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Stability conditions of unperturbed motion governed by the ternary differential system of Lyapunov-Darboux type with nonlinearities of fifth degree

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Abstract. In this paper, there was studied Lyapunov stability of the unperturbed motion for the ternary differential system with nonlinearities of fifth degree on a center-affine variety. The Lyapunov series was constructed and the stability conditions of the unperturbed motion governed by this system were determined in the critical case.

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Keywords: differential system, stability of unperturbed motion, critical equation, non-critical equation, Lyapunov series.

Condiţiile de stabilitate a mişcării neperturbate guvernate de sistemul diferenţial ternar de tip Lyapunov-Darboux cu nelinearităţi de gradul cinci

Rezumat. În lucrare a fost studiată stabilitatea după Lyapunov a mişcării neperturbate pentru sistemul diferenţial ternar cu nelinearităţi de gradul cinci, pe o varietate centro-afină. A fost construită seria Lyapunov şi determinate condiţiile de stabilitate a mişcării neperturbate guvernate de acest sistem în cazul critic.

Cuvinte-cheie: sistem diferenţial, stabilitatea mişcării neperturbate, ecuaţie critică, ecuaţie necritică, serie Lyapunov.

1. Introduction

We examine the three-dimensional differential system $s^3(1,5)$ of unperturbed motion of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta\mu} x^\alpha x^\beta x^\gamma x^\delta x^\mu \quad (j,\alpha,\beta,\gamma,\delta,\mu = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta\gamma\delta\mu}$ is a symmetric tensor in the lower indices, by which a total convolution is done.

Definition 1.1. *According to I. G. Malkin* [1], *we will say that the system (1) is critical if the characteristic equation of this system has one zero root, and all other roots of this equation have negative real parts.*

Lemma 1.1. *The three-dimensional differential system (1) is critical if and only if the following center-affine invariant conditions hold*

$$L_{1,3} > 0, \quad L_{2,3} > 0, \quad L_{3,3} = 0, \tag{2}$$

where

$$L_{1,3} = -\theta_1, \quad L_{2,3} = \frac{1}{2}(\theta_1^2 - \theta_2), \quad L_{3,3} = \frac{1}{6}(-\theta_1^3 + 3\theta_1\theta_2 - 2\theta_3), \tag{3}$$

and

$$\theta_1 = a^\alpha_\alpha, \quad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \quad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta. \tag{4}$$

Lemma 1.2. *In the case of conditions* (2)*, by a center-affine transformation* [2]*, the system* (1) *can be brought to the critical Lyapunov form*

$$\begin{aligned} \frac{dx^1}{dt} &= a^j_{\alpha\beta\gamma\delta\mu} x^\alpha x^\beta x^\gamma x^\delta x^\mu, \\ \frac{dx^j}{dt} &= a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta\mu} x^\alpha x^\beta x^\gamma x^\delta x^\mu \quad (j = 2,3; \alpha,\beta,\gamma,\delta,\mu = \overline{1,3}), \end{aligned} \tag{5}$$

where the first equation from (5) *is called the* ***critical equation*** *and the second one – the* ***non-critical equation****.*

2. The critical differential system of Lyapunov-Darboux type with nonlinearities of fifth degree

In the center-affine condition $\eta = a^\alpha_{\beta\gamma\delta\mu\nu} x^\beta x^\gamma x^\delta x^\mu x^\nu x^\tau y^\xi \varepsilon_{\alpha\tau\xi} \equiv 0$ [3], with notations

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad a^2_1 = p, \quad a^2_2 = q, \quad a^2_3 = r, \quad a^3_1 = s, \quad a^3_2 = m, \quad a^3_3 = n, \tag{6}$$

the system (5), it is a critical system of Lyapunov-Darboux type, of the form

$$\begin{aligned} \frac{dx}{dt} &= 5xR(x,y,z), \\ \frac{dy}{dt} &= px + qy + rz + 5yR(x,y,z), \\ \frac{dz}{dt} &= sx + my + nz + 5zR(x,y,z), \end{aligned} \tag{7}$$

where

$$\begin{aligned} R(x,y,z) &= a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + 6a_8x^2y^2 + \\ &+ 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + 4a_{14}y^3z + 4a_{15}yz^3, \end{aligned} \tag{8}$$

and $m, n, p, q, r, s, a_i (i = \overline{1,15})$ are real arbitrary coefficients.

Remark 2.1. *For the system* (7)*, we have*

$$L_{2,3} = nq - mr,$$

and according to conditions (2) *these values are greater than zero.*

Remark 2.2. *Under the conditions of Remark 2.1, without loss of generality, we can assume that $nq \neq 0$.*

Proof. We consider the center-affine substitution

$$\bar{x} = x, \quad \bar{y} = z, \quad \bar{z} = y. \tag{9}$$

It is easy to verify that in the case of substitution (9), we obtain that in system (7) the expression mr becomes nq. Taking into account Remark 2.1, we obtain that $nq \neq 0$. □

We analyze the noncritical equations

$$\begin{aligned} &px + qy + rz + 5y(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ &+6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ &+4a_{14}y^3z + 4a_{15}yz^3) = 0, \\ &sx + my + nz + 5z(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ &+6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ &+4a_{14}y^3z + 4a_{15}yz^3) = 0. \end{aligned} \tag{10}$$

Then from the first relation of (10) we express y, and from the second relation we express z

$$\begin{aligned} y = &-\frac{p}{q}x - \frac{r}{q}z - \frac{5}{q}y(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ &+6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ &+4a_{14}y^3z + 4a_{15}yz^3), \\ z = &-\frac{s}{n}x - \frac{m}{n}y - \frac{5}{n}z(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ &+6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ &+4a_{14}y^3z + 4a_{15}yz^3). \end{aligned} \tag{11}$$

We seek y and z as a holomorphic functions of x. Then we can write

$$y(x) = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots,$$

$$z(x) = B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots \tag{12}$$

Substituting (12) into (11) we have

$$A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ... = -\frac{p}{q}x - \frac{r}{q}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+B_5x^5 + ...) - \frac{5}{q}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)[a_1x^4 + a_2(A_1x + A_2x^2 +$$
$$+A_3x^3 + A_4x^4 + A_5x^5 + ...)^4 + a_3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^4 +$$
$$+4a_4x^3(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...) + 4a_5x^3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+B_5x^5 + ...) + 4a_6x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^3 + 4a_7x(B_1x + B_2x^2 +$$
$$+B_3x^3 + B_4x^4 + B_5x^5 + ...)^3 + 6a_8x^2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2 +$$
$$+6a_9x^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^2 + 12a_{10}x^2(A_1x + A_2x^2 + A_3x^3 +$$
$$+A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...) + +12a_{11}x(A_1x + A_2x^2 +$$
$$+A_3x^3 + A_4x^4 + A_5x^5 + ...)^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...) +$$
$$+12a_{12}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+B_5x^5 + ...)^2 + 6a_{13}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2(B_1x + B_2x^2 + B_3x^3 +$$
$$+B_4x^4 + B_5x^5 + ...)^2 + 4a_{14}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^3(B_1x + B_2x^2 +$$
$$+B_3x^3 + B_4x^4 + B_5x^5 + ...) + 4a_{15}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)(B_1x +$$
$$+B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^3],$$

$$B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ... = -\frac{s}{n}x - \frac{m}{n}(A_1x + A_2x^2 + A_3x^3 +$$
$$+A_4x^4 + A_5x^5 + ...) - \frac{5}{n}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)[a_1x^4 + a_2(A_1x +$$
$$+A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^4 + a_3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^4 +$$
$$+4a_4x^3(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...) + 4a_5x^3(B_1x + B_2x^2 + B_3x^3 +$$
$$+B_4x^4 + B_5x^5 + ...) + 4a_6x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^3 + 4a_7x(B_1x +$$
$$+B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^3 + 6a_8x^2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2 +$$
$$+6a_9x^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^2 + 12a_{10}x^2(A_1x + A_2x^2 + A_3x^3 +$$
$$+A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...) + 12a_{11}x(A_1x + A_2x^2 +$$
$$+A_3x^3 + A_4x^4 + A_5x^5 + ...)^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...) +$$
$$+12a_{12}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+B_5x^5 + ...)^2 + 6a_{13}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2(B_1x + B_2x^2 + B_3x^3 +$$
$$+B_4x^4 + B_5x^5 + ...)^2 + 4a_{14}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^3(B_1x + B_2x^2 +$$

$$+B_3x^3 + B_4x^4 + B_5x^5 + \dots) + 4a_{15}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots)(B_1x+$$
$$+B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots)^3],$$

This implies that

$$A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots = -\frac{p + rB_1}{q}x - \frac{rB_2}{q}x^2 - \frac{rB_3}{q}x^3 - \frac{rB_4}{q}x^4-$$
$$-\frac{1}{q}(5a_1A_1 + 20a_4A_1^2 + 30a_8A_1^3 + 20a_6A_1^4 + 5a_2A_1^5 + 60a_{10}A_1^2B_1 + 60a_{11}A_1^3B_1+$$
$$+20a_{14}A_1^4B_1 + 20a_5A_1B_1 + 60a_{12}A_1^2B_1^2 + 30a_{13}A_1^3B_1^2 + 30a_9A_1B_1^2 + 20a_{15}A_1^2B_1^3+$$
$$+20a_7A_1B_1^3 + 5a_3A_1B_1^4 + rB_5)x^5 + \dots$$
$$B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots = -\frac{mA_1 + s}{n}x - \frac{mA_2}{n}x^2 - \frac{mA_3}{n}x^3 - \frac{mA_4}{n}x^4-$$
$$-\frac{1}{n}(5a_2A_1^4B_1 + 20a_4A_1B_1 + 20a_6A_1^3B_1 + 30a_8A_1^2B_1 + 60a_{10}A_1B_1^2 + 60a_{11}A_1^2B_1^2+$$
$$+20a_{14}A_1^3B_1^2 + 60a_{12}A_1B_1^3 + 30a_{13}A_1^2B_1^3 + 20a_{15}A_1B_1^4 + mA_5 + 5a_1B_1 + 20a_5B_1^2+$$
$$+30a_9B_1^3 + 20a_7B_1^4 + 5a_3B_1^5)x^5 + \dots$$

From this identity we have

$$A_1 = \frac{rs - np}{nq - mr}, \quad B_1 = \frac{mp - qs}{nq - mr}; \quad A_2 = B_2 = A_3 = B_3 = A_4 = B_4 = 0,$$
$$A_5 = -\frac{5}{nq - mr}(a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1+$$
$$+12a_{11}A_1^2B_1 + 4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3 + a_3B_1^4)(nA_1 - rB_1),$$
$$B_5 = \frac{5}{nq - mr}(a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1+$$
$$+12a_{11}A_1^2B_1 + 4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3 + a_3B_1^4)(mA_1 - qB_1),$$
$$A_6 = B_6 = A_7 = B_7 = A_8 = B_8 = 0,$$
$$A_9 = -\frac{5}{nq - mr}[(a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1+$$
$$+12a_{11}A_1^2B_1 + 4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3 + a_3B_1^4)(nA_5 - rB_5) + 4(a_2A_1^3A_5 + a_4A_5 + 3a_6A_1^2A_5 + 3a_8A_1A_5+$$
$$+3a_{10}A_5B_1 + 6a_{11}A_1A_5B_1 + 3a_{14}A_1^2A_5B_1 + 3a_{12}A_5B_1^2 + 3a_{13}A_1A_5B_1^2+$$
$$+a_{15}A_5B_1^3 + 3a_{10}A_1B_5 + 3a_{11}A_1^2B_5 + a_{14}A_1^3B_5 + a_5B_5 + 6a_{12}A_1B_1B_5 + 3a_{13}A_1^2B_1B_5+$$
$$+3a_9B_1B_5 + 3a_{15}A_1B_1^2B_5 + 3a_7B_1^2B_5 + a_3B_1^3B_5)(nA_1 - rB_1)],$$

$$
\begin{gathered}
B_9 = \frac{5}{nq - mr}[(a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1 + \\
+12a_{11}A_1^2B_1 + 4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + \\
+4a_{15}A_1B_1^3 + 4a_7B_1^3 + a_3B_1^4)(mA_5 - qB_5) + 4(a_2A_1^3A_5 + a_4A_5 + \\
+3a_6A_1^2A_5 + 3a_8A_1A_5 + 3a_{10}A_5B_1 + 6a_{11}A_1A_5B_1 + 3a_{14}A_1^2A_5B_1 + \\
+3a_{12}A_5B_1^2 + 3a_{13}A_1A_5B_1^2 + a_{15}A_5B_1^3 + 3a_{10}A_1B_5 + 3a_{11}A_1^2B_5 + a_{14}A_1^3B_5 + \\
+a_5B_5 + 6a_{12}A_1B_1B_5 + 3a_{13}A_1^2B_1B_5 + 3a_9B_1B_5 + 3a_{15}A_1B_1^2B_5 + \\
+3a_7B_1^2B_5 + a_3B_1^3B_5)(mA_1 - qB_1)], \\
A_{10} = B_{10} = A_{11} = B_{11} = 0, \ldots
\end{gathered} \tag{13}
$$

Substituting (12) into the right-hand sides of the critical differential equations (7), we get the following identity

$$
\begin{gathered}
C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \ldots = 5x(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + \\
+4a_6xy^3 + 4a_7xz^3 + 6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + \\
+6a_{13}y^2z^2 + 4a_{14}y^3z + 4a_{15}yz^3),
\end{gathered}
$$

or in detailed form

$$
\begin{gathered}
C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \ldots = \\
= 5x[a_1x^4 + a_2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^4 + a_3(B_1x + B_2x^2 + B_3x^3 + \\
+B_4x^4 + B_5x^5 + \ldots)^4 + 4a_4x^3(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots) + \\
+4a_5x^3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + 4a_6x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \\
+A_5x^5 + \ldots)^3 + 4a_7xz(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^3 + 6a_8x^2(A_1x + A_2x^2 + \\
+A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2 + 6a_9x^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^2 + \\
+12a_{10}x^2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + \\
+B_5x^5 + \ldots) + 12a_{11}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2(B_1x + B_2x^2 + B_3x^3 + \\
+B_4x^4 + B_5x^5 + \ldots) + 12a_{12}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x + B_2x^2 + \\
+B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^2 + 6a_{13}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2(B_1x + \\
+B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^2 + 4a_{14}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \\
+\ldots)^3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + 4a_{15}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \\
+A_5x^5 + \ldots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^3].
\end{gathered}
$$

From here, we obtain

$$C_1 = C_2 = C_3 = C_4 = 0,$$
$$C_5 = 5(a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1 + 12a_{11}A_1^2B_1+$$
$$+4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3 + a_3B_1^4),$$
$$C_6 = 20(a_2A_1^3A_2 + a_4A_2 + 3a_6A_1^2A_2 + 3a_8A_1A_2 + 3a_{10}A_2B_1 + 6a_{11}A_1A_2B_1+$$
$$+3a_{14}A_1^2A_2B_1 + 3a_{12}A_2B_1^2 + 3a_{13}A_1A_2B_1^2 + a_{15}A_2B_1^3 + 3a_{10}A_1B_2 + 3a_{11}A_1^2B_2+$$
$$+a_{14}A_1^3B_2 + a_5B_2 + 6a_{12}A_1B_1B_2 + 3a_{13}A_1^2B_1B_2 + 3a_9B_1B_2 + 3a_{15}A_1B_1^2B_2+$$
$$+3a_7B_1^2B_2 + a_3B_1^3B_2),$$
$$C_7 = 10(3a_2A_1^2A_2^2 + 2a_2A_1^3A_3 + 2a_4A_3 + 6a_6A_1A_2^2 + 6a_6A_1^2A_3 + 3a_8A_2^2+$$
$$+6a_8A_1A_3 + 6a_{11}A_2^2B_1 + 6a_{14}A_1A_2^2B_1 + 6a_{10}A_3B_1 + 12a_{11}A_1A_3B_1+$$
$$+6a_{14}A_1^2A_3B_1 + 3a_{13}A_2^2B_1^2 + 6a_{12}A_3B_1^2 + 6a_{13}A_1A_3B_1^2 + 2a_{15}A_3B_1^3 + 6a_{10}A_2B_2+$$
$$+12a_{11}A_1A_2B_2 + 6a_{14}A_1^2A_2B_2 + 12a_{12}A_2B_1B_2 + 12a_{13}A_1A_2B_1B_2+$$
$$+6a_{15}A_2B_1^2B_2 + 6a_{12}A_1B_2^2 + 3a_{13}A_1^2B_2^2 + 3a_9B_2^2 + 6a_{15}A_1B_1B_2^2 + 6a_7B_1B_2^2+$$
$$+3a_3B_1^2B_2^2 + 6a_{10}A_1B_3 + 6a_{11}A_1^2B_3 + 2a_{14}A_1^3B_3 + 2a_5B_3 + 12a_{12}A_1B_1B_3+$$
$$+6a_{13}A_1^2B_1B_3 + 6a_9B_1B_3 + 6a_{15}A_1B_1^2B_3 + 6a_7B_1^2B_3 + 2a_3B_1^3B_3), \ldots \tag{14}$$

3. The stability conditions of unperturbed motion for the ternary differential system of Lyapunov-Darboux type with nonlinearities of fifth degree

We will introduce the following notation:

$$M = a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1 + 12a_{11}A_1^2B_1+$$
$$+4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3+ \tag{15}$$
$$+4a_7B_1^3 + a_3B_1^4,$$

According to Lyapunov Theorem [4, §32], we have

Theorem 3.1. *Let the critical system* (7) *be given on the invariant variety. The stability of the unperturbed motion is described by one of the following three possible cases*:

I. *If $M > 0$, then the unperturbed motion is* ***unstable****;*

II. *If $M < 0$, then the unperturbed motion is* ***stable****;*

III. *If $M = 0$, then the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, this motion is asymptotically stable.

Proof. According to Lyapunov Theorem [4], we analyze the coefficients of the series (14). The stability or the instability of the unperturbed motion of the system (7) is determined by the sign of expression C_5, and we get Cases I and II.

Therefore, if $C_5 = 0$, then all $A_i = B_i = 0$ $(\forall i)$, so we get Case III of this theorem. □

Remark 3.1. *Theorem 3.1 was presented at the 31st Conference on Applied and Industrial Mathematics (CAIM-2024)* [6].

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