Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 18(2), 2024, Pages 70–77 E-ISSN: 2587-3644
CZU: 517.925 DOI: 10.36120/2587-3644.v18i2.70-77

On computation of the ordinary Hilbert series for Sibirsky graded algebras of differential system $\mathbf{s}(3,5)$

Lidia Muşinschi and Victor Pricop

Abstract. The generalized and ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of autonomous polynomial differential systems are of particular importance for some problems of qualitative theory of differential systems. In the Republic of Moldova the computation of these series have their beginnings in the works of Professor M. N. Popa and his disciples. But the construction of these series for some complicated differential systems encounters insurmountable computational difficulties, especially, for the generalized Hilbert series, from which the ordinary Hilbert series can be easily obtained. In this paper, it is shown how the adaptation of Molien's formula address to the mentioned problem to overcome the enormous calculations, an ordinary Hilbert series were obtained for Sibirsky graded algebras of comitants and invariants for the differential system $s(3,5)$.

2020 Mathematics Subject Classification: 34C05, 34C07, 34C14.
Keywords: differential systems, Sibirsky graded algebras, Hilbert series, Krull dimension.

Despre calcularea seriilor Hilbert obişnuite pentru algebrele graduate Sibirschi ale sistemului diferențial $\mathbf{s}(3,5)$

Rezumat. Seriile Hilbert generalizate şi obişnuite pentru algebrele graduate Sibirschi ale comitanţilor şi invarianţilor sistemelor diferenţiale polinomiale autonome joacă un rol deosebit pentru unele probleme din teoria calitativă a acestor sisteme. În Republica Moldova calcularea seriilor Hilbert încep în lucrările profesorului M. N. Popa şi ale discipolilor săi. Totuşi construcţia acestor serii pentru unele sisteme diferenţiale complicate întâmpină dificultăţi enorme de calcul, în special pentru seria Hilbert generalizată din care se obţine cu uşurinţă seria Hilbert obişnuită. În această lucrare se arată cum se foloseşte adaptarea formulei lui Molien pentru a depăşi problema calculelor enorme. Au fost obţinute seriile Hilbert obişnuite pentru algebrele graduate Sibirschi ale comitanţilor şi invarianţilor pentru sistemul diferenţial $s(3,5)$.

Cuvinte-cheie: sisteme diferenţiale, algebre graduate Sibirschi, serii Hilbert, dimensiunea Krull.

1. Introduction

Consider a two-dimensional autonomous polynomial system of differential equations

$$\frac{dx}{dt} = \sum_{i=0}^{\ell} P_{m_i}(x,y), \quad \frac{dy}{dt} = \sum_{i=0}^{\ell} Q_{m_i}(x,y), \tag{1}$$

where $\Gamma = \{m_i\}_{i=0}^{\ell}$, $(\ell < \infty)$ is a some finite set of distinct non-negative integers, $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ are homogeneous of degree m_i with respect to the phase variables x and y (i.e., $P_{m_i}(\alpha x, \alpha y) = \alpha^{m_i} P_{m_i}(x,y)$, $Q_{m_i}(\alpha x, \alpha y) = \alpha^{m_i} Q_{m_i}(x,y)$, $\alpha \in \mathbb{R}$). The coefficients and variables in the polynomials $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ take values from the field of real numbers $\mathbb{R}$.

Hereafter for the system of the form (1) we will use the notation $s(m_0, m_1, ..., m_\ell)$ or $s(\Gamma)$ where $\Gamma = \{m_i\}_{i=0}^{\ell}$, m_i are degrees of homogeneities $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ with respect to the phase variables x and y.

One of the methods to study the differential systems of the form (1) is "*The method of algebraic invariants in the theory of differential equations*", which is developed in the works of Academician K. S. Sibirsky [1, 2, 3] and his disciples.

This method generated applications of Lie groups and algebras, graded algebras of invariants and comitants, generating functions and Hilbert series to the study of the system (1) (see, for example [4, 5]).

One of the methods of computation of generalized and ordinary Hilbert series for differential systems is Silvester's generalized method known from [4]. This method for differential systems with high-degree polynomial on the right-hand sides is connected with cumbersome computations with application of supercomputers. In contrast to the mentioned above, using the residues method, were obtained the ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants for the following differential systems $s(1,3,7)$ [6], $s(3,7)$ [7], $s(1,3,5)$ [8], $s(1,3,5,7)$ [9]. So, it is welcome to complete the set of computed Hilbert series with others, for example, the Hilbert series for the system $s(3,5)$.

2. Graded algebras of comitants (invariants) of the system (1.1)

Let A be a set of coefficients of the system (1). Denote by $GL(2,\mathbb{R})$ a group of centro-affine transformations

$$q: \ \overline{x} = \alpha x + \beta y, \ \overline{y} = \gamma x + \delta y \ (\Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0) \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ takes value from the field of real numbers $\mathbb{R}$.

Definition 2.1. *The polynomial $k(x, y, A)$ of phase variables and coefficients of the system* (1) *is called a centro-affine comitant of this system, if the equality*

$$k(\overline{x}, \overline{y}, \overline{A}) = \Delta^{-g} k(x, y, A)$$

holds for any $q \in GL(2, \mathbb{R})$, any coefficients of the system (1)*, any phase variables. If the comitant k does not depend on the phase variables, then it its called the invariant* (*usually denoted by i*) *of the system* (1) *by the centro-affine group $GL(2, \mathbb{R})$.*

The number g is called the weight of the comitant k. If $g = 0$, then k is called the absolute comitant, otherwise the relative comitant.

Definition 2.2. *For any differential system $s(m_0, m_1, ..., m_\ell)$, a centro-affine comitant has a type*

$$(d) = (\delta, d_1, d_2, ..., d_\ell) \tag{3}$$

where d_i is the degree of homogeneity of comitant with respect to the coefficients of homogeneities $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$, δ is the degree of homogeneity of comitant with respect to the phase variables x, y. At the same time the number $d = \sum_{i=1}^{\ell} d_i$ (δ) *is called the degree* (*order*) *of comitant of the type* (3)*.*

Lemma 2.1. [4] *The set of centro-affine comitants of the system* (1) *of the same type* (3) *forms a finite-dimensional linear space, i.e., has a finite maximal system of linearly independent comitants* (*linear basis*) *of a given type through which all others are linearly expressed.*

Remark 2.1. *In* [4] *it is shown that the set of centro-affine comitants generate a finite-determined graded algebra of comitants* (*invariants*) *with respect to the unimodular group $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$, which are denoted by*

$$S_\Gamma = \sum_{(d)} S_\Gamma^{(d)} \quad \left(SI_\Gamma = \sum_{(d)} SI_\Gamma^{(d)}\right).$$

By $dim_{\mathbb{R}} S_\Gamma^{(d)}$ ($dim_{\mathbb{R}} SI_\Gamma^{(d)}$) *are denoted dimensions of linear spaces from the Lemma* 2.1*. These algebras in* [5] *were named Sibirsky graded algebra of comitants* (*invariants*)*, respectively.*

If for Sibirsky graded algebras $S_{m_0, m_1, ..., m_\ell}$ and $SI_{m_0, m_1, ..., m_\ell}$ we introduce a single notation A, then they can be written in the form

$$A =< a_1, a_2, ..., a_m | f_1 = 0, f_2 = 0, ..., f_n = 0 > \quad (m, n < \infty) \tag{4}$$

where a_i are the generators of this algebra, f_j – defining relations (syzygies) between these generators.

One of the problems in studying these algebras is determining the numbers and expressions of its generators. For this, it is necessary to study the type of comitants and invariants that forms these generators. From the paper [10] on Modern Algebra, it follows that generating functions and Hilbert series play an essential role in solving this problem.

3. Hilbert series for Sibirsky graded algebras $S_{m_0,m_1,...,m_\ell}$ and $SI_{m_0,m_1,...,m_\ell}$

From [4] it is known

$$\varphi_\Gamma^{(0)}(u) = (1-u^{-2})\psi_{m_0}^{(0)}(u)\psi_{m_1}^{(0)}(u)...\psi_{m_\ell}^{(0)}(u) \tag{5}$$

where

$$\psi_{m_i}^{(0)}(u) = \begin{cases} \frac{1}{(1-uz_i)(1-u^{-1}z_i)}, & \text{for } m_i = 0, \\ \frac{1}{(1-u^{m_i+1}z_i)(1-u^{-m_i-1}z_i)\prod_{k=1}^{m_i}(1-u^{m_i-2k+1}z_i)^2}, & \text{for } m_i \neq 0 \end{cases} \tag{6}$$

for each $\Gamma = \{m_i\}_{i=0}^{\ell}$.

The expressions (5)–(6) we will call *initial form of the generating function* for centro-affine comitants of the system (1).

In the paper [4], it is shown that if the function (5)–(6) is represented as

$$\varphi_\Gamma(u) - u^{-2}\varphi_\Gamma(u^{-1}) = \varphi_\Gamma^{(0)}(u), \tag{7}$$

then we can restrict ourselves to the study of only rational function $\varphi_\Gamma(u)$.

However, the question arises, how to obtain the function $\varphi_\Gamma(u)$ from (7) for more complicated Γ. This problem was solved by generalizing the Silvester's method by decomposition of the function $\varphi_\Gamma^{(0)}(u)$ in elementary fractions [4].

Following the paper [4], under a generalized Hilbert series of the algebra S_Γ, we will understand

$$H(S_\Gamma, u, z_0, z_1, ..., z_\ell) = \sum_{(d)} dim_{\mathbb{R}} S_\Gamma^{(d)} u^\delta z_0^{d_0} z_1^{d_1} ... z_\ell^{d_\ell},$$

and

$$H(S_\Gamma, u, z_0, z_1, ..., z_\ell) = \varphi_\Gamma(u) \tag{8}$$

where $\varphi_\Gamma(u)$ is from (7).

Note that (according to the same paper [4]) an ordinary Hilbert series is obtained in an obvious way from the generalized

$$H_{S_\Gamma}(u) = H(S_\Gamma, u, u, u, ..., u). \tag{9}$$

If we denote the algebra of invariants for a fixed Γ for the system (1) by SI_Γ, then for generalized Hilbert series of this algebra we have

$$H(SI_\Gamma, z_0, z_1, ..., z_\ell) = H(S_\Gamma, 0, z_0, z_1, ..., z_\ell) = \varphi_\Gamma(0), \tag{10}$$

and for the ordinary Hilbert series we obtain

$$H_{SI_\Gamma}(z) = H(SI_\Gamma, z, z, ..., z). \tag{11}$$

The computation of the generalized Hilbert series for differential systems with high-degree polynomial on the right-hand sides is connected with cumbersome computations with the application of supercomputers. This emphasizes the importance of the calculus of ordinary Hilbert series. From the papers [4], [5], it follows that, we can studying the structures of these algebras using the generalized and the ordinary Hilbert series of Sibirsky graded algebras for the differential system (1). The Hilbert series gives an information about the upper bound of degrees for generators of these algebras.

Remark 3.1. *According to* [4], [5] *the Krull dimension for finitely determined algebras is equal to the order of pole of corresponding ordinary Hilbert series at the unit. The Krull dimension gives us the maximal number of algebraically independent comitants and invariants of corresponding Sibirsky graded algebras of differential systems.*

4. Methods of computation of the generalized and ordinary Hilbert series for Sibirsky graded algebras of differential systems

From the paper [4], it is known the Silvester's generalized method for computation of the generalized and ordinary Hilbert series. Using this method, there were calculated Hilbers series for Sibirsky graded algebras for differential systems $s(1)$, $s(2)$, $s(0,2)$, $s(1,3)$, $s(2,3)$, $s(5)$ [4], $s(1,4)$, $s(1,5)$ [5]. An attempt to obtain the Hilbert series for relatively simple system $s(1,2,3)$ with this method was unsuccessful. For more complicated differential systems this method encounters insurmountable computational difficulties. Use of other methods is welcome.

Definition 4.1. [11] *For a graded vector space $V = \bigoplus_{d=k}^{\infty} V_d$ with V_d finite dimensional for all d we define the Hilbert series ov V as the formal Laurent series*

$$H(V,t) = \sum_{d=k}^{\infty} dim(V_d)t^d.$$

An important tool for computing invariants is the Hilbert series. The Hilbert series of a ring contains a lot of information about the ring itself. For example, the dimension and other geometric invariants can be read from the Hilbert series.

Theorem 4.1. (Molien's formula [11]) *Let G be a finite group acting on a finite-dimensional vector space V over a field K of characteristic not dividing $|G|$. Then*

$$H(K[V]^G, t) = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{det_V^0(1 - t\sigma)}.$$

If K has the characteristic 0, then $det_V^0(1 - t\sigma)$ can be taken as $det_V(1 - t\sigma)$.

We recall the Residue Theorem in complex function theory. This theorem can be applied to compute the Hilbert series of invariant rings [11].

Theorem 4.2. (The Residue Theorem [11]) *Suppose that D is a connected, simply connected compact region in $\mathbb{C}$, whose border is ∂D, and $\gamma : [0, 1] \to \mathbb{C}$ is a smooth curve such that $\gamma([0, 1]) = \partial D$, $\gamma(0) = \gamma(1)$ and circles around D exactly once in the counter clockwise direction. Assume that f is a meromorphic function on $\mathbb{C}$ with no poles in ∂D. Then we have*

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in D} Res(f, a).$$

There are only finitely many points in the compact region D such that f has non-zero residue there.

Theorem 4.3. [11]

$$H(K[V]^G, t) = \frac{1}{2\pi i} \int\limits_{S^1} \frac{1}{det(I - t_{\rho_V}(z))} \frac{dz}{z}$$

where $S^1 \subset \mathbb{C}$ is the unit circle $\{z : |z| = 1\}$.

Using the Residue Theorem and corresponding generating function [4] the last formula was adapted for computation of ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems as follows

Theorem 4.4. [5]

$$H_{SI_\Gamma}(t) = \frac{1}{2\pi i} \int\limits_{S^1} \frac{\varphi_\Gamma^{(0)}(z)}{z} dz$$

where $S^1 \subset \mathbb{C}$ is the unit circle $\{z : |z| = 1\}$, $\varphi_\Gamma^{(0)}(z)$ is the corresponding generating function (5)–(6).

5. Computation of the ordinary Hilbert series for Sibirsky graded algebras of differential system $s(3,5)$

Using Theorem 4.4, we obtain

Theorem 5.1. *For the differential system $s(3,5)$, the following ordinary Hilbert series for Sibirsky graded algebras of comitants $S_{3,5}$ and invariants $SI_{3,5}$ was obtained*

$$H_{S_{3,5}}(t) = \frac{1}{(1+t)^2(1-t^2)^4(1-t^4)^3(1-t^3)^7(1-t^5)^4(1-t^7)} \cdot (1 + 2t + 2t^2 + \\ +8t^3 + 49t^4 + 179t^5 + 533t^6 + 1382t^7 + 3301t^8 + 7356t^9 + 15353t^{10} + \\ +29865t^{11} + 54402t^{12} + 93137t^{13} + 150665t^{14} + 231125t^{15} + 337272t^{16} + \\ +468744t^{17} + 621438t^{18} + 786783t^{19} + 952653t^{20} + 1104296t^{21} + 1226739t^{22} + \\ +1306380t^{23} + 1334077t^{24} + 1306380t^{25} + 1226739t^{26} + 1104296t^{27} + \\ +952653t^{28} + 786783t^{29} + 621438t^{30} + 468744t^{31} + 337272t^{32} + 231125t^{33} + \\ +150665t^{34} + 93137t^{35} + 54402t^{36} + 29865t^{37} + 15353t^{38} + 7356t^{39} + \\ +3301t^{40} + 1382t^{41} + 533t^{42} + 179t^{43} + 49t^{44} + 8t^{45} + 2t^{46} + 2t^{47} + t^{48})$$

$$H_{SI_{3,5}}(t) = \frac{1}{(1-t)^4(1+t)^5(1-t^4)^4(1-t^3)^6(1-t^5)^3} \cdot (1 + t + t^2 + 7t^3 + \\ +36t^4 + 106t^5 + 290t^6 + 672t^7 + 1451t^8 + 2875t^9 + 5322t^{10} + 9053t^{11} + \\ +14398t^{12} + 21263t^{13} + 29463t^{14} + 38314t^{15} + 47076t^{16} + 54444t^{17} + \\ +59516t^{18} + 61259t^{19} + 59516t^{20} + 54444t^{21} + 47076t^{22} + 38314t^{23} + \\ +29463t^{24} + 21263t^{25} + 14398t^{26} + 9053t^{27} + 5322t^{28} + 2875t^{29} + 1451t^{30} + \\ +672t^{31} + 290t^{32} + 106t^{33} + 36t^{34} + 7t^{35} + t^{36} + t^{37} + t^{38})$$

Remark 5.1. *For the Sibirsky graded algebras $S_{3,5}$ $(SI_{3,5})$, the Krull dimensions is equal to* 19 (17) *respectively.*

Remark 5.2. *The Krull dimension gives us the maximal number of algebraically independent comitants and invariants of Sibirsky graded algebras $S_{3,5}$ and $SI_{3,5}$ of differential system $s(3,5)$.*

Remark 5.3. *Note that for Hilbert series of Sibirsky graded algebra of comitants of the system $s(\Gamma)$, where $0 \notin \Gamma$, the following equality holds $H_{S_\Gamma}(t) = H_{SI_{\Gamma\cup\{0\}}}(t)$.*

From [11], a method for computing the ordinary Hilbert series of invariants rings using the residues is known, that was adapted for the ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems. This method is more effective than the generalized Sylvester's method.

References

[1] Sibirsky, K.S. *Introduction to the algebraic theory of invariants of differential equations*. Chişinău, Ştiinţa, 1982, 168 p. (in Russian).

[2] Sibirsky, K.S. *The method of invariants in the qualitative theory of differential equations*. Chişinău, RIO AN MSSR, 1968, 184 p. (in Russian).

[3] Sibirsky, K.S. *Algebraic invariants of differential equations and matrices*. Chişinău, Ştiinţa, 1976, 268 p. (in Russian).

[4] Popa, M.N. *Algebraic methods for differential systems*. Seria Matematica Aplicată şi Industrială, Universitatea din Piteşti, Editura the Flower Power, 2004, vol. 15 (in Romanian).

[5] Popa, M.N, Pricop, V.V. *The Center and Focus Problem: Algebraic Solutions and Hypoteheses*. London, New York, Ed. Taylor&Frances Group, 2022, 215 p.

[6] Pricop, V. The common Hilbert series of the differential system $s(1,3,7)$ and the Krull dimensions of Sibirsky algebras. *The 24th Conference on Applied and Industrial Mathematics*, September 15-18, 2016, Craiova, Romania. Book of Abstracts, p. 39-40.

[7] Pricop, V., Tacu, M. About computing of ordinary Hilbert series for Sibirsky graded algebras of differential system $s(3,7)$. *The 30rd Conference on Applied and Industrial Mathematics*, September 14-17, 2023, Iași, Romania. Book of Abstracts, p. 27-28.

[8] Pricop, V. The differential system $s(1,3,5)$ and corresponding common Hilbert series. *International Conference Mathematics & Information Technologies: Research and Education*, June 24–26, 2016, Chişinău, Republic of Moldova. Abstracts, p. 57-58.

[9] Pricop, V. Computation of common Hilbert series for the differential system $s(1,3,5,7)$ using the residue theorem. *The 4th Conference of Mathematical Society of the Republic of Moldova*, June 28 – July 2, 2017, Chişinău, Republic of Moldova. Proceedings of CMSM4, p. 325-330.

[10] Ufnarovskij, V.A. *Combinatorial and Asymptotic Methods in Algebra*. In Algebra VI, A. I. Kostrikin and I. R. Shafarevich (Eds.), Encyclopaedia of Mathematical Sciences, Vol. 57, Springer, Berlin, New York, 1995, 196 p.

[11] Derksen, H., Kemper, G. *Computational Invariant Theory*. Encyclopaedia of Mathematical Sciences, vol. 130, Springer-Verlag, Berlin, 2002, 268 p.

Received: October 15, 2024 *Accepted: December 20, 2024*

(Lidia Muşinschi) Vladimir Andrunachievici Institute of Mathematics and Computer Science, Moldova State University, 5 Academiei St., Chişinău, Republic of Moldova
E-mail address: `lidia.zdragat@mail.ru`

(Victor Pricop) Technical University of Moldova, 9/8 Studenţilor St., Chişinău;
Vladimir Andrunachievici Institute of Mathematics and Computer Science, Moldova State University, 5 Academiei St., Chişinău, Republic of Moldova
E-mail address: `victor.pricop@mate.utm.md`