta(c+2)y+q\beta-1)/(x-1)],\\
&\mathcal{L}_4 = (c+1)x+1,\ \alpha_1 = (1+c)(1-3q\beta-cq\beta+(2+c)^2\beta^2),\\
&\alpha_2 = -(1+c)(2+c)^2\beta^2,\ \alpha_3 = (1+c)(2+c),\ \alpha_4 = -1-c-q\beta;
\end{aligned}$$

in *Case* (24):

$$\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = \beta x+y-\beta,\ \mathcal{L}_3 = \exp[y/(x-1)],\\
&\mathcal{L}_4 = \exp[(x^2+f\beta y^2-1)/(x-1)^2],\ \alpha_1 = 2(1+\beta^2+f\beta^3),\\
&\alpha_2 = -2\beta^2(1+f\beta),\ \alpha_3 = 2\beta(1+f\beta),\ \alpha_4 = -1;
\end{aligned}$$

in *Case* (25):

$$\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = (\beta-\gamma)x+y-\beta,\\
&\mathcal{L}_4 = ((\beta-\gamma)(\gamma^2-1)-\beta)(x+\gamma y)+2\beta,\\
&\mathcal{L}_3 = \exp[\beta\gamma(\gamma-y)(\gamma+\beta\gamma^2-\gamma^3-2\beta)/(2(x-1)(1-\gamma^2))],\\
&\alpha_1 = \beta(2+\beta\gamma)(1-\beta\gamma+\gamma^2)((\beta-\gamma)(\gamma^2-1)-\beta),\\
&\alpha_2 = \beta^2\gamma(1+\beta\gamma-\gamma^2)((\beta-\gamma)(\gamma^2-1)-\beta),\\
&\alpha_3 = 2(\gamma^2-1)(1-\beta\gamma+\gamma^2),\ \alpha_4 = 4\beta^2.
\end{aligned}$$

□

Lemma 3.4. *The following sets of conditions are sufficient for the origin* $(0, 0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&a = 0, b = 1, c = (3\gamma^2 - 2)/(1 - \gamma^2), d = \gamma^3/(\gamma^2 - 1), f = \gamma/(1 - \gamma^2),\\
&g = (\gamma^2 - 2)/(1 - \gamma^2), k = 0, l = \gamma/(1 - \gamma^2), m = (2\gamma^2 - 1)/(\gamma^2 - 1),\\
&n = -1, p = \gamma/(\gamma^2 - 1), q = 0, r = 0, s = 1/(1 - \gamma^2);
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&a = (4\gamma^3)/((\gamma^2 - 3)(9 + \gamma^2)), b = ((3 + \gamma^2)(5\gamma^2 - 9))/((\gamma^2 - 3)(9 + \gamma^2)),\\
&c = -18/(9 + \gamma^2), d = (\gamma(9 + 16\gamma^2 - \gamma^4))/((\gamma^2 - 3)(9 + \gamma^2)),\\
&f = (6\gamma(3 + \gamma^2))/((3 - \gamma^2)(9 + \gamma^2)), l = (6\gamma(3 + \gamma^2))/((3 - \gamma^2)(9 + \gamma^2)),\\
&k = (4\gamma^3)/((3 - \gamma^2)(9 + \gamma^2)), g = (2(3 + \gamma^2)(2\gamma^2 - 9))/((3 - \gamma^2)(9 + \gamma^2)),\\
&m = (9 - \gamma^2)/(9 + \gamma^2), n = (3((3 + \gamma^2)^2 - 4\gamma^2))/((\gamma^2 - 3)(9 + \gamma^2)),\\
&p = (6\gamma(3 + \gamma^2))/((\gamma^2 - 3)(9 + \gamma^2)), q = (\gamma(3 + \gamma^2)^2)/((3 - \gamma^2)(9 + \gamma^2)),\\
&r = 0, s = (3 + \gamma^2)/(3 - \gamma^2);
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&a = \delta(1 + \beta\delta)/\nu,\ b = (\beta^2\delta^2 - \delta^2 - \beta\delta - 1)/(\beta\delta\nu),\ c = -(1 + 3\beta\delta + 3\delta^2)/\nu,\\
&d = ((1 + \beta\delta)(1 + 2\beta\delta + \beta\delta^3) - \delta^4)/(\beta\delta^2\nu),\ f = -\delta/\nu,\ k = -(\delta(1 + \beta\delta))/\nu,\\
&g = (\nu - \beta\delta - 2\beta^2\delta^2)/(\beta\delta\nu),\ l = -\delta/\nu,\ m = (\delta(\beta + 2\delta))/\nu,\ n = (1 - \delta^2)/\nu,\\
&p = \delta/\nu,\ q = -(1 + \beta\delta)/(\delta\nu),\ r = 0,\ s = (\beta\delta)/\nu,\ \textit{where}\ \nu = 1 + 2\beta\delta + \delta^2;
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&a = (\delta - \beta)/\sigma,\ b = \delta(\beta^2 + \beta\delta - \delta^2 - 1)/(\beta\sigma),\ c = (3 - 3\beta\delta + \delta^2)/\sigma,\\
&d = (1 - (\beta - \delta)(\beta + 2\beta\delta^2 - \delta^3))/(\beta\sigma),\ f = -\delta/\sigma,\ k = (\beta - \delta)/\sigma,\\
&g = -\delta(2\beta^2 + \beta\delta - \delta^2 - 1)/(\beta\sigma),\ l = -\delta/\sigma,\ m = (\beta\delta - 2)/\sigma,\\
&n = (1 - \delta^2)/\sigma,\ p = \delta/\sigma,\ q = (\beta - \delta)\delta^2/\sigma,\ r = 0,\ s = \beta\delta/\sigma,\\
&\textit{where}\ \ \sigma = 2\beta\delta - \delta^2 - 1;
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&b = 2(1 + a^2),\ c = 2a^2 - 3,\ d = (1 + a^2)/a,\ f = -2a,\ g = -2,\ k = -a,\\
&l = -2a,\ m = 2(1 - a^2),\ n = 4a^2 - 1,\ p = 2a,\ q = -4a^3,\ r = 0,\ s = -2a^2;
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&a = (u^2 - 4f^2 - 1)/(8f),\\
&b = (32f^2 + 16f^4 \mp 32f^2u + 2u^2 - u^4 - 1)/(4(1 \mp u)(-1 + 4f^2 + u^2)),\\
&c = (4f^2 \pm 12u - 3u^2 - 9)/(4(1 \mp u)),\ d = ((4f^2 - 1)^2 + u^2(u^2 - 56f^2 - 2)\\
&\qquad \mp u((u^2 - 1)^2 - 8f^2(7 + 6f^2 + u^2)))/(8f(1 \mp u)(4f^2 + u^2 - 1)),\\
&g = (1 - 20f^2 - u^2 \mp u(4f^2 + u^2 - 1))/(2(4f^2 + u^2 - 1)),\\
&k = (1 + 4f^2 - u^2)/(8f),\ l = f,\ m = (3u^2 \mp 8u - 4f^2 + 5)/(4(1 \mp u)),\\
&n = (4f^2 - u^2 - 1 \pm 2u)/(2(1 \mp u)),\ s = (4f^2 + u^2 - 1)/(4(-1 \pm u)),\\
&p = -f,\ q = f(1 + 4f^2 - u^2)/(2(u \mp 1)^2),\ r = 0.
\end{aligned}
\tag{32}
$$

Proof. In each of the sets of conditions (27)–(32), the system (3) has an integrating factor of the form (26) $\mu(x, y) = \mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\mathcal{L}_3^{\alpha_3}\mathcal{L}_4^{\alpha_4}$.

In *Case* (27):

$$\mathcal{L}_1 = x - 1;\ \mathcal{L}_2 = x - \gamma y + \gamma^2 - 1,\ \mathcal{L}_3 = \exp[(1-\gamma^2)(y-\gamma)/(\gamma(x-1))],$$
$$\mathcal{L}_4 = x - \gamma y - 1,\ \alpha_1 = -3,\ \alpha_2 = 1,$$
$$\alpha_3 = -\gamma^2/(\gamma^2-1)^2,\ \alpha_4 = (\gamma^2-2)/(1-\gamma^2);$$

in *Case* (28):

$$\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = (\gamma^2+3)x - 2y\gamma + \gamma^2 - 3,$$
$$\mathcal{L}_3 = \exp[3(\gamma - y)(\gamma^4 - 9)/(2\gamma(x-1)(9+\gamma^2))],$$
$$\mathcal{L}_4 = 3(3+\gamma^2)x - 6y\gamma - \gamma^2 - 9,\ \alpha_1 = -3,\ \alpha_2 = 1,$$
$$\alpha_3 = -2\gamma^2(9+\gamma^2)/(3(\gamma^2-3)^2),\ \alpha_4 = (18+3\gamma^2+\gamma^4)/(3(\gamma^2-3));$$

in *Case* (29):

$$\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = x - y\delta + \beta\delta,\ \mathcal{L}_3 = \exp[1 + \beta\delta + \delta^2 + y\beta\delta^2/(x-1)],$$
$$\mathcal{L}_4 = \beta\delta(x - y\delta) - \delta^2 - 2\beta\delta - 1,\ \alpha_1 = -3,\ \alpha_4 = 1,$$
$$\alpha_2 = (1 + 2\beta\delta + \beta^2\delta^2 - 2\beta\delta^3 - \delta^4)/(\delta^2\nu),\ \alpha_3 = (1+\beta\delta+\delta^2)/(\beta\delta^3\nu);$$

in *Case* (30):

$$\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = y - \beta + x\delta,\ \mathcal{L}_3 = \exp[(1 + y\beta - \beta\delta + \delta^2)/(x-1)],$$
$$\mathcal{L}_4 = \beta(\delta x + y) - \sigma,\ \ \alpha_1 = -3,\ \alpha_2 = -(-1 + 2\beta\delta + \beta^2\delta^2 - 2\beta]\delta^3 + \delta^4)/\sigma,$$
$$\alpha_3 = -(\delta(-1+\beta\delta-\delta^2))/(\beta\sigma),\ \alpha_4 = 1;$$

in *Case* (31):

$$\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = 2a^2x - ay + 1,\ \mathcal{L}_3 = \exp[2(1+2a^2-ay)/(a^2(x-1))],$$
$$\mathcal{L}_4 = ax - y - a,\ \alpha_1 = -3,\ \alpha_2 = 1,\ \alpha_3 = -a^2(1+2a^2),\ \alpha_4 = 4a^4 + 4a^2 - 1;$$

in *Case* (32):

$$\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = (4f^2 + u^2 - 1)(2fx + (1 \mp u)y) + 8f(1 \mp u),$$
$$\mathcal{L}_3 = \exp[\tfrac{(64f^3(2f(3+4f^2+u^2\mp 4u)+(1\mp u)(4f^2+u^2-1)y))}{((1\mp u)(1+4f^2+u^2-1)^3(x-1))}],$$
$$\mathcal{L}_4 = 8f^2x + 4f(1 \mp u)y + 1 - 4f^2 - u^2,\ \alpha_1 = -3,\ \alpha_2 = 1,$$
$$\alpha_3 = -(((-1+4f^2+u^2)^2(3+4f^2 \mp 4u + u^2))/(128f^2(u \mp 1)^2)),$$
$$\alpha_4 = ((4f^2-1)(7+4f^2) \pm 8u(3-4f^2+u^2) + u^2(8f^2+u^2-26))/(8(1\mp u)^3).$$

□

Lemma 3.5. *The following three sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$\begin{aligned} &a = 0, b = 1, c = (3\gamma^2-2)/(1-\gamma^2), d = \gamma^3/(\gamma^2-1), f = \gamma/(1-\gamma^2),\\ &g = (\gamma^2-2)/(1-\gamma^2), k = 0, l = \gamma/(1-\gamma^2), m = (2\gamma^2-1)/(\gamma^2-1),\\ &n = -1, p = \gamma/(\gamma^2-1), q = 0, r = 0, s = 1/(1-\gamma^2); \end{aligned} \tag{33}$$

$$
\begin{aligned}
&a = -k = \gamma\delta(4 + 2c + 3\gamma^2 + 2c\gamma^2),\ b = -\delta(10 + 4c + 7\gamma^2 + 4c\gamma^2),\\
&d = \delta((1 + \gamma^2)(9\gamma^2 + 6c\gamma^2 - 4c) - 8)/\gamma,\ f = l = -p = \gamma\delta,\\
&g = \delta(16 + 6c + 11\gamma^2 + 6c\gamma^2),\ m = -(c + 1),\ n = 2\delta,\\
&q = \delta(24 + 23\gamma^2 + 6\gamma^4 + 2c(1 + \gamma^2)(10 + 2c + 5\gamma^2 + 2c\gamma^2))/\gamma,\ r = 0,\\
&s = -\delta(5 + 2c + 3\gamma^2 + 2c\gamma^2)(6 + 2c + 3\gamma^2 + 2c\gamma^2),\ \delta = -1/(2(1 + \gamma^2));
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&a = -1/(2\gamma),\ b = 1/2,\ c = -3/2,\ d = -(1 + 2\gamma^2)/(2\gamma),\\
&f = l = n = p = r = 0,\ g = -1,\ k = 1/(2\gamma),\ m = 1/2,\ q = \gamma/2,\ s = 1/2.
\end{aligned} \tag{35}
$$

Proof. In this Lemma, the existence of the center is guaranteed by the presence of the integrating factor of the form

$$\mu(x, y) = \mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\Phi^{\alpha_3},$$

where $\mathcal{L}_1$, $\mathcal{L}_2$ are invariant straight lines and Φ is an invariant conic. Indeed,
in Case (33):

$$
\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = y - \beta,\ \Phi = 4(1 + \beta^2)(x - 1) - (x + y\beta)^2,\\
&\alpha_1 = -2,\ \alpha_2 = -1,\ \alpha_3 = -1/2;
\end{aligned}
$$

in Case (34):

$$
\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = \gamma(1 + \gamma^2) + 2y(2 + c + \gamma^2 + c\gamma^2) - x\gamma(5 + 2c + 3\gamma^2 + 2c\gamma]2),\\
&\Phi = 4(1 + \gamma^2) - 8x(3 + c + 2\gamma^2 + c\gamma^2) - 4y\gamma(5 + 2c + 3\gamma^2 + 2c\gamma^2)\\
&\qquad +(6x + 2cx + y\gamma + 3x\gamma^2 + 2cx\gamma^2)^2,\ \alpha_1 = -2,\ \alpha_2 = -1,\ \alpha_3 = -1/2;
\end{aligned}
$$

in Case (35):

$$
\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = \gamma x + y - 2\gamma,\ \Phi = 2(1 + \gamma^2) - (x + \gamma y)(2(1 + \gamma^2) - \gamma(\gamma x + y)),\\
&\alpha_1 = -2,\ \alpha_2 = 1,\ \alpha_3 = -1.
\end{aligned}
$$

□

Lemma 3.6. *The following set of conditions is sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&a = -(f(1-6u^2+u^4))/(2(1-u^2)^2),\\
&b = (f(1+u^2)^2(1-6u^2+u^4)+32u^3(1-u^2))/(4(1-u^2)u(1+u^2)^2),\\
&c = (f(1-14u^2+u^4)+8u(u^2-1))/(4(1-u^2)u),\\
&d = (3-u^2)(3u^2-1)(f(1+u^2)^2+4u(u^2-1))/(2(u^2-1)^2(1+u^2)^2),\\
&g = (2fu(1+u^2)^2+(u^2-1)(1+10u^2+u^4))/((1-u^2)(1+u^2)^2),\\
&k = (f(1-6u^2+u^4))/(2(u^2-1)^2),\ l = f,\\
&m = (f(1-14u^2+u^4)+4u(u^2-1))/(4u(u^2-1)),\\
&n = (4u(f(1+u^2)^2-2u(u^2-1)))/(u^2-1)(1+u^2)^2,\ p=-f,\\
&q = (f(1+u^2)^2(1+10u^2+u^4)+4(u^2-1)u(u^2-3)(3u^2-1))\\
&\qquad /(2(u^2-1)^2(1+u^2)^2),\ r=0,\\
&s = (u(f(1+u^2)^4+8u(u^2-1)^3))/((u^2-1)^3(1+u^2)^2);
\end{aligned}
\tag{36}
$$

Proof. Under the conditions (36), there is an invertible transformation of the form

$$
x = \frac{a_1X+b_1Y}{a_3X+B_3Y-1},\quad y = \frac{a_2X+b_2Y}{a_3X+B_3Y-1}, \tag{37}
$$

in a neighborhood of $O(0,0)$, where $a_j, b_j \in \mathbb{R}$, $j=1,2,3$ and $a_1b_2-b_1a_2 \neq 0$. The transformation (37) brings the system (3) to the polynomial system

$$
\dot{X} = Y + M(X^2,Y),\quad \dot{Y} = -X(1+N(X^2,Y)). \tag{38}
$$

This system has an axis of symmetry $X=0$ and therefore $O(0,0)$ is a center for (38) and for the initial system (3) (see, [2], pp.29-31).

In *Case* (36) the transformation (37) looks as

$$
x = \frac{2uX+(1-u^2)Y}{2uX-u^2-1},\quad y = \frac{(u^2-1)X+2uY}{2uX-u^2-1}
$$

and the system (38) has the form

$$
\begin{aligned}
\dot{X} &= Y + (4fu^2(1+u^2)^4X^4 - X^2(f(1+u^2)^6+16u^3(u^2-1)^2(1+u^2)Y-\\
&\quad 16u^3(u^2-1)^3Y^2) - 4u(u^2-1)^3(1+u^2)^2Y^2)/(4u(u^2-1)^2(1+u^2)^3),\\
\dot{Y} &= X(1+(4X^2fu^2(1+u^2)^4(1+u^2+(1-u^2)Y)+(1-u^2)(1+u^2)^2(f-8u\\
&\quad +4fu^2+8u^3+6fu^4-8u^5+4fu^6+8u^7+fu^8)Y+(u^2-1)^2(1+u^2)(f-4u\\
&\quad +4fu^2-20u^3+6fu^4+20u^5+4fu^6+4u^7+fu^8)Y^2\\
&\quad -16u^3(u^2-1)^4Y^3)/(4u(-1+u^4)^3).
\end{aligned}
$$

□

4. Solution of the problem of the center

4.1. Centers in the conditions (7).

Lemma 4.1. *Under the conditions* (7) *the system* (3) *has a center at the origin* $(0,0)$ *if and only if the first four Lyapunov quantities vanish* $L_1 = L_2 = L_3 = L_4 = 0$.

Proof. The Lemma 4.1 is proved in {[2], pp. 111–116}. □

4.2. Centers in the conditions (12).

Let $\alpha = \gamma - \beta$.

Lemma 4.2. *When conditions* (12) *hold, the system* (3) *has a center at the origin* $(0,0)$ *if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

Proof. In conditions (12) the first Lyapunov quantity is $L_1 = f_0 f_1 f_2$, where $f_0 = \gamma$, $f_1 = c\beta + \beta + \gamma$, $f_2 = q\beta + (1+c)(\beta - \gamma)\gamma - c - 3$. If $f_0 = 0$, then Lemma {3.3, (23)} and if $f_1 = 0$, then Lemma {3.1, (15)}. Assume that $f_0 f_1 \neq 0$ and let $f_2 = 0$. Then $q = (c + 3 + \gamma(1+c)(\gamma - \beta))/\beta$ and $L_2 = (c+2)(2c\beta + 5\beta + \gamma)$. If $c + 2 = 0$, then $\gcd(P, Q) = x - 1$. Let $2c\beta + 5\beta + \gamma = 0 \Rightarrow \gamma = -\beta(5 + 2c) \Rightarrow L_3 = c + 3 = 0 \Rightarrow$ Lemma {3.2, (16)}. □

4.3. Centers in the conditions (13).

Lemma 4.3. *Under the conditions* {(13), $\alpha = -\beta$}, *the system* (3) *has a center at the origin* $(0,0)$ *if and only if the first four Lyapunov quantities vanish* $L_1 = L_2 = L_3 = L_4 = 0$.

Proof. The system {(3), (13), $\alpha = -\beta$ } has the invariant straight lines $\mathcal{L}_1 = x - 1$, $\mathcal{L}_2 = \beta x + y - \beta$ and the exponential factor $\mathcal{L}_3 = \exp[y/(x-1)]$. For {(3), (13), $\alpha = -\beta$ } we calculate at $(0,0)$ the first four Lyapunov quantities L_1, L_2, L_3 and L_4. In the sequel, in expressions of Lyapunov quantities we always neglect the non-zero factors. The first one look as

$$L_1 = ac + (c+2)f.$$

If $c = 0$, then {Lemma 3.2, (17)}. Let $c \neq 0$. Then $L_1 = 0 \Rightarrow a = -f(c+2)/c \Rightarrow L_2 = f(c+2)l_2$, where

$$l_2 = c^3\beta + 2c^2\beta + 6cf - 12f^2\beta + 4cf^2\beta.$$

If $f = 0$, then {Lemma 3.3, (23), $q = 1/\beta$}, and if $c = -2$, then {Lemma 3.3, (24)}. Suppose $cf(c+2) \neq 0$. Reducing L_3 by l_2 with respect to the variable c we obtain

$L_3 = (c - 2f\beta)(12f\beta + cf\beta - c)$. If $c - 2f\beta = 0$, then $l_2 = f(1+\beta^2) + \beta = 0 \Rightarrow f = -\beta/(1+\beta^2) \Rightarrow$ {Lemma 3.2, (18)}. If $12f\beta + c(f\beta - 1) = 0 \Rightarrow c = 12f\beta/(1 - f\beta) \Rightarrow$

$$l_2 = 24\beta^2(5f\beta + 1) + 5(f\beta - 1)^2(f\beta + 1),$$
$$L_4 = 15(f\beta + 1)(17f\beta - 11) + 50\beta^4(24 - 43f^2 + 120f\beta) - 2\beta^2(271 + 2290f\beta).$$

The system $\{l_2 = 0, L_4 = 0\}$ has no real solutions with respect to the unknowns f and β. □

Lemma 4.4. *Let the conditions* {(13)} *hold for the system* (3). *Then the origin* (0,0) *is a center if and only if the first four Lyapunov quantities vanish* $L_1 = L_2 = L_3 = L_4 = L_5 = 0$.

Proof. Denote $\alpha + \beta = \gamma \neq 0$. Then under the conditions $\{(13), \alpha = \gamma - \beta\}$ we calculate the first five Lyapunov quantities. The first one look as $L_1 = A_c \cdot c + B_c$, where

$$A_c = \beta((a + f)(1 - \gamma^2) - \gamma),$$
$$B_c = 2\beta(f - \gamma) - (a + f)\gamma(-1 + 2a\beta - 2f\beta + 3\beta\gamma - \gamma^2).$$

Remark that in conditions $\{(13), \alpha = \gamma - \beta\}$ the system (3) has the invariant straight lines $\mathcal{L}_\infty = x - 1$, $\mathcal{L}_2 = (\beta - \gamma)x + y - \beta]$ and the exponential factor $\mathcal{L}_3 = \exp[(fy\beta^2 + \gamma^2 - \beta\gamma(3 + c + f\beta) - d\beta - 1)/(x - 1)]$.

Let $A_c = 0$. Then $\gamma^2 - 1 \neq 0$ and the system $\{A_c = 0,\ B_c = 0\}$ gives us

$$a = \gamma^2(1 + \beta\gamma - \gamma^2)/(2\beta(1 - \gamma^2)),\ f = \gamma(\gamma - 2\beta + \beta\gamma^2 - \gamma^3)/(2\beta(\gamma^2 - 1)).$$

Substituting the expression of a and f into L_2, L_3, L_4 and L_5 we obtain that $L_2 = \gamma f_0 f_1 f_2$, $L_3 = \gamma f_0 f_1 f_3$, $L_4 = \gamma f_0 f_1 f_4$, $L_5 = \gamma f_0 f_1 f_5$, where

$$f_0 = c\beta(\gamma^2 - 1) + \gamma(\gamma^2 + \beta\gamma - 1),$$
$$f_1 = 4\beta - 2c\beta(\gamma^2 - 1) + \gamma(\gamma^2 - 1)^2 - \beta\gamma^2(\gamma^2 + 1),$$
$$f_2 = (3\gamma - c\beta)(\gamma^2 - 1) - 2\beta(3\gamma^2 - 1)$$

and f_3, f_4, f_5 are polynomials in variables c, β, γ. If $f_0 = 0$, then {Lemma 3.2, (19)}, and if $f_1 = 0$, then {Lemma 3.3, (25)}.

Assume that $(\gamma^2 - 1)f_0 f_1 \neq 0$ and let $f_2 = 0$. From $f_2 = 0$ we calculate $c = (6\beta\gamma^2 - 3\gamma^3 - 2\beta + 3\gamma)/(\beta(1 - \gamma^2))$ and substitute it in L_3, L_4, L_5: $L_3 = \varphi_0\varphi_3, L_4 = \varphi_0\varphi_4, L_5 = \varphi_0\varphi_5$, where

$$\varphi_0 = 1 + \beta\gamma - \gamma^2,\ \varphi_3 = \gamma(14 + 17\gamma^2 - \gamma^4)\beta + (1 - \gamma^2)(2 + 9\gamma^2 - \gamma^4),$$

and φ_4, φ_5 are polynomials in β, γ. If $\varphi_0 = 0$, then {Lemma 3.4, (27)}, and if $\varphi_3 = 0$, then $\beta = (\gamma^2 - 1)(2 + 9\gamma^2 - \gamma^4)/(\gamma(14 + 17\gamma^2 - \gamma^4)) \Rightarrow$

$$\begin{aligned}
\varphi_4 &= 212 - 976\gamma^2 + 149\gamma^4 + 39\gamma^6 - 21\gamma^8 - 3\gamma^{10},\\
\varphi_5 &= 341744 + 8972240\gamma^2 + 20117464\gamma^4 - 330716296\gamma^6 + 161032455\gamma^8\\
&\quad -51877847\gamma^{10} - 26524163\gamma^{12} + 4576265\gamma^{14} + 470531\gamma^{16} - 577165\gamma^{18}\\
&\quad -126177\gamma^{20} - 8877\gamma^{22} - 174\gamma^{24}.
\end{aligned}$$

The polynomials φ_4 and φ_5 have no common solutions.

Let now $\beta\gamma A_c \neq 0$ and express c from $L_1 = 0$. Substituting

$$c = (2\beta(f-\gamma) - \gamma(a+f)(2a\beta - 2f\beta + 3\beta\gamma - \gamma^2 - 1))/(\beta(\gamma + (a+f)(\gamma^2 - 1)))$$

in L_2, L_3, L_4 and L_5 we obtain that $L_2 = \psi_0\psi_2$, $L_3 = \psi_0\psi_3$, $L_4 = \psi_0\psi_4$, $L_5 = \psi_0\psi_5$, where

$$\begin{aligned}
\psi_0 &= ((\beta-\gamma)\gamma^3 - 2\beta(f - \gamma - a\gamma^2) - \gamma^2)(a\beta\gamma + (a+f)(a\beta - \gamma^2 - f\beta\gamma^2)),\\
\psi_2 &= A_\beta \cdot \beta + B_\beta,\\
A_\beta &= 2f + 2(a+f)(2f\gamma - 1) + 2(a+f)^2(f - 4\gamma + f\gamma^2) + (a+f)^3(3 - 5\gamma^2),\\
B_\beta &= (a+f)((a+f)(3-\gamma^2) - 2\gamma)
\end{aligned}$$

and ψ_3, ψ_4, ψ_5 are polynomials in a, f, β, γ. If $\psi_0 = 0$, then {Lemma 3.2, (20)}. Let $\psi_0 \neq 0$. If $a + f = 0$, then $\psi_2 = 0 \Rightarrow a = f = 0 \Rightarrow \gcd(P, Q) = x - 1 \neq 1$. The system $\{A_\beta = 0,\ B_\beta = 0,\ \gamma(a+f) \neq 0\} \Rightarrow$ gives us

$$a = 4\gamma^3/((\gamma^2 - 3)(9 + \gamma^2)),\ f = 6\gamma(3 + \gamma^2)/((3 - \gamma^2)(9 + \gamma^2))$$

$\Rightarrow \psi_3 = \eta_0\eta_3$, $\psi_4 = \eta_0\eta_4$, $\psi_5 = \eta_0\eta_5$, where

$$\eta_0 = 3 + 2\beta\gamma - \gamma^2,\ \eta_3 = 16\gamma(9 + 9\gamma^2 + \gamma^4)\beta - (\gamma^2 + 9)(\gamma^2 - 3)(1 + 3\gamma^2).$$

If $\eta_0 = 0$, then {Lemma 3.4,(28)}. If $\eta_3 = 0$, then $\beta = (\gamma^2 - 3)(9 + \gamma^2)(1 + 3\gamma^2)/(16\gamma(9 + 9\gamma^2 + \gamma^4))$ and

$$\begin{aligned}
\eta_4 &= 1359 - 3582\gamma^2 - 1524\gamma^4 - 178\gamma^6 - 11\gamma^8,\\
\eta_5 &= 399256533 + 7147083924\gamma^2 + 16160765949\gamma^4 - 88245537822\gamma^6\\
&\quad -98340968934\gamma^8 - 42412220400\gamma^{10} - 10500825742\gamma^{12} - 1982042948\gamma^{14}\\
&\quad -323807311\gamma^{16} - 38802452\gamma^{18} - 2858767\gamma^{20} - 107822\gamma^{22}.
\end{aligned}$$

The polynomials η_4 and η_5 have not common roots.

Suppose now that $\beta\gamma(f + a)A_c A_\beta \neq 0$. From $\psi_2 = 0$ we express β : $\beta = -B_\beta/A_\beta$ and substitute it in ψ_3, ψ_4, ψ_5. We obtain: $\psi_3 = \delta_0\delta_3$, $\psi_4 = \delta_0\delta_4$, $\psi_5 = \delta_0\delta_5$, where

$$\delta_0 = a - f(a+f)^2 + (a+f)(3a+f)\gamma + (a+f)^2(2a+f)\gamma^2.$$

First we will examine the equality $\delta_0 = 0$. Each of the following three sets 1) $\{f = 0, a = -(1/\gamma)\}$, 2) $\{f = 0, a = -1/(2\gamma)\}$, 3) $\{f = -2a \neq 0, \gamma = (1+2a^2)/a\}$ vanish δ_0. In the case 1) (respectively, 2), 3)) we have Lemma $\{3.2, (16), \beta = -1/\gamma\}$ (respectively, Lemma 3.5, (35), Lemma 3.4, (31)). Suppose $f(f+2a) \neq 0$ and $a = (u^2-4f^2-1)/(8f)$. Then $\delta_0 = 0 \Rightarrow \gamma = 2f(3+4f^2 \mp 4u + u^2)/((-1 \pm u)(4f^2+u^2-1)) \Rightarrow$ Lemma 3.4, (32).

Let $\delta_0 \neq 0$ and $\delta_3 = \delta_4 = \delta_5 = 0$. Suppose that $\mathcal{R}_0 = 0$, where $\mathcal{R}_0 = (\gamma^2 - 3)(9f + 6\gamma + f\gamma^2) + 36\gamma$. Taking into account that $\gamma \neq 0$, the equality $\mathcal{R}_0 = 0$ gives us $f = 6\gamma(3+\gamma^2)/((3-\gamma^2)(9+\gamma^2))$. Substituting the expression of f in $\delta_3, \delta_4, \delta_5$, we obtain: $\delta_3 = \Delta_0\Delta_3$, $\delta_4 = \Delta_0\Delta_4$, $\delta_5 = \Delta_0\Delta_5$, where

$$
\begin{aligned}
&\Delta_0 = (3-\gamma^2)(9+\gamma^2)a + 4\gamma^3,\\
&\Delta_3 = 5a^2(\gamma^2-3)^2(1+\gamma^2)(3+\gamma^2)(9+\gamma^2)^2-\\
&8a\gamma(\gamma^2-3)(9+\gamma^2)(117+129\gamma^2+49\gamma^4+5\gamma^6)+\\
&6(3+\gamma^2)(81+702\gamma^2+540\gamma^4+186\gamma^6+11\gamma^8)
\end{aligned}
$$

and Δ_4, Δ_5 are polynomial in a, γ (here and in the future we neglect the nonzero factors). If $\Delta_0 = 0$, then $\delta_0 = 0$. We calculate the following two resultants with respect to the variable a:

$$
\begin{aligned}
Ra43 \equiv Resultant[\Delta_4, \Delta_3, a] = {}& -5314410 - 10333575\gamma^2 - 10482291\gamma^4\\
&-7030476\gamma^6 - 3361176\gamma^8 - 757026\gamma^{10} + 22734\gamma^{12} + 52740\gamma^{14} + 11850\gamma^{16}\\
&-5775\gamma^{18} + 605\gamma^{20},\\
Ra53 \equiv Resultant[\Delta_5, \Delta_3, a] = {}& -204368836893016410 - 781248520450475667 45\gamma^2\\
&-16375606609634273122 2\gamma^4 - 96691944562452884637\gamma^6\\
&-65295919868671875114\gamma^8 - 23151483327541830504 3\gamma^{10}\\
&-34981897148362181939 4\gamma^{12} - 22516518347763889041 9\gamma^{14}\\
&+80965486019817254 16\gamma^{16} + 15613396708709900471 4\gamma^{18}\\
&+15740952222206914995 6\gamma^{20} + 75314811652151245182\gamma^{22}\\
&+41695659944998900 92\gamma^{24} - 20155578686541419814\gamma^{26}\\
&-15882244595493905700\gamma^{28} - 6818058573715824678\gamma^{30}\\
&-1850109425790483978\gamma^{32} - 301146961631290581\gamma^{34}\\
&-12297992828350350\gamma^{36} + 8047908867120815\gamma^{38} + 2104984082966110\gamma^{40}\\
&+224322029461865\gamma^{42} - 66037811138170\gamma^{44} - 14404146643895\gamma^{46}\\
&+4307153529500\gamma^{48} + 103033012500\gamma^{50}
\end{aligned}
$$

and the resultant with respect to the variable γ : $Resultant[Ra53, Ra43, \gamma] \neq 0$. Therefore, the system $\{\Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0\}$ is incompatible in rapport with the variables a and γ.

In what follows, in this subsection we will consider $\delta_0\mathcal{R}_0 \neq 0$ and using the resultants we solve the system of polynomials equations in a, f, γ : $\delta_3 = 0, \delta_4 = 0, \delta_5 = 0$. The system $f = 0, \delta_3 = 0, \delta_4 = 0, \delta_5 = 0, \delta_0 \neq 0$ has not real solutions. Eliminating nonzero factors such as $f, \gamma, 1+\gamma^2, 3+\gamma^2, 1+3\gamma^2, 2+\gamma^2, 403+1741\gamma^2+522\gamma^4, 72361+91494574\gamma^2+2288596483\gamma^4+5776070816\gamma^6+5787798803\gamma^8+2797313270\gamma^{10}+652276861\gamma^{12}+63507100\gamma^{14}+2102500\gamma^{16}, 61009+540628\gamma^2+1380646\gamma^4+606612\gamma^6+165105\gamma^8$, we calculate the resultants:

$$\begin{aligned}&\mathcal{R}a43 = Resultant[\delta_4, \delta_3, a],\ \mathcal{R}a53 = Resultant[\delta_5, \delta_3, a],\\ &\mathcal{R}a54 = Resultant[\delta_5, \delta_4, a], \mathcal{R}af_1 = Resultant[\mathcal{R}a53, \mathcal{R}a43, f],\\ &\mathcal{R}af_2 = Resultant[\mathcal{R}a54, \mathcal{R}a43, f].\end{aligned}$$

The polynomials in γ : $\mathcal{R}af_1, \mathcal{R}af_2$ have only the following common real solutions $\gamma = \pm 1,\ \gamma = \pm\sqrt{5}$. If $\gamma = \pm 1$, then

$$\mathcal{R}a43 = 0,\ \mathcal{R}a53 = 0, \mathcal{R}a54 = 0 \tag{39}$$

$\Rightarrow b = 0$, and if $\gamma = \pm\sqrt{5}$, then (39) has not real solutions. □

4.4. Centers in conditions (14).

Lemma 4.5. *In the conditions* (14) *the system* (3) *has a center at the origin* $(0, 0)$ *if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

Proof. Let $\alpha = \gamma - \beta$. For the system $\{(3), (14)\}$ we calculate at $(0, 0)$ the first three Lyapunov quantities L_1, L_2 and L_3. The Lyapunov quantity L_1 looks as $L_1 = f_0 f_1$, where

$$\begin{aligned}&f_0 = 2(c+2)^2\beta\gamma + (c+2)(1+d\beta - f\beta + 4f\beta\gamma^2) + f\beta\gamma(d - 2f + 2f\gamma^2),\\ &f_1 = 1 + d\beta + f\beta + 3\beta\gamma + c\beta\gamma - \gamma^2 + f\beta\gamma^2.\end{aligned}$$

If $f_0 = 0$, then Lemma $\{3.2, (21)\}$. Let $f_1 = 0$. Then

$$d = -(1 + f\beta + 3\beta\gamma + c\beta\gamma - \gamma^2 + f\beta\gamma^2)/\beta \Rightarrow L_2 = A_\beta\beta + B_\beta,$$

where

$$A_\beta = 10 + 9c + 2c^2 - 2f^2 + 17f\gamma + 8cf\gamma + 6f^2\gamma^2,\ B_\beta = \beta + 2\gamma + c\gamma + f\gamma^2 - f.$$

If $A_\beta = 0,\ B_\beta = 0$, then $c = -(5+3\gamma^2)/(2(1+\gamma^2)),\ f = -\gamma/(2(1+\gamma^2)) \Rightarrow L_3 = \beta - \gamma = 0 \Rightarrow$ Lemma $\{3.5, (33)\}$.

Let now $A_\beta \neq 0$. Then $L_2 = 0 \Rightarrow \beta = -B_\beta/A_\beta \Rightarrow L_3 = \varphi_1\varphi_2\varphi_3$, where

$$\begin{aligned}&\varphi_1 = 2f + \gamma + 2f\gamma^2,\ \varphi_2 = 4 + 4c + c^2 - f^2 + 8f\gamma + 4cf\gamma + 3f^2\gamma^2,\\ &\varphi_3 = 6 + 5c + c^2 - f^2 + 9f\gamma + 4cf\gamma + 3f^2\gamma^2.\end{aligned}$$

If $\varphi_1 = 0$, then Lemma $\{3.5, (34)\}$. Denote $\gamma(u) = (1 - 6u^2 + u^4)/(4u(u^2 - 1))$. Then $\varphi_2 = \varphi_{21}\varphi_{22}/(16u^2(u^2-1)^2)$, where $\varphi_{21} = f - 8u - 4cu - 14fu^2 + 8u^3 + 4cu^3 + fu^4$, $\varphi_{22} = 3f - 8u - 4cu - 10fu^2 + 8u^3 + 4cu^3 + 3fu^4$. If $\varphi_{21} = 0$, then Lemma $\{3.6, (36)\}$. The case $\{\gamma(u), \varphi_{22} = 0\}$ is reduced by transformation $u = (v - 1)/(1 + v)$ to the case $\{\gamma(v), \varphi_{21}|_{u=v} = 0\}$.

Let now $\varphi_3 = 0$ and put $\gamma = (f - \delta - f\delta^2)/(2f\delta)$. Then $\varphi_3 = \varphi_{31}\varphi_{32}/(4\delta^2)$, where $\varphi_{31} = f + 3\delta + 2c\delta - 3f\delta^2$ and $\varphi_{32} = 3f + 3\delta + 2c\delta - f\delta^2$. From $\varphi_{31} = 0$ we calculate $c: \ c = (3f\delta^2 - 3\delta - f)/(2\delta) \Rightarrow \beta = -(f + \delta + f\delta^2)/(2f\delta) \Rightarrow f = -\delta/(1 + 2\beta\delta + \delta^2) \Rightarrow$ Lemma $\{3.4, (29)\}$. In the case $\varphi_{32} = 0$ we have $c = (f\delta^2 - 3f - 3\delta)/(2\delta) \Rightarrow \beta = (f - \delta + f\delta^2)/(2f\delta) \Rightarrow f = \delta/(1 - 2\beta\delta + \delta^2) \Rightarrow$ Lemma $\{3.4, (30)\}$.

□

The statement of the Main Theorem follows from Lemmas 4.1 − 4.5.

References

[1] Christopher, C., Llibre, J. and Pereira, J.V. Multiplicity of invariant algebraic curves in polynomial vector fields. *Pacific J. of Math.*, 2007, vol. 229, no. 1, 63–117.

[2] Cozma, D. *Integrability of cubic systems with invariant straight lines and invariant conics.* Chişinău: Ştiinţa, 2013.

[3] Cozma, D. and Şubă, A. The solution of the problem of center for cubic differential systems with four invariant straight lines. *Scientific Annals of the "Al.I.Cuza" University, Mathematics*, 1998, vol. XLIV, s.I.a., 517–530.

[4] Dulac, H. Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre. *Bull. Sciences Math.*, 1908, vol. 32, 230–252.

[5] Puţuntică, V., Şubă, A. Cubic differential systems with affine real invariant straight lines of total parallel multiplicity six and configurations (3(m),1,1,1). *Acta et Commentationes. Exact and Natural Sciences*, 2018, vol. 6, no. 2, 93–116.

[6] Repeşco, V. Qualitative analysis of polynomial differential systems with the line at infinity of maximal multiplicity: exploring linear, quadratic, cubic, quartic, and quintic cases. *Acta et Commentationes. Exact and Natural Sciences*, 2023, vol. 16, no. 2, 111–117.

[7] Romanovski, V.G., Shafer, D.S. *The center and cyclicity problems: a computational algebra approach*. Boston, Basel, Berlin: Birkhäuser, 2009.

[8] Sibirski, K. The number of limit cycles in the neighborhood of a singular point. *Diff. Uravneniya,* 1965, vol. 1, no. 1, 51–66 (in Russian).

[9] Şubă, A. and Cozma, D. Solution of the problem of the center for cubic differential system with three invariant straight lines in generic position. *Qualitative Theory of Dyn. Systems*, 2005, vol. 6, 45–58.

[10] Şubă, A., Repeşco, V. and Puţuntică, V. Cubic systems with invariant affine straight lines of total parallel multiplicity seven. *Electronic J. Diff. Equ.*, 2013, vol. 2013, no. 274, 1–22. http://ejde.math.txstate.edu/

[11] Şubă, A., Vacaraş, O. Cubic differential systems with an invariant straight line of maximal multiplicity. *Annals of the University of Craiova. Mathematics and Computer Science Series*, 2015, vol. 42, no. 2, 427–449.

[12] Şubă, A., Vacaraş, O. Quartic differential systems with an invariant straight line of maximal multiplicity. *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2018, vol. 86, no. 1, 76–91.

[13] Şubă, A., Vacaraş, O. Quartic differential systems with a non-degenerate monodromic critical point and multiple line at infinity. *Acta et Commentationes. Exact and Natural Sciences*, 2023, vol. 16, no. 2, 25–34.

[14] Vacaraş, O. Cubic differential systems with two affine real non-parallel invariant straight lines of maximal multiplicity. *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2015, vol. 79, no. 3, 79–101.

Received: October 10, 2024 *Accepted: December 20, 2024*

(Alexandru Şubă) "Vladimir Andrunachievici" Institute of Mathematics and Computer Sciences, Moldova State University, 5 Academiei St., Chişinău, Republic of Moldova;
"Ion Creangă" State Pedagogical University, 1 Ion Creangă St., Chişinău, Republic of Moldova
E-mail address: alexandru.suba@math.md