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Center problem for quartic differential systems with an affine invariant straight line of maximal multiplicity

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Abstract. In this paper the quartic differential systems with a center-focus critical point and non-degenerate infinity are examined. We show that in this family the maximal multiplicity of an affine invariant straight line is six. Modulo the affine transformation and time rescaling the classes of systems with an affine invariant straight line of multiplicity two, three,..., six are determined. In the cases when the quartic systems has an affine invariant straight line of maximal multiplicity the problem of the center is solved.

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Keywords: quartic differential system, multiple invariant line, center-focus critical point.

Problema centrului pentru sistemele diferenţiale cuartice cu o dreaptă invariantă afină de multiplicitate maximală

Rezumat. În această lucrare sunt examinate sistemele diferenţiale cuartice cu punct critic de tip centru-focar şi infinitul nedegenerat. Se arată că în această familie de sisteme multiplicitatea maximală a unei drepte invariante afine este egală cu şase. Cu exactitatea unei transformări afine de coordonate şi rescalarea timpului sunt determinate clasele de sisteme cu o dreaptă invariantă afină de multiplicitatea doi, trei, ..., şase. În cazurile când sistemele cuartice au o dreaptă invarinată de multiplicitate maximală problema centrului este rezolvată.

Cuvinte-cheie: sistem diferenţial cuartic, dreaptă invariantă multiplă, punct critic de tip centru-focar.

1. Introduction

We consider the real polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $\dot{x} = dx/dt,\ \dot{y} = dy/dt$.

Let $n = max\{deg(p),\ deg(q)\}$. If $n = 2$ (respectively, $n = 3,\ n = 4$) then the system (1) is called quadratic (respectively, cubic, quartic). Via an affine transformation of coordinates and time rescaling each non-degenerate quartic system with a non-degenerate infinity and a center-focus critical point, i.e. a critical point with pure imaginary eigenvalues, can be written in the form

$$\begin{cases} \dot{x} = y + p_2(x, y) + p_3(x, y) + p_4(x, y) \equiv p(x, y), \\ \dot{y} = -(x + q_2(x, y) + q_3(x, y) + q_4(x, y)) \equiv q(x, y), \end{cases} \tag{2}$$

where $p_i(x, y) = \sum_{j=0}^{i} a_{i-j,j}x^{i-j}y^j$, $q_i(x, y) = \sum_{j=0}^{i} b_{i-j,j}x^{i-j}y^j$, $i = 2, 3, 4$ are homogeneous polynomials in x and y of degree i with real coefficients.

The critical point $(0, 0)$ of system (2) is either a focus or a center. The problem of distinguishing between a center and a focus is called the center problem.

Suppose that

$$yp_4(x, y) - xq_4(x, y) \not\equiv 0, \gcd(p, q) = 1, \tag{3}$$

i.e. at infinity the system (2) has at most five distinct singular points and the right-hand sides of (2) do not have the common divisors of degree greater than 0.

Denote $\mathbb{X} = p\,(x, y)\,\frac{\partial}{\partial x} + q\,(x, y)\,\frac{\partial}{\partial y}$.

An algebraic curve $f(x, y) = 0,\ f \in \mathbb{C}[x, y]$ (a function $f = \exp(g/h), g, h \in \mathbb{C}[x, y]$) is called invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K) \le n-1$ such that the identity $\mathbb{X}(f) \equiv f(x, y)K_f(x, y)$, $(x, y) \in \mathbb{R}^2$ $((x, y) \in \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) = 0\})$ holds. In particular, a straight line $l \equiv \alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ is invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x, y]$ such that the identity $\alpha P(x, y) + \beta Q(x, y) \equiv (\alpha x + \beta y + \gamma)K_l(x, y)$, $(x, y) \in \mathbb{R}^2$ holds.

The invariant straight line $\alpha x + \beta y + \gamma = 0$ has *multiplicity* ν if ν is the greatest positive integer such that $(\alpha x + \beta y + \gamma)^\nu$ divides $\mathbb{E} = p \cdot \mathbb{X}(q) - q \cdot \mathbb{X}(p)$ [1].

The quartic differential systems of the form (2) with multiple line at infinity were examined in [8]. In this paper, we establish that in the class of systems (2) the maximal multiplicity of an affine invariant line is six. The coefficient conditions when (2) has an affine invariant line of multiplicity two, three, four, five and six are determined and in the cases of multiplicity six, the center problem is solved.

2. Classification of the quartic systems with a multiple affine invariant straight line

Let the quartic system (2) have an affine real invariant straight line l_1. By a transformation of the form

$$x \to \nu \cdot (x\cos\varphi + y\sin\varphi),\ \ y \to \nu \cdot (y\cos\varphi - x\sin\varphi),\ \ \nu \neq 0$$

we can do l_1 to be described by the equation $x = 1$. Then,

$$\begin{aligned} &a_{40} = -(a_{20} + a_{30}), a_{31} = -(1 + a_{11} + a_{21}),\\ &a_{22} = -(a_{02} + a_{12}), a_{13} = -a_{03}, a_{04} = 0, \end{aligned} \tag{4}$$

and (2) is reduced to the system

$$\begin{cases} \dot{x} = (1-x)(y + a_{20}x^2 + xy + a_{11}xy + a_{02}y^2 + a_{20}x^3 + a_{30}x^3 + x^2y+\\ \qquad +a_{11}x^2y + a_{21}x^2y + a_{02}xy^2 + a_{12}xy^2 + a_{03}y^3) \equiv p(x,y),\\ \dot{y} = -(x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2+\\ \qquad +b_{03}y^3 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4)) \equiv q(x,y). \end{cases} \tag{5}$$

Next, we will determine the conditions when the invariant line $x = 1$ for the system (5) has maximal multiplicity.

For (5) we have

$$\mathbb{E} = (x-1)\Big(Y_2(y) + Y_3(y)\cdot(x-1) + \cdots + Y_{10}(y)\cdot(x-1)^{10}\Big),$$

where $Y_j(y),\ j = 2, \ldots 10$, are polynomial in y. The invariant line $x-1 = 0$ has multiplicity at least j if the system of identity $\{Y_2(y) \equiv 0, \ldots, Y_j(y) \equiv 0\}$ holds. In particular, the line $x - 1 = 0$ has multiplicity at least two if $Y_2(y) \equiv 0$. The polynomial $Y_2(y)$ look as: $Y_2(y) = Y_{21}(y)\cdot Y_{22}(y)$, where

$Y_{21}(y) = 1+b_{20}+b_{30}+b_{40}+(b_{11}+b_{21}+b_{31})y+(b_{02}+b_{12}+b_{22})y^2+(b_{03}+b_{13})y^3+b_{04}y^4$,

$Y_{22}(y) = 3 + 2a_{11} + 4a_{20}^2 + a_{21} + 4a_{20}a_{30} + a_{30}^2 - 2a_{20}b_{11} - a_{30}b_{11} + 3b_{20} + 2a_{11}b_{20} + a_{21}b_{20} - 2a_{20}b_{21} - a_{30}b_{21} + 3b_{30} + 2a_{11}b_{30} + a_{21}b_{30} - 2a_{20}b_{31} - a_{30}b_{31} + 3b_{40} + 2a_{11}b_{40} + a_{21}b_{40} + 2(2a_{02} + a_{12} + 6a_{20} + 4a_{11}a_{20} + 2a_{20}a_{21} + 3a_{30} + 2a_{11}a_{30} + a_{21}a_{30} - 2a_{20}b_{02} - a_{30}b_{02} - 2a_{20}b_{12} - a_{30}b_{12} + 2a_{02}b_{20} + a_{12}b_{20} - 2a_{20}b_{22} - a_{30}b_{22} + 2a_{02}b_{30} + a_{12}b_{30} + 2a_{02}b_{40} + a_{12}b_{40})y + (9 + 3a_{03} + 12a_{11} + 4a_{11}^2 + 8a_{02}a_{20} + 4a_{12}a_{20} + 6a_{21} + 4a_{11}a_{21} + a_{21}^2 + 4a_{02}a_{30} + 2a_{12}a_{30} - 3b_{02} - 2a_{11}b_{02} - a_{21}b_{02} - 6a_{20}b_{03} - 3a_{30}b_{03} + 2a_{02}b_{11} + a_{12}b_{11} - 3b_{12} - 2a_{11}b_{12} - a_{21}b_{12} - 6a_{20}b_{13} - 3a_{30}b_{13} + 3a_{03}b_{20} + 2a_{02}b_{21} + a_{12}b_{21} - 3b_{22} - 2a_{11}b_{22} - a_{21}b_{22} + 3a_{03}b_{30} + 2a_{02}b_{31} + a_{12}b_{31} + 3a_{03}b_{40})y^2 + 2(6a_{02} + 4a_{02}a_{11} + 3a_{12} + 2a_{11}a_{12} + 2a_{03}a_{20} + 2a_{02}a_{21} + a_{12}a_{21} + a_{03}a_{30} - 3b_{03} - 2a_{11}b_{03} - a_{21}b_{03} - 4a_{20}b_{04} - 2a_{30}b_{04} + a_{03}b_{11} - 3b_{13} - 2a_{11}b_{13} - a_{21}b_{13} + a_{03}b_{21} + a_{03}b_{31})y^3 + (4a_{02}^2 + 6a_{03} + 4a_{03}a_{11} +$

$4a_{02}a_{12}+a_{12}^2+2a_{03}a_{21}+a_{03}b_{02}-2a_{02}b_{03}-a_{12}b_{03}-9b_{04}-6a_{11}b_{04}-3a_{21}b_{04}+a_{03}b_{12}-2a_{02}b_{13}-a_{12}b_{13}+a_{03}b_{22})y^4+2(2a_{02}+a_{12})(a_{03}-b_{04})y^5+a_{03}(a_{03}-b_{04})y^6.$

If $Y_{21}(y)\equiv 0$, then the system (5) is degenerate, i.e. $\deg\Big(\gcd(p,q)\Big)>0$, therefore we require $Y_{22}(y)$ to be identically equal to zero. Solving the identity $Y_{22}(y)\equiv 0$ we obtain the following result:

Lemma 2.1. *The invariant straight line $x=1$ has for quartic system* (5) *the multiplicity at least two if and only if the coefficients of* (5) *verify the following five series of conditions:*

$$a_{03}=0, b_{04}=0, a_{12}=-2a_{02}, b_{13}=-b_{03}, a_{21}=-3-2a_{11},\\ b_{22}=-b_{02}-b_{12}, b_{31}=2a_{20}+a_{30}-b_{11}-b_{21}; \tag{6}$$

$$a_{03}=0, b_{04}=0, a_{12}=-2a_{02}, b_{13}=-b_{03}, b_{22}=3+2a_{11}+a_{21}-b_{02}-b_{12},\\ b_{40}=(-3-2a_{11}-4a_{20}^2-a_{21}-4a_{20}a_{30}-a_{30}^2+2a_{20}b_{11}+a_{30}b_{11}-3b_{20}-\\ -2a_{11}b_{20}-a_{21}b_{20}+2a_{20}b_{21}+a_{30}b_{21}-3b_{30}-2a_{11}b_{30}-a_{21}b_{30}+2a_{20}b_{31}+\\ +a_{30}b_{31})/(3+2a_{11}+a_{21}); \tag{7}$$

$$a_{03}=0,\ b_{04}=0,\ b_{13}=2a_{02}+a_{12}-b_{03},\ \ b_{31}=(-9-12a_{11}-4a_{11}^2+\\ +4a_{02}a_{20}+2a_{12}a_{20}-6a_{21}-4a_{11}a_{21}-a_{21}^2+2a_{02}a_{30}+a_{12}a_{30}+3b_{02}+\\ +2a_{11}b_{02}+a_{21}b_{02}-2a_{02}b_{11}-a_{12}b_{11}+3b_{12}+2a_{11}b_{12}+a_{21}b_{12}-\\ -2a_{02}b_{21}-a_{12}b_{21}+3b_{22}+2a_{11}b_{22}+a_{21}b_{22})/(2a_{02}+a_{12}),\\ b_{40}=-(2a_{02}+a_{12}+6a_{20}+4a_{11}a_{20}+2a_{20}a_{21}+3a_{30}+2a_{11}a_{30}+\\ a_{21}a_{30}-2a_{20}b_{02}-a_{30}b_{02}-2a_{20}b_{12}-a_{30}b_{12}+2a_{02}b_{20}+a_{12}b_{20}-\\ -2a_{20}b_{22}-a_{30}b_{22}+2a_{02}b_{30}+a_{12}b_{30})/(2a_{02}+a_{12}); \tag{8}$$

$$a_{03}=0, a_{12}=-2a_{02}, a_{21}=-3-2a_{11}, a_{30}=-2a_{20}; \tag{9}$$

$$b_{04}=a_{03},\ \ b_{22}=(-4a_{02}^2+3a_{03}+2a_{03}a_{11}-4a_{02}a_{12}-a_{12}^2+a_{03}a_{21}-\\ -a_{03}b_{02}+2a_{02}b_{03}+a_{12}b_{03}-a_{03}b_{12}+2a_{02}b_{13}+a_{12}b_{13})/a_{03},\\ b_{31}=(-6a_{02}-4a_{02}a_{11}-3a_{12}-2a_{11}a_{12}+2a_{03}a_{20}-2a_{02}a_{21}-a_{12}a_{21}+\\ a_{03}a_{30}+3b_{03}+2a_{11}b_{03}+a_{21}b_{03}-a_{03}b_{11}+3b_{13}+2a_{11}b_{13}+a_{21}b_{13}-\\ -a_{03}b_{21})/a_{03},\quad b_{40}=-(a_{03}+4a_{02}a_{20}+2a_{12}a_{20}+2a_{02}a_{30}+a_{12}a_{30}-\\ -2a_{20}b_{03}-a_{30}b_{03}-2a_{20}b_{13}-a_{30}b_{13}+a_{03}b_{20}+a_{03}b_{30})/a_{03}. \tag{10}$$

The multiplicity of the invariant straight line $x=1$ is at least three if $\{Y_2(y)\equiv 0,\ Y_3(y)\equiv 0\}$. Taking into account the condition (3) the identity $Y_3(y)\equiv 0$ gives, in each of the five cases (6)-(10) of Lemma 2.1 the following series of conditions:

1) (6) $\Rightarrow$

$$a_{30}=-2a_{20},\ a_{02}=0,\ a_{11}=-3; \tag{11}$$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (2a_{02} + 3a_{20} + a_{11}a_{20} - 2a_{20}b_{02} + a_{02}b_{20} - a_{02}b_{40})/a_{20},\\
&b_{21} = (6 + 2a_{11} + a_{20}^2 - 2a_{20}b_{11} + 3b_{20} + a_{11}b_{20} - 3b_{40} - a_{11}b_{40})/a_{20},\\
&b_{30} = (3a_{20} + 2a_{30} + a_{20}b_{20} + a_{30}b_{20} - 3a_{20}b_{40} - a_{30}b_{40})/a_{20};
\end{aligned} \tag{12}$$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (3a_{02} + 3a_{30} + a_{11}a_{30} - 2a_{30}b_{02} + 2a_{02}b_{20} + a_{02}b_{30})/a_{30},\\
&b_{21} = 9 + 3a_{11} - 2a_{30}b_{11} + 6b_{20} + 2a_{11}b_{20} + 3b_{30} + a_{11}b_{30})/a_{30},\\
&a_{20} = 0,\ b_{40} = b_{20} + 2;
\end{aligned} \tag{13}$$

2) (7) $\Rightarrow$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (9 + 9a_{11} + 2a_{11}^2 + 2a_{02}a_{20} + 3a_{21} + a_{11}a_{21} + a_{02}a_{30}-\\
&-6b_{02} - 4a_{11}b_{02} - 2a_{21}b_{02})/(3 + 2a_{11} + a_{21}),\\
&b_{21} = (15a_{20} + 8a_{11}a_{20} + 3a_{20}a_{21} + 6a_{30} + 3a_{11}a_{30} + a_{21}a_{30} - 6b_{11}-\\
&-4a_{11}b_{11} - 2a_{21}b_{11})/(3 + 2a_{11} + a_{21}),\\
&b_{31} = (-3a_{20} + a_{20}a_{21} + a_{11}a_{30} + a_{21}a_{30} + 3b_{11} + 2a_{11}b_{11}+\\
&+a_{21}b_{11})/(3 + 2a_{11} + a_{21});
\end{aligned} \tag{14}$$

$$\begin{aligned}
&b_{03} = a_{02},\quad b_{12} = (9 + 6a_{11} + a_{11}^2 + a_{02}a_{20} + a_{02}a_{30} - 6b_{02} - 2a_{11}b_{02}+\\
&+a_{02}b_{11} - a_{02}b_{31})/(3 + a_{11}),\\
&b_{21} = (9a_{20} + 4a_{11}a_{20} + a_{20}a_{21} + 6a_{30} + 3a_{11}a_{30} + a_{21}a_{30} + a_{11}b_{11}+\\
&+a_{21}b_{11} - 6b_{31} - 3a_{11}b_{31} - a_{21}b_{31})/(3 + a_{11}),\\
&b_{30} = (9a_{20} + 4a_{11}a_{20} + a_{20}a_{21} + 6a_{30} + 3a_{11}a_{30} + a_{21}a_{30} + a_{11}b_{11}+\\
&+a_{21}b_{11} - 6b_{31} - 3a_{11}b_{31} - a_{21}b_{31})/(3 + a_{11});
\end{aligned} \tag{15}$$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (a_{02}a_{20} - 6b_{02} + 2a_{21}b_{02} - 2a_{02}b_{11} - a_{02}b_{21})/(3 - a_{21}),\\
&a_{11} = -3,\quad b_{31} = a_{20} + a_{30} + b_{11},\quad b_{30} = (-9 + 3a_{20}^2 + 3a_{21} + a_{20}a_{30}-\\
&-6a_{20}b_{11} - 2a_{30}b_{11} - 6b_{20} + 2a_{21}b_{20} - 3a_{20}b_{21} - a_{30}b_{21})/(3 - a_{21});
\end{aligned} \tag{16}$$

3) (8) $\Rightarrow$

$$\begin{aligned}
&b_{03} = a_{02},\quad b_{12} = (3a_{02} + 4a_{02}a_{11} + a_{11}a_{12} + 3a_{02}a_{21} + a_{12}a_{21}+\\
&+a_{02}b_{02} + a_{12}b_{02} - 3a_{02}b_{22} - a_{12}b_{22})/a_{02},\\
&b_{30} = -(3a_{02} - 3a_{11}a_{20} - 3a_{20}a_{21} - a_{11}a_{30} - a_{21}a_{30} - 3a_{20}b_{02}-\\
&-a_{30}b_{02} + 2a_{02}b_{20} + 3a_{20}b_{22} + a_{30}b_{22})/a_{02},\\
&b_{21} = (6a_{11} + 3a_{11}^2 + a_{02}a_{20} + 6a_{21} + 4a_{11}a_{21} + a_{21}^2 + 6b_{02}+\\
&+3a_{11}b_{02} + a_{21}b_{02} - 2a_{02}b_{11} - 6b_{22} - 3a_{11}b_{22} - a_{21}b_{22})/a_{02};
\end{aligned} \tag{17}$$

$$\begin{aligned}
&b_{03} = a_{02},\quad b_{12} = (3a_{02} + 4a_{02}a_{11} + a_{11}a_{12} + 3a_{02}a_{21} + a_{12}a_{21}+\\
&a_{02}b_{02} + a_{12}b_{02} - 3a_{02}b_{22} - a_{12}b_{22})/a_{02},\\
&b_{30} = -(6a_{02}^4 + 18a_{02}^2a_{11} + 3a_{02}^2a_{11}^2 + 3a_{02}^3a_{12} - 9a_{02}a_{11}^2a_{12} - a_{02}a_{11}^3a_{12}+
\end{aligned} \tag{18}$$

$$a_{11}^3a_{12}^2 + 3a_{02}^3a_{20} - 2a_{02}^3a_{11}a_{20} - 2a_{02}^2a_{11}a_{12}a_{20} + 18a_{02}^2a_{21} - 6a_{02}^2a_{11}a_{21}-$$
$$a_{02}^2a_{11}^2a_{21} - 18a_{02}a_{11}a_{12}a_{21} + 3a_{11}^2a_{12}^2a_{21} - 3a_{02}^3a_{20}a_{21} - 2a_{02}^2a_{12}a_{20}a_{21}-$$
$$9a_{02}^2a_{21}^2 - 9a_{02}a_{12}a_{21}^2 + 3a_{02}a_{11}a_{12}a_{21}^2 + 3a_{11}a_{12}^2a_{21}^2 + a_{02}^2a_{21}^3 + 2a_{02}a_{12}\cdot$$
$$a_{21}^3 + a_{12}^2a_{21}^3 + 18a_{02}^2b_{02} - 15a_{02}^2a_{11}b_{02} - 2a_{02}^2a_{11}^2b_{02} - 18a_{02}a_{11}a_{12}b_{02}+$$
$$2a_{02}a_{11}^2a_{12}b_{02} + 3a_{11}^2a_{12}^2b_{02} - 4a_{02}^3a_{20}b_{02} - 2a_{02}^2a_{12}a_{20}b_{02} - 27a_{02}^2a_{21}b_{02}+$$
$$a_{02}^2a_{11}a_{21}b_{02} - 18a_{02}a_{12}a_{21}b_{02} + 10a_{02}a_{11}a_{12}a_{21}b_{02} + 6a_{11}a_{12}^2a_{21}b_{02}+$$
$$5a_{02}^2a_{21}^2b_{02} + 8a_{02}a_{12}a_{21}^2b_{02} + 3a_{12}^2a_{21}^2b_{02} - 18a_{02}^2b_{02}^2 + 2a_{02}^2a_{11}b_{02}^2 - 9a_{02}\cdot$$
$$a_{12}b_{02}^2 + 7a_{02}a_{11}a_{12}b_{02}^2 + 3a_{11}a_{12}^2b_{02}^2 + 8a_{02}^2a_{21}b_{02}^2 + 10a_{02}a_{12}a_{21}b_{02}^2+$$
$$3a_{12}^2a_{21}b_{02}^2 + 4a_{02}^2b_{02}^3 + 4a_{02}a_{12}b_{02}^3 + a_{12}^2b_{02}^3 - 6a_{02}^3b_{11} + 2a_{02}^2a_{11}a_{12}b_{11}+$$
$$2a_{02}^3a_{21}b_{11} + 2a_{02}^2a_{12}a_{21}b_{11} + 4a_{02}^3b_{02}b_{11} + 2a_{02}^2a_{12}b_{02}b_{11} + 4a_{02}^4b_{20}+$$
$$2a_{02}^3a_{12}b_{20} - 3a_{02}^3b_{21} + a_{02}^2a_{11}a_{12}b_{21} + a_{02}^3a_{21}b_{21} + a_{02}^2a_{12}a_{21}b_{21}+$$
$$2a_{02}^3b_{02}b_{21} + a_{02}^2a_{12}b_{02}b_{21} - 18a_{02}^2b_{22} + 15a_{02}^2a_{11}b_{22} + 2a_{02}^2a_{11}^2b_{22}+$$
$$18a_{02}a_{11}a_{12}b_{22} - 2a_{02}a_{11}^2a_{12}b_{22} - 3a_{11}^2a_{12}^2b_{22} + 4a_{02}^3a_{20}b_{22} + 2a_{02}^2\cdot$$
$$a_{12}a_{20}b_{22} + 27a_{02}^2a_{21}b_{22} - a_{02}^2a_{11}a_{21}b_{22} + 18a_{02}a_{12}a_{21}b_{22} - 10a_{02}\cdot$$
$$a_{11}a_{12}a_{21}b_{22} - 6a_{11}a_{12}^2a_{21}b_{22} - 5a_{02}^2a_{21}^2b_{22} - 8a_{02}a_{12}a_{21}^2b_{22} - 3a_{12}^2\cdot$$
$$a_{21}^2b_{22} + 36a_{02}^2b_{02}b_{22} - 4a_{02}^2a_{11}b_{02}b_{22} + 18a_{02}a_{12}b_{02}b_{22} - 14a_{02}a_{11}a_{12}\cdot$$
$$b_{02}b_{22} - 6a_{11}a_{12}^2b_{02}b_{22} - 16a_{02}^2a_{21}b_{02}b_{22} - 20a_{02}a_{12}a_{21}b_{02}b_{22} - 6a_{12}^2\cdot$$
$$a_{21}b_{02}b_{22} - 12a_{02}^2b_{02}^2b_{22} - 12a_{02}a_{12}b_{02}^2b_{22} - 3a_{12}^2b_{02}^2b_{22} - 4a_{02}^3b_{11}b_{22}-$$
$$2a_{02}^2a_{12}b_{11}b_{22} - 2a_{02}^3b_{21}b_{22} - a_{02}^2a_{12}b_{21}b_{22} - 18a_{02}^2b_{22}^2 + 2a_{02}^2a_{11}b_{22}^2-$$
$$9a_{02}a_{12}b_{22}^2 + 7a_{02}a_{11}a_{12}b_{22}^2 + 3a_{11}a_{12}^2b_{22}^2 + 8a_{02}^2a_{21}b_{22}^2 + 10a_{02}a_{12}\cdot$$
$$a_{21}b_{22}^2 + 3a_{12}^2a_{21}b_{22}^2 + 12a_{02}^2b_{02}b_{22}^2 + 12a_{02}a_{12}b_{02}b_{22}^2 + 3a_{12}^2b_{02}b_{22}^2-$$
$$4a_{02}^2b_{22}^3 - 4a_{02}a_{12}b_{22}^3 - a_{12}^2b_{22}^3))/(a_{02}^3(2a_{02}+a_{12})),$$
$$a_{30} = (3a_{02}a_{11} - a_{11}^2a_{12} - 2a_{02}^2a_{20} + 3a_{02}a_{21} - a_{02}a_{11}a_{21} - 2a_{11}a_{12}\cdot$$
$$a_{21} - a_{02}a_{21}^2 - a_{12}a_{21}^2 + 3a_{02}b_{02} - 2a_{02}a_{11}b_{02} - 2a_{11}a_{12}b_{02} - 3a_{02}\cdot$$
$$a_{21}b_{02} - 2a_{12}a_{21}b_{02} - 2a_{02}b_{02}^2 - a_{12}b_{02}^2 - 3a_{02}b_{22} + 2a_{02}a_{11}b_{22}+$$
$$2a_{11}a_{12}b_{22} + 3a_{02}a_{21}b_{22} + 2a_{12}a_{21}b_{22} + 4a_{02}b_{02}b_{22} + 2a_{12}b_{02}b_{22}-$$
$$2a_{02}b_{22}^2 - a_{12}b_{22}^2)/a_{02}^2;$$

$$b_{03} = a_{02},\quad a_{02} = 0,\quad b_{22} = a_{11} + a_{21} + b_{02},$$
$$b_{30} = (162 + 207a_{11} + 87a_{11}^2 + 12a_{11}^3 - 3a_{12}^2 - 9a_{12}a_{20} - 4a_{11}a_{12}a_{20}+$$
$$45a_{21} + 36a_{11}a_{21} + 7a_{11}^2a_{21} - a_{12}a_{20}a_{21} + 3a_{21}^2 + a_{11}a_{21}^2 - 198b_{02}-$$
$$174a_{11}b_{02} - 38a_{11}^2b_{02} + 4a_{12}a_{20}b_{02} - 42a_{21}b_{02} - 18a_{11}a_{21}b_{02} - 2a_{21}^2b_{02}+$$
$$72b_{02}^2 + 32a_{11}b_{02}^2 + 8a_{21}b_{02}^2 - 8b_{02}^3 + 12a_{12}b_{11} + 6a_{11}a_{12}b_{11} + 2a_{12}a_{21}b_{11}- \qquad (19)$$
$$4a_{12}b_{02}b_{11} - 99b_{12} - 87a_{11}b_{12} - 19a_{11}^2b_{12} + 2a_{12}a_{20}b_{12} - 21a_{21}b_{12}-$$
$$9a_{11}a_{21}b_{12} - a_{21}^2b_{12} + 72b_{02}b_{12} + 32a_{11}b_{02}b_{12} + 8a_{21}b_{02}b_{12} - 12b_{02}^2b_{12}-$$
$$2a_{12}b_{11}b_{12} + 18b_{12}^2 + 8a_{11}b_{12}^2 + 2a_{21}b_{12}^2 - 6b_{02}b_{12}^2 - b_{12}^3 - 2a_{12}^2b_{20}+$$

$$6a_{12}b_{21} + 3a_{11}a_{12}b_{21} + a_{12}a_{21}b_{21} - 2a_{12}b_{02}b_{21} - a_{12}b_{12}b_{21})/a_{12}^2,$$
$$a_{30} = -(18 + 15a_{11} + 3a_{11}^2 + 2a_{12}a_{20} + 3a_{21} + a_{11}a_{21} - 18b_{02}-$$
$$8a_{11}b_{02} - 2a_{21}b_{02} + 4b_{02}^2 - 9b_{12} - 4a_{11}b_{12} - a_{21}b_{12} + 4b_{02}b_{12}+$$
$$b_{12}^2)/a_{12};$$

$$\begin{aligned}&b_{03} = a_{02}, \quad a_{02} = 0, \quad b_{22} = a_{11} + a_{21} + b_{02},\\&b_{30} = -(3a_{12} + 9a_{20} + 3a_{11}a_{20} + 3a_{30} + a_{11}a_{30} - 6a_{20}b_{02} - 2a_{30}b_{02}-\\&-3a_{20}b_{12} - a_{30}b_{12} + 2a_{12}b_{20})/a_{12},\\&b_{21} = (-18 - 15a_{11} - 3a_{11}^2 + a_{12}a_{20} - 3a_{21} - a_{11}a_{21} + 12b_{02} + 6a_{11}b_{02}+\\&2a_{21}b_{02} - 2a_{12}b_{11} + 6b_{12} + 3a_{11}b_{12} + a_{21}b_{12})/a_{12};\end{aligned} \tag{20}$$

4) (9) $\Rightarrow$

$$a_{02} = 0, \; a_{11} = -3, \; a_{20} = 0; \tag{21}$$

5) (10) $\Rightarrow$

$$\begin{aligned}&b_{13} = a_{12} + a_{02}, \quad b_{21} = (-6a_{02} - 3a_{02}a_{11} + a_{03}a_{20} - a_{02}a_{21} + 6b_{03}+\\&+3a_{11}b_{03} + a_{21}b_{03} - 2a_{03}b_{11})/a_{03},\\&b_{30} = -(3a_{03} + 3a_{02}a_{20} + a_{02}a_{30} - 3a_{20}b_{03} - a_{30}b_{03} + 2a_{03}b_{20})/a_{03},\\&b_{12} = (-3a_{02}^2 + 3a_{03} + a_{03}a_{11} - a_{02}a_{12} - 2a_{03}b_{02} + 3a_{02}b_{03} + a_{12}b_{03})/a_{03};\end{aligned} \tag{22}$$

$$\begin{aligned}&b_{13} = a_{12} + a_{02}, \quad b_{21} = (9a_{02}^3 - 15a_{02}a_{03} - 6a_{02}a_{03}a_{11} + 6a_{02}^2a_{12}-\\&3a_{03}a_{12} - a_{03}a_{11}a_{12} + a_{02}a_{12}^2 + a_{03}^2a_{20} - a_{02}a_{03}a_{21} + 6a_{02}a_{03}b_{02}+\\&2a_{03}a_{12}b_{02} - 12a_{02}^2b_{03} + 9a_{03}b_{03} + 4a_{03}a_{11}b_{03} - 7a_{02}a_{12}b_{03} - a_{12}^2b_{03}+\\&a_{03}a_{21}b_{03} - 2a_{03}b_{02}b_{03} + 3a_{02}b_{03}^2 + a_{12}b_{03}^2 - 2a_{03}^2b_{11} + 3a_{02}a_{03}b_{12}+\\&a_{03}a_{12}b_{12} - a_{03}b_{03}b_{12})/a_{03}^2,\\&b_{30} = (12a_{02}^4 + 3a_{02}^2a_{03} - 9a_{03}^2 - 3a_{03}^3 + 5a_{02}^2a_{03}a_{11} - 9a_{03}^2a_{11} - 2a_{03}^2a_{11}^2+\\&7a_{02}^3a_{12} + a_{02}a_{03}a_{11}a_{12} + a_{02}^2a_{12}^2 - a_{02}a_{03}^2a_{20} + 4a_{02}^2a_{03}a_{21} - 3a_{03}^2a_{21}-\\&a_{03}^2a_{11}a_{21} + a_{02}a_{03}a_{12}a_{21} + 6a_{02}^2a_{03}b_{02} + 6a_{03}^2b_{02} + 4a_{03}^2a_{11}b_{02} + 2a_{02}a_{03}\cdot\\&a_{12}b_{02} + 2a_{03}^2a_{21}b_{02} - 31a_{02}^3b_{03} - 3a_{02}a_{03}b_{03} - 6a_{02}a_{03}a_{11}b_{03} - 16a_{02}^2a_{12}\cdot\\&b_{03} - a_{03}a_{11}a_{12}b_{03} - 2a_{02}a_{12}^2b_{03} + a_{03}^2a_{20}b_{03} - 5a_{02}a_{03}a_{21}b_{03} - a_{03}a_{12}a_{21}\cdot\\&b_{03} - 8a_{02}a_{03}b_{02}b_{03} - 2a_{03}a_{12}b_{02}b_{03} + 27a_{02}^2b_{03}^2 + a_{03}a_{11}b_{03}^2 + 11a_{02}a_{12}\cdot\\&b_{03}^2 + a_{12}^2b_{03}^2 + a_{03}a_{21}b_{03}^2 + 2a_{03}b_{02}b_{03}^2 - 9a_{02}b_{03}^3 - 2a_{12}b_{03}^3 + b_{03}^4+\\&3a_{02}^2a_{03}b_{12} + 3a_{03}^2b_{12} + 2a_{03}^2a_{11}b_{12} + a_{02}a_{03}a_{12}b_{12} + a_{03}^2a_{21}b_{12}-\\&4a_{02}a_{03}b_{03}b_{12} - a_{03}a_{12}b_{03}b_{12} + a_{03}b_{03}^2b_{12} - 2a_{03}^3b_{20})/a_{03}^3,\\&a_{30} = -(3a_{02}^3 + 3a_{02}a_{03} + 2a_{02}a_{03}a_{11} + a_{02}^2a_{12} + 2a_{03}^2a_{20} + a_{02}a_{03}a_{21}-\\&7a_{02}^2b_{03} - 3a_{03}b_{03} - 2a_{03}a_{11}b_{03} - 2a_{02}a_{12}b_{03} - a_{03}a_{21}b_{03} + 5a_{02}b_{03}^2+\\&a_{12}b_{03}^2 - b_{03}^3)/a_{03}^2.\end{aligned} \tag{23}$$

Lemma 2.2. *The invariant straight line* $x = 1$ *has for quartic system* (5) *the multiplicity at least three if and only if the coefficients of* (5) *verify the following series of conditions:*
1) $\{(6), (11)\}$; *2)* $\{(6), (12)\}$; *3)* $\{(6), (13)\}$; *4)* $\{(7), (14)\}$;
5) $\{(7), (15)\}$; *6)* $\{(7), (16)\}$; *7)* $\{(8), (17)\}$; *8)* $\{(8), (18)\}$;
9) $\{(8), (19)\}$; *10)* $\{(8), (20)\}$; *11)* $\{(9), (21)\}$; *12)* $\{(10),(22)\}$;
13) $\{(10), (23)\}$.

The multiplicity of the invariant straight line $x = 1$ is at least four if $\{Y_2(y) \equiv 0,\ Y_3(y) \equiv 0,\ Y_4(y) \equiv 0\}$. Taking into account the condition (3) the identity $Y_4(y) \equiv 0$ gives, in each of the cases 1)-13) of Lemma 2.2 the following series of conditions:
1) $\{(6), (11)\} \Rightarrow$

$$b_{03} = 0,\ \ b_{12} = -2b_{02},\ \ b_{40} = -1 - 2a_{20}^2 + 2a_{20}b_{11} - b_{20} + a_{20}b_{21} - b_{30}; \tag{24}$$

2) $\{(6), (12)\} \Rightarrow$

$$b_{02} = (3a_{02} + a_{20} + a_{02}b_{20})/a_{20},\ \ b_{11} = (4 + a_{11})(3 + b_{20})/a_{20},\ b_{40} = -1; \tag{25}$$

3) $\{(6), (13)\} \Rightarrow$

$$b_{02} = (-3a_{02} + a_{30} + a_{02}b_{30})/a_{30},\ b_{11} = (4 + a_{11})(-3 + b_{30})/a_{30},\ b_{20} = -3; \tag{26}$$

4) $\{(7), (14)\} \Rightarrow$

$$\begin{aligned} b_{02} &= (3 + 2a_{11} + 2a_{02}a_{20} + a_{21} + a_{02}a_{30})/(3 + 2a_{11} + a_{21}),\\ b_{11} &= (4 + a_{11})(2a_{20} + a_{30})/(3 + 2a_{11} + a_{21}),\\ b_{20} &= -(9 + 6a_{11} - 2a_{20}^2 + 3a_{21} - a_{20}a_{30})/(3 + 2a_{11} + a_{21}),\\ b_{30} &= (9 + 6a_{11} + 2a_{20}^2 + 3a_{21} + 3a_{20}a_{30} + a_{30}^2)/(3 + 2a_{11} + a_{21}); \end{aligned} \tag{27}$$

5) $\{(7), (15)\} \Rightarrow$

$$\begin{aligned} &b_{02} = 1 - a_{02}a_{20} - a_{02}a_{30} + a_{02}b_{31},\ \ b_{11} = (4 + a_{11})(b_{31} - a_{20} - a_{30}),\\ &b_{20} = -3 - a_{20}^2 - a_{20}a_{30} + a_{20}b_{31}; \end{aligned} \tag{28}$$

6) $\{(7), (16)\} \Rightarrow$

$$b_{02} = 1 + a_{02}b_{11},\ \ b_{21} = a_{20} - 5b_{11} + a_{21}b_{11},\ \ b_{20} = -3 + a_{20}b_{11}; \tag{29}$$

7) $\{(8), (17)\} \Rightarrow$

$$\begin{aligned} &b_{22} = 1 + a_{11} + a_{21},\ \ b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02},\\ &b_{20} = -(3a_{02} + a_{20} - a_{20}b_{02})/a_{02}; \end{aligned} \tag{30}$$

$$\begin{aligned}
&b_{22} = (-6a_{02} - a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02},\\
&a_{21} = (-7a_{02} - 2a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&a_{30} = (2a_{02} + a_{12} - 2a_{02}^2a_{20} - 4a_{02}b_{02} - 2a_{12}b_{02} + 2a_{02}b_{02}^2 + a_{12}b_{02}^2)/a_{02}^2;
\end{aligned} \tag{31}$$

8) $\{(8), (18)\} \Rightarrow$

$$\begin{aligned}
&b_{22} = 1 + a_{11} + a_{21}, \quad b_{20} = -(3a_{02} + a_{20} - a_{20}b_{02})/a_{02},\\
&b_{21} = (2 - a_{11} + a_{02}a_{20} - a_{21} - 2b_{02} + a_{11}b_{02} + a_{21}b_{02})/a_{02},\\
&b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02};
\end{aligned} \tag{32}$$

$$\begin{aligned}
&b_{22} = -(6a_{02} + a_{02}a_{11} + 2a_{12} - 4a_{02}b_{02} - 2a_{12}b_{02})/a_{02},\\
&b_{21} = (9a_{02} + a_{02}a_{11} + 2a_{12} + a_{02}^2a_{20} - 13a_{02}b_{02} - a_{02}a_{11}b_{02}-\\
&4a_{12}b_{02} + 4a_{02}b_{02}^2 + 2a_{12}b_{02}^2)/a_{02}^2, \quad b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02},\\
&a_{21} = -(7a_{02} + 2a_{02}a_{11} + 2a_{12} - 4a_{02}b_{02} - 2a_{12}b_{02})/a_{02};
\end{aligned} \tag{33}$$

$$\begin{aligned}
&b_{22} = (-6a_{02} - a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&b_{21} = (2a_{02}^2 - 2a_{02}^2a_{11} + 9a_{02}a_{12} + 2a_{12}^2 - 2a_{02}^3a_{20} - 6a_{02}^2b_{02}+\\
&2a_{02}^2a_{11}b_{02} - 16a_{02}a_{12}b_{02} - 4a_{12}^2b_{02} + 4a_{02}^2b_{02}^2 + 7a_{02}a_{12}b_{02}^2+\\
&2a_{12}^2b_{02}^2)/(a_{02}^2(2a_{02} + a_{12})),\\
&a_{21} = (-7a_{02} - 2a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&b_{11} = (4a_{02} + a_{02}a_{11} + 3a_{02}^2a_{20} + a_{02}a_{12}a_{20} - 7a_{02}b_{02} - a_{02}a_{11}b_{02}-\\
&a_{12}b_{02} + 3a_{02}b_{02}^2 + a_{12}b_{02}^2)/(a_{02}(2a_{02} + a_{12}));
\end{aligned} \tag{34}$$

9) $\{(8), (19)\} \Rightarrow$

$$\begin{aligned}
&b_{02} = 1, \quad b_{21} = (2 + a_{11} - a_{11}^2 + a_{12}a_{20} - a_{21} - a_{11}a_{21} - 2b_{12}+\\
&a_{11}b_{12} + a_{21}b_{12})/a_{12}, \quad b_{20} = -(3a_{12} + a_{20} + a_{11}a_{20} - a_{20}b_{12})/a_{12},\\
&b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12};
\end{aligned} \tag{35}$$

$$\begin{aligned}
&b_{02} = 1, \ b_{21} = 2(a_{20} - b_{11})(1 + a_{12}a_{20} - a_{12}b_{11}),\\
&a_{21} = -3 - 2a_{11} - 2a_{12}a_{20} + 2a_{12}b_{11}, \ b_{12} = 1 + a_{11} - a_{12}a_{20} + a_{12}b_{11};
\end{aligned} \tag{36}$$

$$\begin{aligned}
&b_{02} = 1, \ b_{21} = (7 + 10a_{11} + 3a_{11}^2 + a_{12}a_{20} - 9b_{12} - 5a_{11}b_{12} + 2b_{12}^2)/a_{12},\\
&a_{21} = -5 - 4a_{11} + 2b_{12}, \ b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12};
\end{aligned} \tag{37}$$

10) $\{(8), (20)\} \Rightarrow$

$$\begin{aligned}
&b_{02} = 1, \ b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12}, \ a_{21} = -5 - 4a_{11} + 2b_{12},\\
&a_{30} = (1 + 2a_{11} + a_{11}^2 - 2a_{12}a_{20} - 2b_{12} - 2a_{11}b_{12} + b_{12}^2)/a_{12};
\end{aligned} \tag{38}$$

$$\begin{aligned}
&b_{02} = 1, \ b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12},\\
&b_{20} = -(3a_{12} + a_{20} + a_{11}a_{20} - a_{20}b_{12})/a_{12};
\end{aligned} \tag{39}$$

11) $\{(9),(21)\} \Rightarrow$

$$b_{04}=0,\ b_{13}=-b_{03},\ b_{22}=-b_{12}-b_{02},\ b_{40}=-b_{30}-b_{20}-1; \quad (40)$$

12) $\{(10),(22)\} \Rightarrow$

$$b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03}, \quad b_{20}=(16a_{02}^2-3a_{03}^2+4a_{02}^2a_{11}+ 4a_{02}a_{12}+a_{02}a_{11}a_{12}-a_{02}a_{03}a_{20}-24a_{02}b_{03}-6a_{02}a_{11}b_{03}-4a_{12}b_{03}- a_{11}a_{12}b_{03}+a_{03}a_{20}b_{03}+8b_{03}^2+2a_{11}b_{03}^2+4a_{02}a_{03}b_{11}+a_{03}a_{12}b_{11}- 2a_{03}b_{03}b_{11})/a_{03}^2, \quad a_{21}=(-7a_{02}^2-3a_{03}-2a_{03}a_{11}-2a_{02}a_{12}+ 10a_{02}b_{03}+2a_{12}b_{03}-3b_{03}^2)/a_{03}, \quad a_{30}=(4a_{02}^3+a_{02}^2a_{12}-2a_{03}^2a_{20}- 10a_{02}^2b_{03}-2a_{02}a_{12}b_{03}+8a_{02}b_{03}^2+a_{12}b_{03}^2-2b_{03}^3)/a_{03}^2; \quad (41)$$

$$b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03},\ b_{20}=-(3a_{03}+a_{02}a_{20}-a_{20}b_{03})/a_{03}, b_{11}=(4+a_{11})(b_{03}-a_{02})/a_{03}; \quad (42)$$

13) $\{(10),(23)\} \Rightarrow$

$$b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03},\ b_{20}=-(3a_{03}+a_{02}a_{20}-a_{20}b_{03})/a_{03}, b_{11}=-(4+a_{11})(a_{02}-b_{03})/a_{03}, \quad b_{12}=(-a_{02}^2+a_{03}+a_{03}a_{11}- a_{02}a_{12}+a_{02}b_{03}+a_{12}b_{03})/a_{03}; \quad (43)$$

$$b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03}, \quad a_{21}=(-7a_{02}^2-3a_{03}-2a_{03}a_{11}- 2a_{02}a_{12}+10a_{02}b_{03}+2a_{12}b_{03}-3b_{03}^2)/a_{03}, \quad b_{20}=(16a_{02}^2-3a_{03}^2+ 4a_{02}^2a_{11}+4a_{02}a_{12}+a_{02}a_{11}a_{12}-a_{02}a_{03}a_{20}-24a_{02}b_{03}-6a_{02}a_{11}b_{03}- 4a_{12}b_{03}-a_{11}a_{12}b_{03}+a_{03}a_{20}b_{03}+8b_{03}^2+2a_{11}b_{03}^2+4a_{02}a_{03}b_{11}+ a_{03}a_{12}b_{11}-2a_{03}b_{03}b_{11})/a_{03}^2, \quad b_{12}=(-a_{02}^2+a_{03}+a_{03}a_{11}-a_{02}a_{12}+ a_{02}b_{03}+a_{12}b_{03})/a_{03}; \quad (44)$$

$$b_{02}=(-2a_{02}^2+2a_{03}+a_{03}a_{11}-a_{02}a_{12}+2a_{02}b_{03}+a_{12}b_{03}-a_{03}b_{12})/a_{03}, a_{21}=-(7a_{02}^2+3a_{03}+2a_{03}a_{11}+2a_{02}a_{12}-10a_{02}b_{03}-2a_{12}b_{03}+ 3b_{03}^2)/a_{03}, b_{20}=(24a_{02}^4-6a_{02}^2a_{03}+2a_{03}^2-3a_{03}^3-18a_{02}^2a_{03}a_{11}+2a_{03}^2a_{11}+ 32a_{02}^3a_{12}-7a_{02}a_{03}a_{12}-8a_{02}a_{03}a_{11}a_{12}+10a_{02}^2a_{12}^2-a_{03}a_{12}^2- a_{03}a_{11}a_{12}^2+a_{02}a_{12}^3-50a_{02}^3b_{03}-4a_{02}a_{03}b_{03}+16a_{02}a_{03}a_{11}b_{03}- 60a_{02}^2a_{12}b_{03}+2a_{03}a_{12}b_{03}+3a_{03}a_{11}a_{12}b_{03}-14a_{02}a_{12}^2b_{03}-a_{12}^3b_{03}+ 34a_{02}^2b_{03}^2+4a_{03}b_{03}^2-4a_{03}a_{11}b_{03}^2+35a_{02}a_{12}b_{03}^2+4a_{12}^2b_{03}^2-8a_{02}b_{03}^3- 7a_{12}b_{03}^3+4a_{02}a_{03}^2b_{11}+a_{03}^2a_{12}b_{11}-2a_{03}^2b_{03}b_{11}+22a_{02}^2a_{03}b_{12}- a_{03}^2b_{12}+a_{03}^2a_{11}b_{12}+8a_{02}a_{03}a_{12}b_{12}+a_{03}a_{12}^2b_{12}-23a_{02}a_{03}b_{03}b_{12}- 3a_{03}a_{12}b_{03}b_{12}+7a_{03}b_{03}^2b_{12}-a_{03}^2b_{12}^2)/a_{03}^3, \quad (45)$$

$a_{20} = (-6a_{02}^3 + 2a_{02}a_{03} + 4a_{02}a_{03}a_{11} - 6a_{02}^2a_{12} + a_{03}a_{12} + a_{03}a_{11}a_{12} -$
$a_{02}a_{12}^2 + 10a_{02}^2b_{03} - 2a_{03}a_{11}b_{03} + 9a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 4a_{02}b_{03}^2 - 3a_{12}b_{03}^2 -$
$5a_{02}a_{03}b_{12} - a_{03}a_{12}b_{12} + 3a_{03}b_{03}b_{12})a_{03}^2.$

It is easy to see that the set of conditions $\{(7), (16), (29)\}$ is a particular case for the set of conditions $\{(7), (15), (28)\}$ and the set of conditions $\{(8), (19), (35)\}$ is a particular case for $\{(8), (20), (39)\}$, the set of conditions $\{(10), (23), (43)\}$ is a particular case for the set of conditions $\{(10), (22), (42)\}$. The conditions $\{(8), (19), (37)\}$ and $\{(8), (20), (38)\}$ are the same.

Lemma 2.3. *The invariant straight line $x = 1$ has for quartic system* (5) *the multiplicity at least four if and only if the coefficients of* (5) *verify the following series of conditions:*
1) $\{(6), (11), (24)\}$; *2)* $\{(6), (12), (25)\}$; *3)* $\{(6), (13), (26)\}$;
4) $\{(7), (14), (27)\}$; *5)* $\{(7), (15), (28)\}$; *6)* $\{(8), (17), (30)\}$;
7) $\{(8), (17), (31)\}$; *8)* $\{(8), (18), (32)\}$; *9)* $\{(8), (18), (33)\}$;
10) $\{(8), (18), (34)\}$; *11)* $\{(8), (19), (36)\}$; *12)* $\{(8), (19), (37)\}$;
13) $\{(8), (20), (39)\}$; *14)* $\{(9), (21), (40)\}$; *15)* $\{(10), (22), (41)\}$;
16) $\{(10), (22), (42)\}$; *17)* $\{(10), (23), (44)\}$; *18)* $\{(10), (23), (45)\}$.

The multiplicity of the invariant straight line $x = 1$ is at least five if in each of the cases 1)–18) of Lemma 2.3 the identity $Y_5(y) \equiv 0$ holds. Proceeding as in the previous case and taking into account (3), we will examine each case separately.

1) $\{(6), (11), (24)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{30} &= (3 - 2a_{20}^2 - 3b_{02} + a_{20}b_{11} + 2b_{20} - 2b_{02}b_{20})/(b_{02} - 1), \\ b_{21} &= (-5a_{20} + 2a_{20}b_{02} + 4b_{11} - 2b_{02}b_{11})/(b_{02} - 2); \end{aligned} \tag{46}$$

$$b_{21} = -a_{20}, \quad b_{11} = 2a_{20}, \quad b_{02} = 1. \tag{47}$$

2) $\{(6), (12), (25)\}$ − the identity $Y_5(y) \equiv 0$ and the conditions (3) are not compatible.

3) $\{(6), (13), (26)\} \Rightarrow Y_5(y) = -9 + 3a_{30}^2 + 6b_{30} - b_{30}^2 + 4a_{30}(3 - b_{30})y \not\equiv 0$, because in this case $a_{30} \neq 0$.

4) $\{(7), (14), (27)\} \Rightarrow Y_5(y) = (2a_{20} + a_{30} + (3 + 2a_{11} + a_{21})y)^2(2(3 + 2a_{11} + a_{21})^2 - (2a_{20} + a_{30})^2 - 3(3 + 2a_{11} + a_{21})(2a_{20} + a_{30})y)/(3 + 2a_{11} + a_{21})^2 \not\equiv 0$.

In the conditions 5) $\{(7), (15), (28)\}$, 6) $\{(8), (17), (30)\}$, 7) $\{(8), (17), (31)\}$, 8) $\{(8), (18), (32)\}$ − the identity $Y_5(y) \equiv 0$ and the conditions (3) are not compatible.

9) $\{(8), (18), (33)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$b_{02} = 1, \quad a_{20} = 0, \quad a_{12} = -(7a_{02} + 3a_{02}b_{20})/(2 + b_{20}). \tag{48}$$

10) $\{(8), (18), (34)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$a_{20} = 0, \quad b_{02} = 1, \quad b_{20} = -(7a_{02} + 2a_{12})/(3a_{02} + a_{12}). \tag{49}$$

In the conditions 11) $\{(8), (19), (36)\}$, 12) $\{(8), (19), (37)\}$ − the identity $Y_5(y) \equiv 0$ and the conditions (3) are not compatible.

13) $\{(8), (20), (39)\} \Rightarrow Y_5(y) \not\equiv 0$.

14) $\{(9), (21), (40)\} \Rightarrow Y_5(y) = (b_{11} + b_{21} + b_{31})y(-3 - 2b_{20} - b_{30} + (2b_{02} + b_{12})y^2 + 2b_{03}y^3) \equiv 0 \Rightarrow$

$$b_{03} = 0, \quad b_{12} = -2b_{02}, \quad b_{30} = -3 - 2b_{20}. \tag{50}$$

15) $\{(10), (22), (41)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$\begin{aligned} &b_{11} = (-23a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 46a_{02}^2b_{03} - \\ &3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 29a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \\ &a_{11} = (23a_{02}^3 - 3a_{02}a_{03} - 5a_{02}a_{03}^2 + 10a_{02}^2a_{12} - a_{03}^2a_{12} + a_{02}a_{12}^2 - \\ &46a_{02}^2b_{03} + 3a_{03}b_{03} + 3a_{03}^2b_{03} - 15a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 29a_{02}b_{03}^2 + \\ &5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}(a_{02} - b_{03})), \\ &a_{20} = (-24a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 48a_{02}^2b_{03} - \\ &3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 30a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2; \end{aligned} \tag{51}$$

$$b_{11} = 0, \quad a_{12} = -2a_{02}, \quad b_{03} = a_{02}. \tag{52}$$

16) $\{(10), (22), (42)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$b_{03} = a_{02}, \quad a_{12} = -2a_{02}, \quad a_{21} = -3 - 2a_{11}, \quad a_{30} = -2a_{20}. \tag{53}$$

17) $\{(10), (23), (44)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$b_{11} = 0, \quad b_{03} = a_{02}, \quad a_{12} = -2a_{02}; \tag{54}$$

$$\begin{aligned} &b_{11} = (-23a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 46a_{02}^2b_{03} - \\ &3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 29a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \\ &a_{20} = (-24a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 48a_{02}^2b_{03} - \\ &3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 30a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2 \\ &a_{11} = (23a_{02}^3 - 3a_{02}a_{03} - 5a_{02}a_{03}^2 + 10a_{02}^2a_{12} - a_{03}^2a_{12} + \\ &a_{02}a_{12}^2 - 46a_{02}^2b_{03} + 3a_{03}b_{03} + 3a_{03}^2b_{03} - 15a_{02}a_{12}b_{03} - \\ &a_{12}^2b_{03} + 29a_{02}b_{03}^2 + 5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}(a_{02} - b_{03})). \end{aligned} \tag{55}$$

18) $\{(10), (23), (45)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$\begin{aligned} &b_{03} = (5a_{02} + a_{12})/3, \\ &b_{11} = 2(2a_{02} + a_{12})(-2a_{02}^2 + 9a_{03} + 3a_{03}a_{11} - a_{02}a_{12})/(9a_{03}^2), \\ &b_{12} = (8a_{02}^2 - 3a_{03} + 3a_{03}a_{11} + 8a_{02}a_{12} + 2a_{12}^2)/(6a_{03}); \end{aligned} \tag{56}$$

$$b_{11} = 0,\ b_{03} = a_{02},\ a_{12} = -2a_{02},\ b_{12} = 1 + a_{11}; \tag{57}$$

$$\begin{aligned} &b_{11} = (-23a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 46a_{02}^2b_{03} - \\ &3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 29a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \\ &a_{11} = (23a_{02}^3 - 3a_{02}a_{03} - 5a_{02}a_{03}^2 + 10a_{02}^2a_{12} - a_{03}^2a_{12} + a_{02}a_{12}^2 - \\ &46a_{02}^2b_{03} + 3a_{03}b_{03} + 3a_{03}^2b_{03} - 15a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 29a_{02}b_{03}^2 + \\ &5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}(a_{02} - b_{03})), \\ &b_{12} = (22a_{02}^3 - 2a_{02}a_{03} - 5a_{02}a_{03}^2 + 9a_{02}^2a_{12} - a_{03}^2a_{12} + a_{02}a_{12}^2 - \\ &44a_{02}^2b_{03} + 2a_{03}b_{03} + 3a_{03}^2b_{03} - 13a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 28a_{02}b_{03}^2 + \\ &4a_{12}b_{03}^2 - 6b_{03}^3)(a_{03}(a_{02} - b_{03})). \end{aligned} \tag{58}$$

The sets of conditions {(10), (22), (41) (52)}, {(10), (22), (42) (53)}, {(10), (23), (44) (54)} and {(6), (11), (24), (46)}, {(10), (23), (44), (55)}, {(10), (23), (45), (58)} are the same. The set of conditions {(10), (23), (45), (57)} is a particular case for {(10), (22), (41), (52)}.

Lemma 2.4. *The invariant straight line $x = 1$ has for quartic system* (5) *the multiplicity at least five if and only if the coefficients of* (5) *verify the following series of conditions:*
1) {(6), (11), (24), (46)}*;* *2)* {(6), (11), (24), (47)}*;*
3) {(8), (18), (33), (48)}*;* *4)* {(8), (18), (34), (49)}*;*
5) {(9), (21), (40), (50)}*;* *6)* {(10), (22), (41), (51)}*;*
7) {(10), (22), (41), (52)}*;* *8)* {(10), (23), (45), (56)}.

The multiplicity of the invariant straight line $x = 1$ is at least six if in each of the cases 1)–8) of Lemma 2.4 the identity $Y_6(y) \equiv 0$ holds. Taking into account (3), we will examine each case separately:
1) {(6), (11), (24), (46)} $\Rightarrow \{Y_6(y) \equiv 0, \gcd(p, q) = 1\} \Rightarrow$

$$b_{02} = 3/2, \quad b_{11} = 2a_{20} \neq 0. \tag{59}$$

In the cases 2) {(6), (11), (24), (47)}, 3) {(8), (18), (33), (48)}, 4) {(8), (18), (34), (49)}, 6) {(10), (22), (41), (51)}, 7) {(10), (22), (41), (52)} − the identity $Y_6(y) \equiv 0$ and the conditions (3) are not compatible.

5) {(9), (21), (40), (50)} $\Rightarrow Y_6(y) \equiv 0 \Rightarrow$

$$b_{02} = 3, \quad b_{20} = -3. \tag{60}$$

8) {(10), (23), (45), (56)} $\Rightarrow Y_6(y) \not\equiv 0$.

Lemma 2.5. *The invariant straight line $x = 1$ has for quartic system* (5) *the multiplicity at least six if and only if the coefficients of* (5) *verify the following series of conditions:*
1) (6), (11), (24), (46), (59)*;* *2)* (9), (21), (40), (50), (60).

In the conditions 1) of Lemma 2.5 we have $Y_7(y) = a_{20}(7a_{20} + 2a_{20}b_{20} - 6y + 6a_{20}^2 y - 2b_{20}y - 6a_{20}y^2 + y^3) \not\equiv 0$, $a_{20} \neq 0$, otherwise ($a_{20} = 0$) the right-hand sides of (5) have the common divisors of degree greater than 0. Thus the multiplicity of the invariant straight line $x = 1$ is exactly six.

In the conditions 2) of Lemma 2.5 we have $Y_7(y) = -y(b_{11} + b_{21} + b_{31} + b_{11}y^2 + 2b_{21}y^2 + 3b_{31}y^2)$. The identity $Y_7(y) \equiv 0$ and the conditions (3) are not compatible (the right-hand sides of (5) have the common divisors of degree greater than 0), therefore the multiplicity of the invariant straight line $x = 1$ is exactly six.

In this way we have proved the following theorem.

Theorem 2.1. *In the class of quartic differential systems with a center-focus critical point and non-degenerate infinity the maximal multiplicity of an affine real invariant straight line is equal to six.*

3. Solution of the center problem for quartic systems with an affine invariant straight line of maximal multiplicity.

It is known that a critical point $(0,0)$ is a center for (2) if and only if in a neighborhood of $(0,0)$ the system has a nonconstant analytic first integral $F(x,y)$ (an analytic integrating factor of the form $\mu(x,y) = 1 + \sum \mu_j(x,y)$). If $F(x,y)$ ($\mu(x,y)$) has the form $f_1^{\alpha_1} \cdots f_s^{\alpha_s}$, where f_j, $1 \leq j \leq p$ are invariant straight lines and f_j, $p+1 \leq j \leq s$ are exponential factors, then the system (2) is called Darboux integrable.

Let $F(x,y) = x^2 + y^2 + F_3(x,y) + F_4(x,y) + \cdots + F_n(x,y) + \cdots$, be a function such that

$$\frac{\partial F}{\partial x} p(x,y) + \frac{\partial F}{\partial y} q(x,y) \equiv \sum_{j=1}^{\infty} L_j (x^2+y^2)^{j+1}, \tag{61}$$

where $F_k(x,y) = \sum_{i+j=k} f_{ij}x^i y^j$, $f_{0j} = 0$ if j is even. The L_j are polynomials in the coefficients of (2) and are called the Lyapunov quantities.

For example, the first two quantities look as

$L_1 = (a_{12} - a_{02}a_{11} - a_{11}a_{20} + 3a_{30} + 2a_{02}b_{02} - 3b_{03} + b_{02}b_{11} - 2a_{20}b_{20} + b_{11}b_{20} - b_{21})/4$,

$L_2 = (10a_{02}^3 a_{11} + 41a_{02}a_{03}a_{11} - 12a_{04}a_{11} - a_{02}a_{11}^3 - 10a_{02}^2 a_{12} - 21a_{03}a_{12} + a_{11}^2 a_{12} - 20a_{02}a_{13} + 124a_{02}^2 a_{11}a_{20} + 37a_{03}a_{11}a_{20} - a_{11}^3 a_{20} - 94a_{02}a_{12}a_{20} - 28a13a_{20} + 238a_{02}a_{11}a_{20}^2 - 112a_{12}a_{20}^2 + 124a_{11}a_{20}^3 + 19a_{02}a_{11}a_{21} - 15a_{12}a_{21} + 23a_{11}a_{20}a_{21} - 4a_{11}a_{22} - 90a_{02}^2 a_{30} - 27a_{03}a_{30} - 5a_{11}^2 a_{30} - 378a_{02}a_{20}a_{30} - 372a_{20}^2 a_{30} - 33a_{21}a_{30} - 12a_{02}a_{31} - 36a_{20}a_{31} + 20a_{11}a_{40} - 20a_{02}^3 b_{02} - 82a_{02}a_{03}b_{02} + 24a_{04}b_{02} - 39a_{02}a_{11}^2 b_{02} + 33a_{11}a_{12}b_{02} - 228a_{02}^2 a_{20}b_{02} - 32a_{03}a_{20}b_{02} - 37a_{11}^2 a_{20}b_{02} - 288a_{02}a_{20}^2 b_{02} - 24a_{20}^3 b_{02} + 2a_{02}a_{21}b_{02} + 40a_{20}a_{21}b_{02} - 16a_{22}b_{02} + 71a_{11}a_{30}b_{02} - 88a_{40}b_{02} + 158a_{02}a_{11}b_{02}^2 - 100a_{12}b_{02}^2 + 96a_{11}a_{20}b_{02}^2 - 248a_{30}b_{02}^2 -$

$152a_{02}b_{02}^3 + 24a_{20}b_{02}^3 + 30a_{02}^2b_{03} + 63a_{03}b_{03} + 9a_{11}^2b_{03} + 322a_{02}a_{20}b_{03} + 392a_{20}^2b_{03} + 21a_{21}b_{03} - 87a_{11}b_{02}b_{03} + 228b_{02}^2b_{03} + 80a_{02}b_{04} + 88a_{20}b_{04} - 37a_{02}^2a_{11}b_{11} - 8a_{03}a_{11}b_{11} + 37a_{02}a_{12}b_{11} + 8a13b_{11} - 138a_{02}a_{11}a_{20}b_{11} + 89a_{12}a_{20}b_{11} - 101a_{11}a_{20}^2b_{11} + 147a_{02}a_{30}b_{11} + 303a_{20}a_{30}b_{11} + 64a_{02}^2b_{02}b_{11} - 5a_{03}b_{02}b_{11} - 3a_{11}^2b_{02}b_{11} + 68a_{02}a_{20}b_{02}b_{11} - 144a_{20}^2b_{02}b_{11} - 7a_{21}b_{02}b_{11} + 29a_{11}b_{02}^2b_{11} - 76b_{02}^3b_{11} - 131a_{02}b_{03}b_{11} - 287a_{20}b_{03}b_{11} - 20b_{04}b_{11} + 27a_{02}a_{11}b_{11}^2 - 27a_{12}b_{11}^2 + 27a_{11}a_{20}b_{11}^2 - 81a_{30}b_{11}^2 + 3a_{02}b_{02}b_{11}^2 + 109a_{20}b_{02}b_{11}^2 + 77b_{03}b_{11}^2 - 23b_{02}b_{11}^3 - 21a_{02}a_{11}b_{12} + 21a_{12}b_{12} - 17a_{11}a_{20}b_{12} + 51a_{30}b_{12} + 2a_{02}b_{02}b_{12} - 40a_{20}b_{02}b_{12} - 39b_{03}b_{12} + b_{02}b_{11}b_{12} + 36b_{02}b13 - 29a_{02}a_{11}^2b_{20} + 29a_{11}a_{12}b_{20} + 60a_{02}^2a_{20}b_{20} + 18a_{03}a_{20}b_{20} - 27a_{11}^2a_{20}b_{20} + 252a_{02}a_{20}^2b_{20} + 248a_{20}^3b_{20} + 8a_{02}a_{21}b_{20} + 46a_{20}a_{21}b_{20} - 8a_{22}b_{20} + 59a_{11}a_{30}b_{20} - 80a_{40}b_{20} + 136a_{02}a_{11}b_{02}b_{20} - 86a_{12}b_{02}b_{20} + 28a_{11}a_{20}b_{02}b_{20} - 178a_{30}b_{02}b_{20} - 156a_{02}b_{02}^2b_{20} + 192a_{20}b_{02}^2b_{20} - 75a_{11}b_{03}b_{20} + 234b_{02}b_{03}b_{20} - 30a_{02}^2b_{11}b_{20} - 9a_{03}b_{11}b_{20} - 3a_{11}^2b_{11}b_{20} - 232a_{02}a_{20}b_{11}b_{20} - 350a_{20}^2b_{11}b_{20} - 3a_{21}b_{11}b_{20} + 42a_{11}b_{02}b_{11}b_{20} - 142b_{02}^2b_{11}b_{20} + 53a_{02}b_{11}^2b_{20} + 159a_{20}b_{11}^2b_{20} - 23b_{11}^3b_{20} - 8a_{02}b_{12}b_{20} - 50a_{20}b_{12}b_{20} + 5b_{11}b_{12}b_{20} + 12b13b_{20} + 30a_{02}a_{11}b_{20}^2 - 30a_{12}b_{20}^2 - 16a_{11}a_{20}b_{20}^2 - 30a_{30}b_{20}^2 - 60a_{02}b_{02}b_{20}^2 + 132a_{20}b_{02}b_{20}^2 + 90b_{03}b_{20}^2 + 13a_{11}b_{11}b_{20}^2 - 76b_{02}b_{11}b_{20}^2 + 20a_{20}b_{20}^3 - 10b_{11}b_{20}^3 + 30a_{02}^2b_{21} + 9a_{03}b_{21} + 3a_{11}^2b_{21} + 134a_{02}a_{20}b_{21} + 148a_{20}^2b_{21} + 3a_{21}b_{21} - 17a_{11}b_{02}b_{21} + 64b_{02}^2b_{21} - 53a_{02}b_{11}b_{21} - 105a_{20}b_{11}b_{21} + 23b_{11}^2b_{21} - 9b_{12}b_{21} - 13a_{11}b_{20}b_{21} + 46b_{02}b_{20}b_{21} + 10b_{20}^2b_{21} + 8a_{02}b_{22} + 16a_{20}b_{22} + 4b_{11}b_{22} - 15a_{02}a_{11}b_{30} + 15a_{12}b_{30} - 19a_{11}a_{20}b_{30} + 9a_{30}b_{30} + 30a_{02}b_{02}b_{30} + 32a_{20}b_{02}b_{30} - 45b_{03}b_{30} + 8a_{11}b_{11}b_{30} - 13b_{02}b_{11}b_{30} + 34a_{20}b_{20}b_{30} - 17b_{11}b_{20}b_{30} - 3b_{21}b_{30} - 8a_{11}b_{31} + 28b_{02}b_{31} + 20b_{20}b_{31} - 24a_{20}b_{40} + 12b_{11}b_{40})/96.$

The critical point $(0,0)$ is a center if all Lyapunov quantities L_j vanish. (see [2]).

In the following we will solve the center problem for the system (5) under the conditions 1) and 2) of Lemma 2.5, i.e. when the affine line $x-1=0$ is of maximal multiplicity.

The conditions 1) of Lemma 2.5 are

$$\begin{gathered} a_{11}=-3,\ a_{02}=0,\ a_{30}=-2a_{20},\ a_{21}=3,\ a_{12}=0,\ a_{03}=0,\\ b_{11}=2a_{20},\ b_{02}=3/2,\ b_{30}=-2b_{20}-3,\ b_{21}=0,\ b_{12}=-3, b_{03}=0,\\ b_{40}=2a_{20}^2+b_{20}+2,\ b_{31}=-2a_{20},\ b_{22}=3/2, b_{13}=0,\ b_{04}=0;\ a_{20}\neq 0. \end{gathered} \tag{62}$$

The quartic system (5) takes the form:

$$\begin{aligned} \dot{x} &= (x-1)^2(a_{20}x^2+y-xy),\quad a_{20}\neq 0,\\ \dot{y} &= (-2x-2b_{20}x^2+2(3+2b_{20})x^3-2(2+2a_{20}^2+b_{20})x^4-\\ &\quad -4a_{20}xy+4a_{20}x^3y-3y^2+6xy^2-3x^2y^2)/2. \end{aligned} \tag{63}$$

We remark that the system (63) has the following integrating factor

$$\mu(x,y)=\frac{1}{(x-1)^6}.$$

The conditions 2) of Lemma 2.5 are

$$a_{20} = 0,\ a_{11} = -3,\ a_{02} = a_{30} = 0,\ a_{21} = 3,\ a_{12} = a_{03} = 0,\ b_{20} = -3,\\ b_{02} = 3,\ b_{30} = 3,\ b_{12} = -6,\ b_{03} = 0,\ b_{40} = -1,\ b_{22} = 3,\ b_{13} = 0,\ b_{04} = 0. \tag{64}$$

The quartic system (5) takes the form:

$$\begin{aligned} \dot{x} &= -y(x-1)^3,\\ \dot{y} &= -x + 3x^2 - 3x^3 + x^4 - b_{11}xy - b_{21}x^2y - b_{31}x^3y-\\ &\quad -3y^2 + 6xy^2 - 3x^2y^2, \quad b_{11}^2 + b_{21}^2 + b_{31}^2 \neq 0. \end{aligned} \tag{65}$$

The first two Lyapunov quantities of the system (65) are $L_1 = -b_{21}/4$ and $L_2 = b_{31}/2$. If $L_1 = L_2 = 0$, i.e. $b_{21} = b_{31} = 0$, then the system (65) has the following integrating factor

$$\mu(x,y) = \frac{1}{(x-1)^9} \exp\left(\frac{-b_{11}(b_{11}x^2(x^3 - 5x^2 + 10x - 10) + 20(x-1)^2y)}{20(x-1)^5}\right).$$

Theorem 3.1. *The quartic differential system* (5) *with an affine invariant straight line of maximal multiplicity six has a center at the origin* $(0,0)$ *if and only if its coefficients verify the following sets of conditions: 1)* (62)*; 2)* $\{(64),\ b_{21} = b_{31} = 0\}$.

Theorem 3.2. *The quartic differential system* (5) *with an affine invariant straight line of maximal multiplicity six has a center at the origin* $(0,0)$ *if and only if the first two Lyapunov quantities vanish* $L_1 = L_2 = 0$.

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