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On the symbol of singular operators in the case of contour with corner points

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Abstract. This paper proposes a method for constructing a symbol for singular integral operators in the case of a piecewise Lyapunov contour. The definition of the symbol function involves numbers that characterize the space in which the research is being carried out, as well as the values of the corner points of the contour, which makes it possible to obtain formulas for calculating the essential norms of singular operators and conditions for the solvability of singular equations with a shift and complex conjugation. In obtaining these results, we will essentially rely on the well-known results of I. Gelfand concerning maximal ideals of commutative Banach algebras [7]. In the absence of corner points on the integration contour, the results of this work are consistent with the results from [1].

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Asupra simbolului operatorilor singulari în cazul conturului cu puncte unghiulare

Rezumat. În această lucrare se propune o metodă de construire a simbolului operatorilor integrali singulari în cazul unui contur Lyapunov pe porţiuni. Definiţia funcţiei-simbol conţine parametrii, care caracterizează spaţiul în care se desfăşoară cercetarea, precum şi mărimile punctelor unghiulare ale conturului, ceea ce face posibilă obţinerea de formule de calcul a normelor esenţiale ale operatorilor singulari şi condiţiilor de rezolvabilitate a ecuaţiilor singulare cu translaţii şi conjugare complexă. În obţinerea acestor rezultate, ne vom baza în esenţă pe rezultatele binecunoscute ale lui I. Ghelfand privitoare la idealele maximale ale algebrelor Banach comutative [7]. În absenţa punctelor unghiulare pe conturul de integrare, rezultatele din această lucrare sunt în concordanţă cu rezultatele din [1].

Cuvinte-cheie: operator singular, algebre Banach, contur Lyapunov pe porţiuni, simbol, condiţii Noether.

1. Introduction

A great number of works are devoted to singular integral operators and Riemann boundary value problems in the case of a Lyapunov contour; it is enough to point out the monograph by I. Gokhberg and N. Krupnik [1], which contains an extensive bibliography on this issue. In papers [2], [3] and others, it was shown that the presence of corner points on the integration contour affects some properties of singular operators. In particular, if the integration contour contains one corner point with an angle equal to $\frac{\pi}{2}$, then the essential norm of the operator with the Cauchy kernel in the space L_2 is equal to $1 + \sqrt{2}$, and in the case of the Lyapunov contour this norm is equal to 1. The conditions for the Noetherian property of singular operators with shift or with complex conjugation also depend on the presence of corner points on the integration contour. As usual, by the Noether conditions of the operator A we mean, firstly, obtaining conditions under which the set of values of the operator A is a subspace, or the equality holds

$$ImA = \cap_{f \in KerA^*} Kerf,$$

and, secondly, the equations $Ax = 0$ and $A^{*}\varphi = 0$ have a finite number of linearly independent solutions. As it is known, a linear bounded Noetherian operator is true if and only if it has right and left regularizers. Obtaining the conditions for Noetherianity, as a rule, leads to the concept of an operator symbol, first introduced by S. Mikhlin, and which turned out to be fruitful in many branches of mathematics, including the construction of the Noetherian theory of singular integral operators [4], [5].

Note that Gelfand's theory of maximal ideals also played an important role in obtaining the criterion for the Noether property of one-dimensional singular integral operators with continuous coefficients, Wiener-Hopf operators, multidimensional singular operators, and Toeplitz matrices. The results presented in this paper are a generalization of known results to the case where the integration contour has corner points. Thus, in the case of the absence of corner points on the integration contour, the proposed results of this work agree with the results from [1].

Let us present some facts from the theory of Banach commutative algebras, which will be used below.

Definition 1.1. *A normed space X is called a normed algebra if it is an algebra with unity e and two more axioms are satisfied:*

$$\|e\| = 1; \quad \|xy\| \leq \|x\|\|y\| \quad \forall x, y \in X.$$

If the normed algebra X is also complete, then it is called a Banach algebra.

Let X be a commutative Banach algebra. An ideal M is called maximal if M is not contained in any other nontrivial ideal. Any ideal I (nontrivial) consists only of non-invertible elements. Any ideal is contained in a maximal ideal. According to I. Gelfand's Theorem [7], a Banach algebra over the field of complex numbers, which is a field, is isometrically isomorphic to the field $\mathbb{C}$.

A linear continuous functional f defined on a Banach algebra X is called multiplicative if for any x and y the equality holds

$$f(xy) = f(x) \cdot f(y).$$

The zero subspace of the functional f (i.e. the totality of those $x \in X$ for which $f(x) = 0$) is denoted by *Kerf* and is called the kernel of f.

Theorem 1.1. *The kernel Kerf for any multiplicative functional f is a maximal ideal.*

Theorem 1.2. *For any maximal ideal M, one can construct a unique multiplicative functional f such that Kerf=M.*

Conclusion. Thus, there is a one-to-one correspondence between the set of maximal ideals $\{M\}$ and the set of multiplicative functionals f defined on the algebra X. Therefore, the corresponding functionals are denoted f_M, $(f \leftrightarrow M)$.

Theorem 1.3. *(Gelfand (see* [7]. *An element $x \in X$ is invertible in X if and only if it is not contained in any maximal ideal (equivalent to $f(x) \neq 0$ for any multiplicative functional).*

Thus, the problem of invertibility in the algebra X can be reduced to determining all maximal ideals or to determining all multiplicative functionals defined on X.

2. Algebra $\mathcal{U}_{p\beta}$

Let $\mathcal{U}$ be some algebra (commutative or non-commutative). Recall that a set $\{f_M\}$ of multiplicative functionals is called sufficient if an element x is invertible in $\mathcal{U}$ if and only if $f_M(x) \neq 0$ for any M. According to I. Gelfand's theorem, every commutative Banach algebra has a sufficient set of multiplicative functionals. The set of functionals of the form $\{f_M\}$, where M runs over the set of maximal ideals, forms a sufficient set of functionals.

A simple example of a non-commutative Banach algebra that has a sufficient set of multiplicative functionals is the algebra of upper triangular numerical matrices

$$\mathcal{U} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \right\} \quad (a_{jk} \in C).$$

Two functionals $f_1(a) = a_{11}$ and $f_2(a) = a_{22}$ form a sufficient set.

Let $E = L_2(a,b)$ and let $\mathcal{U}$ be a subalgebra of $L(E)$, generated by one singular operator S:

$$\left(S_\varphi\right)(t) = \frac{1}{\pi i}\int_a^b \frac{\varphi(\tau)}{\tau - t}d\tau \quad (t \in [a,b]).$$

Since $S^* = S$, then $\mathcal{U}$ is a c^* subalgebra of $L(E)$ and, in particular, it is symmetric. The spectrum of the element S in the algebra $\mathcal{U}$ coincides with its spectrum in the algebra $L(E)$, i.e. with the segment $[-1,1]$. Each multiplicative functional is defined by a point $\tau \in [-1,1]$.

$$f_\tau\left(\sum_{k=0}^{n}\alpha_k S^k\right) = \sum_{k=0}^{n}\alpha_k\tau^k.$$

In particular, the operator $A = \alpha I + \beta S$ $(\alpha, \beta \in C)$ is invertible in $\mathcal{U}$ if and only if $\alpha + \beta\tau \neq 0, \forall\tau \in [-1,1]$.

Consider the operator B, defined by the equality

$$(B\varphi)(t) = \alpha\varphi(t) + \frac{\beta}{\pi i}\int_a^b \frac{\varphi(\tau)}{\tau - t}d\tau + \frac{\gamma}{(\pi i)^2}\int_a^b Ln\frac{(b-t)(\tau-a)}{(t-a)(d-\tau)}\frac{\varphi(\tau)\,d\tau}{\tau - t}.$$

The operator B belongs to the algebra $\mathcal{U}$. Indeed, using the Poincaré-Bertrand formula, it is easy to deduce that $B = \alpha I + \beta S + \gamma\left(S^2 - I\right)$. This implies:

Theorem 2.1. *The operator B is invertible if and only if the inequality* $\gamma\tau^2 + \beta\tau + (\alpha - \gamma) \neq 0$ *holds for all* $\tau \in [-1,1]$.

Let us introduce the following notation. We denote by $L(\mathcal{B})$ the algebra of all linear bounded operators acting in a Banach space $\mathcal{B}$. Let $\mathcal{U}_{p\beta}$ be the smallest Banach subalgebra with algebra unit $L\left(L_p\left(R^+, t^\beta\right)\right)$ $(R^+ = [0, +\infty))$, containing the operator

$$\left(S_\varphi\right)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - t}d\tau \quad \left(t \in R^+\right).$$

We will assume that $1 < p < \infty$ and $-1 < \beta < p - 1$. Let δ be a number from the interval $\left(0, \frac{1}{2}\right)$. Let us denote by $l(\delta)$ an arc of a circle containing points -1 and 1 having the following property: from point z $(z \neq \pm 1)$ of the arc $l(\delta)$ the segment $[-1,1]$ is visible at an angle of $2\pi\delta$ and when going around the arc $l(\delta)$ from point -1 to 1 this segment remains to the left. For numbers δ from the interval $\left(\frac{1}{2}, 1\right)$ we set $l(\delta) = -l(1-\delta)$. Let $l\left(\frac{1}{2}\right)$ denote the segment $[-1,1]$. As, it is known [1], the spectrum of the operator S in the space $L_p\left(R^+, |t|^\beta\right)$ coincides with the arc $l\left(\frac{1+\beta}{p}\right)$. Since the algebra $\mathcal{U}_{p\beta}$ is generated by one element, then [1] takes place.

Theorem 2.2. *The set of maximal ideals of the algebra* $\mathcal{U}_{p\beta}$ *is homeomorphic to the arc* $l = l\left(\frac{1+\beta}{p}\right)$. *If* M_z *is the maximal ideal corresponding to the point* $z\,(\in l)$, *then the Gelfand transformation* $S\left(M_z\right) = z$.

This theorem can be significantly expanded (see [6]).

Theorem 2.3. *The algebra $\mathcal{U}_{p\beta}$ is an algebra without a radical with a symmetric involution $A \to \overleftarrow{A}$. In particular,*

$$\overleftarrow{S} = (\cos 2\pi\gamma S - i\sin 2\pi\gamma I)\,(\cos 2\pi\gamma I - i\sin 2\pi\gamma S)^{-1}\left(\gamma = \frac{1+\beta}{p}\right).$$

For $p = 2$, the Gelfand transformation $A(z) = A(M_z)$ satisfies the equality

$$\|A\| = \max_{z\in l(\gamma)} |A(z)|, \tag{1}$$

and for $p \neq 2$, the following estimates hold:

$$\max_{z\in l(\gamma)} |A(z)| \leq \|A\| \leq c\cdot\max\left(\max_{z\in l(\gamma)} |A(z)|,\ \max_{z\in l(\gamma)}\left|\left(1-z^2\right) Ln\frac{1-z}{1+z}\frac{dA(z)}{dz}\right|\right) \tag{2}$$

where the constant c depends only on p and β.

Proof. Let $\gamma = \frac{1+\beta}{p}$. The operator B, defined by the equality $\left(B_\varphi\right)(t) = e^{\gamma t}\varphi\left(e^t\right)$, isometrically maps the space $L_p\left(R^+, t^\beta\right)$ onto $L_p(R)$. It is directly verified that the operator $\tilde{S} = BSB^{-1}$ has the form

$$\left(\tilde{S}_\varphi\right)(t) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{e^{(t-s)\gamma}\varphi(s)}{1-e^{t-s}}ds.$$

Thus, the algebra $\mathcal{U}_{p\beta}$, generated by one operator S, is isometric to some subalgebra of the convolution algebra and, therefore, has no radical [7]. Let $\pi i\hat{S}(\xi)$ be the Fourier transform of the function $\frac{e^{t\gamma}}{1-e^t}$. It can be shown (we will not go into details) that

$$\hat{S}(\xi) = \frac{e^{2\pi(\xi+i\gamma)}+1}{e^{2\pi(\xi+i\gamma)}-1}\ (-\infty \leq \xi \leq +\infty). \tag{3}$$

The set of values of the function $\hat{S}(\xi)$ runs along the arc $l(\gamma)$. We set $z = \hat{S}(\xi)$, then the operator $A \in \mathcal{U}_{p\beta}$ satisfies the equality

$$\boldsymbol{A}\left(\hat{S}(\xi)\right) = \left(FBAB^{-1}F^{-1}(\xi)\right),$$

where F is the Fourier transform. This, in particular, implies equality (1) for $p = 2$. For $p \neq 2$, a lower estimation for the norms of the operator A follows from Theorem 2.2. The upper estimation is obtained using theorem on multipliers of S. Mikhlin [4], in which it is established that

$$\|BAB^{-1}\| \leq \tilde{c}_p\cdot\max\left(\max_{\xi\in R} A\left(\hat{S}(\xi)\right)\right),\ \max_{\xi\in R}\left|\xi\cdot\frac{dA\left(\hat{S}(\xi)\right)}{d\xi}\right|,$$

where the number $\tilde{c}_p$ depends only on p. The theorem has been proven. □

Remark 2.1. *Let us define the functional over $L_2\left(R^+, t^\beta\right)$ by the equality*

$$f(\varphi) = \int_0^\infty \varphi(t) f(t) t^\beta dt,$$

then $S^* = t^{-\beta} S t^\beta I$. It is directly verified that $FBS^*B^{-1}F^{-1} = FB\bar{S}B^{-1}F^{-1}$. Therefore, for $p = 2$, we have $\bar{S} = S^*$.

Corollary 2.1. *Let the function f be differentiable at each point $z \in l(\gamma) \setminus \{-1, 1\}$. If there exists a sequence of polynomials P_n such that*

$$\max_{z \in l(\gamma)} \left|P_n(z) - f(z)\right| \to 0; \quad \max_{z \in l(\gamma)} \left|(1 - z^2) Ln \frac{1 - z}{1 + z} (P_n'(z) - f'(z)\right| \to 0$$

as $n \to \infty$, then $f(S) \in \mathcal{U}_{p\beta}$.

A more general corollary is the following.

Corollary 2.2. *Let $A_0 \in \mathcal{U}_{p\beta}$ and let $\varphi(z)$ be the Gelfand's transform of operator A_0 and h be differentiable at each point $z \in l(\gamma) \setminus \{-1, 1\}$. If there exists a sequence of polynomials P_n such that*

$$\underset{z \in l(\gamma)}{max} \left|P_n(z) - h(z)\right| \to 0; \quad \underset{z \in l(\gamma)}{max} \left|(1 - z^2) \frac{d}{dz} Ln \frac{1 - z}{1 + z} (P_n(\varphi(z)) - h(z)\right| \to 0$$

as $n \to \infty$, then $h(A_0) \in \mathcal{U}_{p\beta}$.

In what follows, we will need the following theorem.

Theorem 2.4. *Let $\omega = e^{\pi i \alpha}$, where α is some complex number. If $-1 < Re\alpha < 1$, then the operator N_ω, defined by the equality*

$$(N_\omega \varphi)(x) = \frac{1}{\pi i} \int_{R^+} \frac{\varphi(y)}{y + \omega x} dy, \left(x \in R^+\right),$$

belongs to the algebra $\mathcal{U}_{p\beta}$ and its Gelfand transformation has the form

$$N_\omega(z) = (z - 1)^{\frac{1+\alpha}{2}} (z + 1)^{\frac{1-\alpha}{2}} \quad (z \in l(\gamma)). \tag{4}$$

The branch of this function is chosen so that at $z = -ictg\pi\gamma$ it takes the value

$$-\frac{iexp(-\pi i \gamma \alpha)}{\sin \pi \gamma}.$$

Proof. It is directly verified that

$$\pi i B N_\omega B^{-1} \varphi = \left(e^{\gamma t} \left(1 + \omega e^t\right)\right) * \varphi.$$

It follows that

$$f_z(N_\omega) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{e^{\gamma t - i\xi t}}{t + \omega e^t} dt = \frac{-ie^{i\xi - \gamma}}{\sin((\gamma - i\xi)\pi)} = (z - 1)^{\frac{1+\alpha}{2}} (z + 1)^{\frac{1-\alpha}{2}}.$$

Let us show that the function $h(z) = f_z(N_\omega)$ satisfies the conditions of Corollary 2.1 or 2.2. Let first $\left|\gamma - \frac{1}{2}\right| \leq \frac{1}{4}$, then $|z| \leq 1$. In this case, for any $\delta (Re\delta > 0)$, the function $(z+1)^\delta$ satisfies the condition of Corollary 2.2 (for example, partial sums of the Taylor series can be taken as the polynomials $P_n(z)$. If $|\gamma - \frac{1}{2}| > \frac{1}{4}$, then the function

$$f_z(N_\omega) = z\left(1 - z^{-1}\right)^{\frac{1+\alpha}{2}} \left(1 + z^{-1}\right)^{\frac{1-\alpha}{2}}$$

satisfies the conditions of Corollary 2.1. The role of the operator A_0 is played by the operator S^{-1}. The invertibility of the operator S follows from the condition $\gamma \neq \frac{1}{2}$. The theorem is proved. □

3. Symbol of the operator $aI + bS_\Gamma$

Let the contour Γ_α consist of two semi-axes starting from the point $z = 0$. We denote by α $(0 < \alpha \leq \pi)$ the angle formed by these half-lines. We will assume that one of these semi-straight lines coincides with the semi-axis $R^+ = [0, +\infty)$ and that the contour Γ_α is oriented in such a way that on $\Gamma_\alpha \cap R^+$ the orientation coincides with that on R^+.

Let $B = L_p\left(\Gamma_\alpha, |t|^\beta\right)$ $(-1 < \beta < p-1)$ and denote by $\lambda_0\left(\Gamma_\alpha\right)$ the set of constant functions on portions that receive two values on Γ_α: one value on R^+ and another value on $\Gamma_\alpha \backslash R^+$. If $h \in \lambda_0(\Gamma)$, then we write

$$h(t) = \begin{cases} h_1, & for \quad t \in R^+ \\ h_2, & for \quad t \in \Gamma_\alpha \backslash R^+ \end{cases}, \quad h_j \in \mathbb{C}.$$

So, $h(0) = h_2$, $h(0+0) = h_1$, $h(\infty - 0) = h_1$, $h(\infty + 0) = h_2$.

We will consider the contour Γ_α compactified with a point at infinity, whose neighborhoods are complementary to the neighborhoods of $z_0 = 0$. Obviously, the contour Γ_α is homeomorphic to a bounded contour $\tilde{\Gamma}$, which has two angular points.

We denote by K_α the Banach algebra generated by the singular integration operator S_Γ and by all multiplication operators on the functions $h \in \lambda_0(\Gamma_\alpha)$. By K^+ we denote the subalgebra of the algebra $L\left(L_p\left(R^+, |t|^\beta\right)\right)$ generated by the singular integral operators $aI + bS$ $(S = S_{R^+})$ with constant coefficients on R^+. As K^+ is commutative, then it possesses [5] a sufficient system of multiplicative functionals. The operator ν,

$$(\nu\varphi)(x) = \left(\varphi(x), \varphi\left(e^{i\alpha}x\right)\right) \left(x \in R^+\right),$$

is linear and bounded and acts from the space $L_p\left(\Gamma_\alpha, |t|^\beta\right)$ to the space $L_p^2\left(R^+, t^\beta\right)$. Let $\varphi \in L_p\left(\Gamma_\alpha, |t|^\beta\right)$ and consider the equation

$$A\varphi = a\varphi + bS_{\Gamma_\alpha}\varphi = \psi,$$

$$a(t)=\begin{cases} a_1, & for \quad t\in R^+ \\ a_2, & for \quad t\in \Gamma_\alpha\backslash R^+ \end{cases}, \qquad b(t)=\begin{cases} b_1, & for \quad t\in R^+ \\ b_2, & for \quad t\in \Gamma_\alpha\backslash R^+ \end{cases}, \qquad a_j,b_j\in\mathbb{C}.$$

This equation can be written as a system of equations: in one equation $t\in R^+$, and in the second equation $t\in\Gamma_\alpha\backslash R^+$. We get,

$$\begin{cases} a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b(\tau)}{\pi i}\int_{\Gamma_\alpha\backslash R^+}\frac{\varphi(\tau)}{\tau-t}d\tau=\psi(t), & t\in R^+, \\ a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b(\tau)}{\pi i}\int_{\Gamma_\alpha\backslash R^+}\frac{\varphi(\tau)}{\tau-t}d\tau=\psi(t), & t\in\Gamma_\alpha\setminus R^+. \end{cases}$$

In the integral

$$\int_{\Gamma_\alpha\backslash R^+}\frac{\varphi(\tau)}{\tau-t}d\tau$$

we change the variable $\tau\to e^{i\alpha}\tau$ and in the second equation of the obtained system, we change t by $e^{i\alpha}t$. Then, we obtain

$$\begin{cases} a_1\varphi_1(t)+\frac{b_1}{\pi i}\int_{R^+}\frac{\varphi_1(\tau)}{\tau-t}d\tau-\frac{b_1}{\pi i}\int_{R^+}\frac{\varphi_2(\tau)}{\tau-e^{-i\alpha}t}d\tau=\psi_1(t), & t\in R^+, \\ a_2\varphi_2(t)+\frac{b_2}{\pi i}\int_{R^+}\frac{\varphi_1(\tau)}{\tau-e^{i\alpha}t}d\tau-\frac{b_2}{\pi i}\int_{R^+}\frac{\varphi_2(\tau)}{\tau-t}d\tau=\psi_2(t), & t\in R^+. \end{cases}$$

in which the notations were used: $f_1(t)=f(t)$, $f_2(t)=f\left(e^{i\alpha}t\right)$ $(t\in R^+)$.

Thus, the operator $\nu A\nu^{-1}$ has the form

$$\nu A\nu^{-1}=\begin{Vmatrix} a_1I+b_1S, & -b_1M \\ b_2N, & a_2I-b_2S \end{Vmatrix},$$

where

$$(S\varphi)(t)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-t}d\tau,\quad (M\varphi)(t)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-e^{-i\alpha}t}d\tau,$$

$$(N\varphi)(t)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-e^{i\alpha}t}d\tau \quad \left(t\in R^+\right).$$

From Theorems 2.2 and 2.3 it follows that operators M and N belong to the algebra K^+ generated by the operator $S(=S_{R^+})$ and the multiplication operators to the constant functions. Therefore, $\nu K_\alpha\nu^{-1}\subset (K^+)^{2\times 2}$. Let $\{\gamma_M\}$ be the homeomorphism system that determines the symbol on the algebra K^+. For any operator $A\in K_\alpha$ we put

$$\widetilde{\gamma}_M(A)=\left\|\gamma_M\left(A_{jk}\right)\right\|_{j,k=1}^2,\ \text{where}\ \left\|A_{jk}\right\|_{j,k=1}^2=\nu A\nu^{-1}.$$

4. Conditions for Noetherianity

Theorem 4.1. *The operator $A\in K_\alpha$ is Noetherian in the space $L_p\left(\Gamma_\alpha,|t|^\beta\right)$ if and only if*

$$det\widetilde{\gamma}_M(A)\neq 0.$$

Indeed, the factor algebra $\widehat{K}^{+}$ with respect to all compact operators in $L\left(L_p\left(R^+, t^\beta\right)\right)$ is commutative, therefore, the elements of the matrix operator $\left\|A_{jk}\right\|_{j,k=1}^{2} = \nu A \nu^{-1}$ commute up to compact. Then, according to [5], the operator $\left\|A_{jk}\right\|_{j,k=1}^{2}$ is Noetherian in $L_p\left(R^+, t^\beta\right)$, if and only if the operator $\Delta = det\left\|A_{jk}\right\|$ is Noetherian in $L_p\left(R^+, t^\beta\right)$. But the operator $det\left\|A_{jk}\right\|$ is Noetherian if and only if $\gamma_M\left(det\left\|(A_{jk})\right\|\right)$. As $\gamma_M\left(det\left\|(A_{jk})\right\|\right) = det\left\|\gamma_M\left(A_{jk}\right)\right\|$, it follows that A is Noetherian if and only if $det\widetilde{\gamma}_M(A) \neq 0$.

The theorem is proved.

Conclusion. Theorem 4.1 allows us to define a symbol on the algebra K. Namely, it is natural to call the matrix $\widetilde{\gamma}_M(A)$ a symbol of the operators $A \in K$. Taking into account formulas (3) and (4), the symbol of the operators $H = hI$, $h \in \lambda_0(\Gamma)$ and S_Γ will have the form:

$$\widetilde{\gamma}_M(H) = \left\| \begin{matrix} h_1 & 0 \\ 0 & h_2 \end{matrix} \right\|, \ \widetilde{\gamma}_M(S_\Gamma) = \left\| \begin{matrix} z & (z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} \\ (z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} & -z \end{matrix} \right\|. \quad (5)$$

We will write the symbol of the operator S_Γ in a more convenient form. For this let us put

$$z = \frac{e^{2\pi(\xi+i\gamma)}+1}{e^{2\pi(\xi+i\gamma)}-1} = cth\left(\pi\left(\xi+i\gamma\right)\right) \left(-\infty \le \xi \le +\infty,\ \gamma = \frac{1+\beta}{p}\right).$$

Then

$$(z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} = 2\frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{e^{\pi(\xi+i\gamma)}-e^{-\pi(\xi+i\gamma)}} = \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)},$$

$$(z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} = 2\frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{e^{\pi(\xi+i\gamma)}-e^{-\pi(\xi+i\gamma)}} = \frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)}.$$

Therefore the symbol of the operator S_Γ takes the form

$$\widetilde{\gamma}_M(S_\Gamma) = \left\| \begin{matrix} cth\left(\pi(\xi+i\gamma)\right) & \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)} \\ \frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)} & -cth\left(\pi(\xi+i\gamma)\right) \end{matrix} \right\|. \quad (6)$$

Remark 4.1. *If $\alpha = \pi$, that is, the contour Γ satisfies the Lyapunov conditions at the point $z_0 = 0$, then the symbol of the operator $H = hI$ remains the same, and the symbol of the operator S_Γ has the form*

$$\widetilde{\gamma}_M(S_\Gamma) = \left\| \begin{matrix} z & \sqrt{z^2-1} \\ \sqrt{z^2-1} & -z \end{matrix} \right\| = \left\| \begin{matrix} cth\pi\left(\xi+i\gamma\right) & \left(sh\pi(\xi+i\gamma)\right)^{-1} \\ \left(sh\pi(\xi+i\gamma)\right)^{-1} & -cth\pi\left(\xi+i\gamma\right) \end{matrix} \right\|. \quad (7)$$

Now we have what it is needed to define the symbol of the singular integral operators with coefficients in $CP(\Gamma)$ in the case of the piecewise Lyapunov contour.

So, let Γ be a piecewise closed Lyapunov contour. We denote by $t_1, \dots, t_n$ all angular points with angles α_k $(0 < \alpha_k < \pi)$ $(k = 1, \dots, n)$ and

$$p(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \ (1 < p < \infty, \ -1 < \beta_k < p - 1) .$$

We denote by $\Sigma(\Gamma, p)(\subset L(L_p(\Gamma, p)))$ the algebra generated by the operators $(H\varphi)(t) = h(t)\varphi(t)$, $h(t) \in CP(\Gamma)$ and the operator S_Γ. We mention, that the ideal formed by the compact operators acting in the space $L_p(\Gamma, p)$ is contained in the algebra $\Sigma(\Gamma, p)$.

$$H(t, \xi) = \left\| \begin{array}{cc} h(t+0) & 0 \\ 0 & h(t-0) \end{array} \right\| . \tag{8}$$

We define the symbol $S_\Gamma(t, \xi)$ of the operator S_Γ as follows:

$$S(t, \xi) = \left\| \begin{array}{cc} cth\pi(\xi + i\gamma(t)) & -\frac{exp((\alpha(t)-\pi)(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma(t))} \\ \frac{exp((\pi-\alpha(t))(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma(t))} & -cth\pi(\xi + i\gamma(t)) \end{array} \right\| , \tag{9}$$

where

$$\alpha(t) = \begin{cases} \alpha_k, & if \quad t = t_k (k = 1, 2, \dots, n) \\ \pi, & if \quad t \in \Gamma \setminus \{t_1, t_2, \dots, t_n\} \end{cases}$$

and

$$\gamma(t) = \begin{cases} \frac{1+\beta_k}{p}, & if \quad t = t_k (k = 1, 2, \dots, n) \\ \frac{1}{p}, & if \quad t \in \Gamma \setminus \{t_1, t_2, \dots, t_n\} \end{cases}$$

Theorem 4.2. *Let and $A \in \Sigma(\Gamma, \rho)$ and $A(t, \xi)$ be its symbol. The operator A is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if*

$$det A(t, \xi) \neq 0 \quad (t \in \Gamma, -\infty \leq \xi \leq +\infty).$$

The proof of Theorem 4.2 follows from Theorem 4.1, using the results from [8].

Theorems 4.1 and 4.2 can be generalized to the case where the integration contour is complex. More precisely, let Γ consist of n rays: $\Gamma = \bigcup_{m=1}^{n} \Gamma_m$, where $\Gamma_m = (\varepsilon_m x : x \in R^+, \ \varepsilon_m \in \mathbb{C}, ||\varepsilon_m| = 1)$, $PC_0(\Gamma)$ is the set of functions continuous on $\Gamma \setminus \{0\}$ and having finite limits as $t \to 0$ and $t \to \infty$ along each ray Γ_m and $K_p(\subset L(L_p(\Gamma)))$ is the algebra generated by singular operators with coefficients from $PC_0(\Gamma)$. We assume that $\varepsilon_1 = 1$, i.e. that $\Gamma_1 = R^+$. Let μ denote the isometry $L_p(\Gamma) \to L_p^n(\Gamma_1)$, defined by the equality $\mu\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$, where $\varphi_k(t) = \varphi(\varepsilon_k t)$ $(k = 1, 2, \dots, n; t \geq 0)$. In this case

$$\mu H \mu^{-1} = \left\| \begin{array}{cccc} H_1 & 0 & \dots & 0 \\ 0 & H_2 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & H_n \end{array} \right\| , \quad \mu S_\Gamma \mu^{-1} = \left\| R_{jk} \right\|_{j,k=1}^{n} .$$

Here

$$(H\varphi)(t) = h(t)\varphi(t),\ (H_k\varphi)(t) = h(\varepsilon_k t)\varphi(t)\ (t \in \Gamma_1)$$

and

$$(R_{jk}\varphi) = \frac{1}{\pi i}\int_0^\infty \frac{\varphi(\tau)d\tau}{\tau - \varepsilon_j^{-1}\varepsilon_k t}.$$

It follows from Theorem 2.4 that $R_{jk} \in K^+$, hence $\mu K_p \mu^{-1} \subset (K^+)^{n\times n}$. As in Theorem 4.1, it can be shown that the operator $A \in K_p$ is Noetherian if and only if the condition

$$det\left\|\widetilde{\gamma}_M\left(A_{jk}\right)\right\|_{j,k=1}^n \neq 0,$$

where $\mu A\mu^{-1} = \left\|A_{jk}\right\|_{j,k=1}^n$. Thus, $\left\|\widetilde{\gamma}_M\left(A_{jk}\right)\right\|_{j,k=1}^n$ defines a matrix symbol on K_p.

5. Calculation of essential norms of singular operators

Recall (see [9]) that for any operator A from some Banach algebra $\mathcal{U}$ with symmetric symbol the following relation holds:

$$\inf_{T\in\mathcal{T}}\|A+T\| = \max_x S_1\left(\mathcal{A}(x)\right), \tag{10}$$

where $\mathcal{A}(x)$ is the symbol of the operator A, and $S_1^2(\mathcal{A}(x))$ denotes the largest eigenvalue of the matrix $\mathcal{A}(x)\cdot(\mathcal{A}(x))^*$. Equality (10) is equivalent to the following equality

$$\inf_{T\in\mathcal{J}}\|A+T\|^2 = \max_{\lambda\in\hat{\sigma}(AA^*)}\lambda, \tag{11}$$

where $\hat{\sigma}(AA^*)$ denotes the spectrum of the residue class $\{AA^* + T\}$ in the quotient algebra $\mathcal{U}/\mathcal{T}$. The set $\hat{\sigma}(AA^*)$ coincides with the set of numbers λ for which the operator $AA^* - \lambda I$ is not Noetherian.

Applying equality (11) to the operator S_{Γ_α}, taking into account formula (7), we obtain,

$$|S_{\Gamma_\alpha}|_\beta^2 = \lim_{\xi\in R}\left(f(\xi) + \sqrt{f^2(\xi)-1}\right), \tag{12}$$

where

$$f(\xi) = \frac{e^{4\pi\xi} + 2\left(e^{(4\pi-2\alpha)\xi} + e^{2\alpha\xi} - \cos\pi\beta e^{2\pi\xi}\right) + 1}{e^{4\pi\xi} + 2\cos\pi\beta e^{2\pi\xi} + 1}.$$

Let us give some examples. Suppose $\alpha = \pi$, i.e. Γ_α is the real axis $\mathbb{R}$, then from equality (12) we obtain

$$|S_{\Gamma_\alpha}|_\beta = \text{ctg}\,\frac{\pi(1-|\beta|)}{4}.$$

Assume that $\beta = 0$ and let

$$z = \frac{1-e^{2\pi\xi}}{1+e^{2\pi\xi}}\quad(-\infty \leq \xi \leq +\infty),$$

then from equality (12) follows the following formula for the essential norm of the operator S_{Γ_α}:

$$|S_{\Gamma_\alpha}|_0 = \operatorname{ctg}\left(\frac{\theta(\alpha)}{2}\right),$$

where

$$2\operatorname{ctg}\theta(\alpha) = \max_{-1\le z\le 1}\left|(1+z)\left(\frac{1-z}{1+z}\right)^{\frac{\alpha}{2\pi}} + (1-z)\left(\frac{1+z}{1-z}\right)^{\frac{\alpha}{2\pi}}\right|.$$

In particular, for $\alpha = \frac{\pi}{3}$, $\alpha = \frac{\pi}{2}$, we obtain $|S_{\Gamma_\alpha}|_0 = \frac{1+\sqrt{5}}{2}$, $|S_{\Gamma_\alpha}|_0 = \sqrt{2}$.

Thus, in the case of a contour with corner points, the essential norm of the singular operator also depends on the values of the angles formed by the contour at its corner points. We also note that for any α $(0 < \alpha \le \pi)$, the inequalities hold

$$1 \le |S_{\Gamma_\alpha}|_0 < 1 + \sqrt{2}. \tag{13}$$

Next, we will consider the case where the integration contour Γ has a finite number of corner points.

Let Γ be a piecewise Lyapunov contour, $\tau_1, \tau_2, \dots, \tau_s$ be all corner points of the contour Γ, and $\alpha_1, \alpha_2, \dots, \alpha_s$ be the angles between the one-sided tangents to Γ at the points $\tau_1, \tau_2, \dots, \tau_s$, respectively. In the space $L_2(\Gamma)$, we will consider the operator A defined by the equality

$$A = S_\Gamma S_\Gamma^* - \lambda I.$$

The symbol of the operator A is the matrix function $A(t,\xi)$ $(t \in \Gamma, -\infty \le \xi \le \infty)$ of the second order, defined as follows:

At points t that do not coincide with any of the points $\tau_1, \tau_2, \dots, \tau_s$, we have

$$A(t,\xi) = (1-\lambda)E_2, \tag{14}$$

where E_2 is the identity matrix of the second order. But, at the points τ_k $(k = 1, 2, \dots, s)$ we obtain

$$A(\tau_k,\xi) = S_k(\xi)(S_k(\xi))^* - \lambda E_2, \tag{15}$$

where $S_k(\xi)$ coincides with the right-hand side of equality (9), in which $p = 2$ and $\beta_k = 0$.

Theorem 5.1. *An operator $A = S_\Gamma S_\Gamma^* - \lambda I$ is Noetherian in the space $L_2(\Gamma)$ if and only if the determinant of its symbol is nonzero:*

$$det A(t,\xi) \ne 0 (t \in \Gamma, -\infty \le \xi \le \infty).$$

To prove this theorem, we need the following lemma.

Lemma 5.1. *An operator* $A_\alpha = S_\alpha S_\alpha^* - \lambda I$ $(S_\alpha = S_{\Gamma_\alpha})$, *acting in the space* $L_2(\Gamma_\alpha)$, *is a local Noetherian operator** *at* $t = 0$ *if and only if it is a local Noetherian operator at* $t = \infty$.

Proof. Let the operator A_α be local Noetherian at $t = 0$. This means (see [9]) that it has left and right local regularizers at this point, i.e. there exist operators R_1, R_2 and a neighborhood $U_0(\ni 0)$ such that. □

$$R_1 A_\alpha P_{U_0} = P_{U_0} + T_1, \quad P_{U_0} A_\alpha R_2 = P_{U_0} + T_2, \tag{16}$$

where T_1 and T_2 are compact operators and P_{U_0} is an operator acting according to the rule

$$\left(P_{U_0}\varphi\right)(t) = \begin{cases} \varphi(t), & if \quad t \in U_o \\ 0, & if \quad t \in U_0 \backslash \Gamma_\alpha \end{cases}.$$

Let us consider the operator M defined by the equality

$$\left(M_\varphi\right)(t) = \frac{e^{i\alpha}}{t}\varphi\left(\frac{e^{i\alpha}}{t}\right) \quad (t \in \Gamma_\alpha).$$

It is easy to prove that the operator M acts in the space $L_2(\Gamma_\alpha)$, $\|M\| = 1$ and the following equalities holds:

$$M S_\alpha M^{-1} = S_\alpha, \quad M S_\alpha^* M^{-1} = S_\alpha^*. \tag{17}$$

Applying the operator M to the equality (15) on the left and M^{-1} on the right and taking into account the equality (16), we obtain

$$\widetilde{R}_1 A_\alpha P_{U_\infty} = P_{U_\infty} + \widetilde{T}_1, \quad P_{U_\infty} A_\alpha \widetilde{R}_2 = P_{U_\infty} + \widetilde{T}_2, \tag{18}$$

where $\widetilde{R}_i = M P_i M^{-1}$ and $\widetilde{T}_i = M P_i M^{-1}$ $(i = 1, 2)$, and U_∞ is a neighborhood of the point $t = \infty$. The equality (18) means that the operator A_α is locally Noetherian at the point $t = \infty$. The converse statement of the lemma is proved similarly. The lemma is proved.

Proof of the Theorem 5.1. Let A be a Noetherian operator and U_τ be some neighborhood of a point $\tau(\in \Gamma)$ that does not contain points $\tau_k \neq \tau$. By φ_τ we denote a function defined on U_τ as follows. If $\tau \neq \tau_k$, then we set $\varphi_\tau(t) \equiv t$ $(t \in U_\tau)$. If $\tau = \tau_k$ $(k = 1, 2, \ldots, s)$, then φ_{τ_k} is a function that maps one-to-one the neighborhood U_{τ_k} onto some neighborhood $V_k(\Gamma_{\alpha_k})$ of the point $t = 0$, where $\varphi_{\tau_k} = 0$ $(k = 1, 2, \ldots, s)$. Since Γ is a piecewise Lyapunov contour, it is possible to achieve that the derivatives $\varphi'_{\tau_k}(t)(t \in U_{\tau_k})$ satisfy the Hölder. condition.

*For the definition of φ - equivalence, see [9] on page 576.

At each point $\tau \neq \tau_k$ the operator A is φ_τ equivalent to the operator $C = (1-\lambda)I$ acting in the space $L_2(\Gamma)$. Since A is Noetherian, then (see [9] Theorem 1.4) the operator C is locally Noetherian at the point τ, hence $\lambda \neq 1$.

At the point τ_k, the operator A is φ_{τ_k} equivalent to the operator $A_k = S_{\alpha_k} S^*_{\alpha_k} - \lambda I$, acting in the space $L_2(\Gamma_{\alpha_k})$. It also follows that A_k is a local Noetherian operator at the point $t = 0$. By Lemma 5.1, A_k is a local Noetherian operator at the point $t = \infty$. At points $t \in \Gamma_{\alpha_k}$ other than zero and infinity, the operator A_k is equivalent to the operator $(1-\lambda)I$. Since $\lambda \neq 1$, A_k is local Noetherian at these points as well. Hence, by Theorem 1.6, it follows from [9] that A_k is Noetherian in $L_2(\Gamma_{\alpha_k})$. It follows from Theorem 4.2 that $det A_k(t,\xi) \neq 0$ $(t \in \Gamma_{\alpha_k}, -\infty \leq \xi \leq \infty)$. It is easy to see that $A_k(0,\xi) = A(\tau_k,\xi)$. Therefore, $det A(t,\xi) \neq 0$ $(t \in \Gamma, -\infty \leq \xi \leq \infty)$.

The necessity of the theorem is proved.

Sufficiency. Let $det A(t,\xi) \neq 0$ $(t \in \Gamma, -\infty \leq \xi \leq \infty)$. Then $\lambda \neq 1$ and

$$det\,(S_k(0,\xi)(S_k(0,\xi))^* - \lambda E_2) \neq 0 \ \ (k = 1, 2, \ldots, s).$$

From this and Lemma 5.1 it follows that the operators $A_k (k = 1, 2, \ldots, s)$ and $C = (1-\lambda)I$ are Noetherian. Since the operator A at each point τ is φ_τ equivalent to one of these operators, it follows (see [9], Theorem 2.4) that A is Noetherian. The theorem is proved.

From Theorem 5.1 follows

Corollary 5.1. *The operator S^* does not belong to the algebra $\Sigma(\Gamma)$ generated by the operators aI $(a \in C(\Gamma))$ and S_Γ.*

Indeed, let us assume that S^* belongs to the algebra $\Sigma(\Gamma)$. Since the symbols of the operators from $\Sigma(\Gamma)$ commute, the symbol of the operator $R = \lambda I - (S^*_\Gamma S_\Gamma - S_\Gamma S^*_\Gamma)$ is equal to λ. Consequently, for all $\lambda \neq 0$ the operator R is Noetherian. It is easy to verify that this contradicts Theorem 5.1.

From Theorem 5.1 and equality (10) it is easy to deduce that the essential norm $|S_\Gamma|$ of the operator S_Γ in the space $L_2(\Gamma)$ is defined by the equality

$$|S_\Gamma| = \max_{1 \leq k \leq s} \left| S_{\alpha_k} \right|. \tag{19}$$

From this and from equality (12) we conclude that the essential norm of the operator S_Γ in the space $L_2(\Gamma)$ satisfies the conditions

$$1 \leq |S_\Gamma| < 1 + \sqrt{2}.$$

Note that similarly, using the symbol and equality (13), we can calculate the essential norms of the Riesz operators $P_\Gamma = (I + S_\Gamma)/2$ and $Q_\Gamma = (I - S_\Gamma)/2$. It turns out that for these operators the following relation holds:

$$|P_\Gamma| = |Q_\Gamma| = \frac{|S_\Gamma|^2 + 1}{2|S_\Gamma|^2}. \tag{20}$$

Remark 5.1. *The equality (20) confirms the following hypothesis of the mathematician S. Marcus: let B be some Banach space and L_1, L_2 subspaces from B such that $L_1 \bigcap L_2 = 0$ and $B = L_1 + L_2$, then equality*

$$|P| = |Q| = \frac{|S_\Gamma|^2 + 1}{2|S_\Gamma|^2}$$

takes place, where P and Q are projectors projecting the space B onto L_1, respectively, on L_2 and $S = P + Q$.

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