Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 18(2), 2024, Pages 97–113 E-ISSN: 2587-3644
CZU: 517.925 DOI: 10.36120/2587-3644.v18i2.97-113

# Affine invariant conditions for a class of differential polynomial cubic systems

Cristina Bujac

**Abstract.** In this article the affine invariant criteria constructed in terms of algebraic polynomials with coefficients $\tilde{a} \in \mathcal{R}^{20}$ for a class of cubic systems are established. We are focused on non-degenerate real cubic systems with 7 invariant straight lines, considering the line at infinity and their multiplicities and possesing four real singularities at infinity. Additionally, the only configurations of the type $(3, 3)$ of mentioned systems are considered and we denote this class by $CSL^{4r\infty}_{(3,3)}$. In [5] the existence of exactly 14 configurations of invariant straight lines for systems in $CSL^{4r\infty}_{(3,3)}$ was proved. Here we complete this classification by determining necessary and sufficient conditions for the realization of each one of the 14 configurations in terms of affine invariant polynomials.
**2020 Mathematics Subject Classification:** 34C23, 34A34.
**Keywords:** polynomial cubic system, invariant straight line, finite/infinite singular point, configuration of invariant straight lines, affine invariant conditions.

# Condiţii afin invariante pentru o clasă de sisteme polinomiale diferenţiale cubice

**Rezumat.** În acest articol sunt stabilite criterii invariante construite în termeni de polinoame algebrice cu coeficienţi $\tilde{a} \in \mathcal{R}^{20}$ pentru o clasă de sisteme cubice. Ne concentrăm pe sisteme cubice reale, nedegenerate, cu 7 drepte invariante, luând în consideraţie dreapta de la infinit şi multiplicităţile acesteia, care posedă patru singularităţi reale la infinit. În plus, sunt analizate doar configuraţiile de tipul $(3, 3)$ ale sistemelor menţionate, iar această clasă este notată cu $CSL^{4r\infty}_{(3,3)}$. În [5] a fost demonstrată existenţa exact a 14 configuraţii de drepte invariante pentru sistemele din $CSL^{4r\infty}_{(3,3)}$. În acest articol, completăm această clasificare prin determinarea condiţiilor necesare şi suficiente afin-invarinate pentru realizarea fiecăreia dintre cele 14 configuraţii depistate.
**Cuvinte-cheie:** sistem cubic polinomial, dreaptă invariantă, punct singular finit/infinit, configuraţie de drepte invariante, condiţii afin-invariante.

## 1. Introduction and preliminary results

Consider the family $\mathbb{CS}$ of real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{1}$$

with variables $x$ and $y$ and real coefficients such that $\gcd(P,Q) = 1$ and $\max(\deg(P,Q)) = 3$. The polynomials $p_i(x,y)$ and $q_i(x,y)$ for $i = 0,1,2,3$ are homogeneous polynomials of degree $i$ in variables $x$ and $y$:

$$\begin{aligned} &p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \\ &p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ &q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \\ &q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2, \\ &q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3. \end{aligned}$$

Let $a \in R^{20}$, i.e. $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (1). We denote

$$\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y].$$

The set $\mathbb{CS}$ of *cubic differential systems* (1) depends on 20 parameters, and therefore mathematicians began studying particular families of $\mathbb{CS}$. Among these families, there are cubic systems with invariant straight lines, and we denote such families of systems by $\mathbb{CSL}$.

*A line* $f(x,y) = w + ux + vy = 0$ over $\mathbb{C}$ *is an invariant* line for a system (1) if and only if there exists $K(x,y) \in \mathbb{C}[x,y]$, which satisfies the following identity in $\mathbb{C}[x,y]$:

$$uP(x,y) + vQ(x,y) = (w + ux + vy)K(x,y).$$

According to [1] the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic differential systems with a finite number of infinite singularities is 9. In paper [17], all the possible configurations of invariant lines are obtained in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. If the total multiplicity of these lines (including the line at infinity) equals eight, then all possible configurations of invariant lines are found in [7, 8, 9, 10, 11].

We continue our investigation on $\mathbb{CSL}$ with invariant lines of total multiplicity 7 (the line at infinity is considered). To each system in $\mathbb{CSL}$, we associate its *configuration of* *invariant lines*, i.e. the set of its invariant lines together with the real singular points of the system located on the union of these lines.

More precisely, we call *configuration of invariant straight lines* of a real planar polynomial differential system (1), the set of (complex) invariant straight lines (which may have real coefficients) including the line at infinity of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

The notion of configuration of invariant lines for a polynomial differential system was first introduced in [15].

It is known that on $\mathbb{CS}$ (1), the group $Aff(2,\mathbb{R})$ of affine transformations of the plane acts [14]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of $G$ on $\mathbb{CS}$. We can identify the set $\mathbb{CS}$ of cubic systems (1) with a subset of $\mathbb{R}^{20}$ via the map $\mathbb{CS} \longrightarrow \mathbb{R}^{20}$, which associates to each cubic system (1) the 20-tuple $\tilde{a} = (a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03})$ of its coefficients.

The definitions of an affine or $GL$-comitant or invariant as well as the definitions of a $T$-comitant and $CT$-comitant can be found in [15] (see also [2]).

Here, we construct the necessary invariant polynomials ($T$-comitants) that we need for detecting the existence of invariant lines for the family of cubic systems having four distinct real singularities and exactly seven invariant straight lines including the line at infinity and counting multiplicities.

We consider the polynomials

$$C_i(a,x,y) = yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 0,1,2,3,$$
$$D_i(a,x,y) = \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 1,2,3.$$

In [16] it was shown that the following polynomials

$$\big\{C_i(a,x,y),\ D_1(a),\ D_2(a,x,y),\ D_3(a,x,y), i = 0,1,2,3\big\} \tag{2}$$

of degree one in the coefficients of systems (1) are $GL$-comitants of these systems.

**Notation 3.** Let $f,\, g \in R[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called *the transvectant of index k* of $(f,g)$ (cf. [12, 18]).

To define the invariant polynomials, we first construct the comitants of the second degree, with respect to the coefficients of the initial systems (1), of the form:

$$
\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}
$$

Next we determine the conditions for the existence of the couples of parallel invariant straight lines which a cubic system can have (see Theorem 1.1). For this we use the following invariant polynomials constructed in [17] and [8]:

$$
\begin{aligned}
\mathcal{V}_1(a,x,y) &= S_{23} + 2D_3^2, \\
\mathcal{V}_2(a,x,y) &= S_{26}, \\
\mathcal{V}_3(a,x,y) &= 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a,x,y) &= C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right], \\
\mathcal{V}_5(a,x,y) &= 6C_3(9A_5 - 7A_6) + 2D_3(4T_{16} - T_{17}) - 3T_3(3A_1 + 5A_2) + \\
&\quad + 3A_2T_4 + 36T_5^2 - 3T_{44}, \\
\mathcal{U}_1(a,x,y) &= S_{24} - 4S_{27}, \\
\mathcal{U}_2(a,x,y) &= 6(S_{23} - 3S_{25}, S_{26})^{(2)} - 3S_{23}(S_{24} - 8S_{27}) - \\
&\quad - 24S_{26}^2 + 2C_3(C_3, S_{23})^{(4)} + 24D_3(D_3, S_{26})^{(1)} + 24D_3^2S_{27}.
\end{aligned}
$$

In order to construct the needed affine invariant conditions, we will use the following polynomials:

$$
\begin{aligned}
\mathcal{H}_1 = {} & 48D_1^4S_{24}\big[2D_1^2+3S_6\big]+192D_1^5(S_{18},D_2)^{(1)}+12S_6^2S_{24}\big[6D_1^2+S_6\big]\\
& +216S_3S_{24}\big[4D_1^4-S_6^2-16S_3^2\big]-108S_{24}(S_5,C_0)^{(1)}\big[8D_1^3-12D_1S_6\\
& +72D_1S_3-9(S_5,C_0)^{(1)}\big]-216S_{24}(S_8,C_0)^{(1)}\big[4D_1^3+2D_1S_6+9(S_5,C_0)^{(1)}\big]\\
& -192\left[(S_{18},C_0)^{(1)}\right]^2\left[13D_1^2+9S_6+24S_3\right]-24(S_{18},C_0)^{(1)}(S_{14},C_1)^{(2)}\big[66D_1^2\\
& +17S_6-72S_3\big]+16(S_{18},D_2)^{(1)}\big[12D_1^3S_6+3D_1S_6^2+104D_1^3S_3-45D_1S_3S_6\\
& +288D_1S_3^2+360D_1^2(S_5,C_0)^{(1)}+189S_6(S_5,C_0)^{(1)}+24S_6(S_8,C_0)^{(1)}-144S_3(S_8,C_0)^{(1)}\big]\\
& +216S_{24}\big((S_{11},C_0)^{(1)},C_0\big)^{(1)}\big[6D_1^2-S_6+9S_3\big]+36\big((S_{14},C_0)^{(1)},C_0\big)^{(1)}\\
& \times\big[15D_1^2S_{24}+12S_3S_{24}+(S_{18},D_2)^{(1)}\big]+1152D_1(S_{18},C_0)^{(1)}\big((S_{18},C_1)^{(1)},C_0\big)^{(1)}\\
& -768\big[\big((S_{14},C_0)^{(1)},D_2\big)^{(1)}\big]^2+24\big((S_{18},C_2)^{(1)},C_1\big)^{(2)}\big[4D_1^4+4D_1^2S_6+S_6^2\\
& +96D_1^2S_3-33D_1(S_8,C_0)^{(1)}-63\big((S_{11},C_0)^{(1)},C_0\big)^{(1)}\big]+3\big((S_{14},C_2)^{(1)},C_2\big)^{(3)}\times\\
& \big[4D_1^4+4D_1^2S_6+S_6^2+32D_1^2S_3-16S_3S_6-32D_1(S_8,C_0)^{(1)}\\
& -64\big((S_{14},C_0)^{(1)},C_0\big)^{(1)}\big]-144\big[9D_1S_{24}+16(S_{18},D_2)^{(1)}\big]\times\\
& \big(((S_{17},C_0)^{(1)},C_0)^{(1)},C_0\big)^{(1)}-64\big(((S_{18},C_2)^{(1)},C_2)^{(2)},C_0\big)^{(1)}\big[D_1^3\\
& -18(S_8,C_0)^{(1)}\big]+243S_{24}\big((((S_{25},C_0)^{(1)},C_0)^{(1)},C_0)^{(1)},C_0\big)^{(1)};
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_2 = {} & -3S_{24}\big[4D_1^3-18(S_5,C_0)^{(1)}+9(S_8,C_0)^{(1)}+2(S_{18},D_2)^{(1)}\big[6D_1^2+16S_3-3S_6\big]\\
& +18D_1S_{24}\big[3S_3-S_6\big]-12D_1\big((S_{18},C_2)^{(1)},C_1\big)^{(2)}+32\big(((S_{18},C_2)^{(1)},C_2)^{(2)},C_0\big)^{(1)};
\end{aligned}
$$

$$\mathcal{H}_3 = 72T_{136}(2307T_{140}-607T_{141})+T_{74}(13T_{144}+264T_{145});$$

$$\mathcal{H}_4 = T_{74};$$

$$
\begin{aligned}
\mathcal{H}_5 = {} & 12D_1^4S_{24}-18D_1S_6(S_{18},D_2)^{(1)}+128D_1S_3(S_{18},D_2)^{(1)}\\
& -48(S_8,C_0)^{(1)}(S_{18},D_2)^{(1)}+27S_{24}((S_{11},C_0)^{(1)},C_0)^{(1)}\\
& -9S_{24}((S_{14},C_0)^{(1)},C_0)^{(1)}+18D_1^2((S_{18},C_2)^{(1)},C_1)^{(2)}\\
& -7S_6((S_{18},C_2)^{(1)},C_1)^{(2)}+2D_1^2((S_{14},C_2)^{(1)},C_2)^{(3)}-S_6((S_{14},C_2)^{(1)},C_2)^{(3)}\\
& +8S_3((S_{14},C_2)^{(1)},C_2)^{(3)}-3S_6^2S_{24}-16D_1(((S_{18},C_2)^{(1)},C_2)^{(2)},C_0)^{(1)}\\
& +54D_1^2S_3S_{24}+27S_6S_3S_{24}-36S_3^2S_{24}-54D_1S_{24}(S_5,C_0)^{(1)}\\
& -48(S_{18},C_0)^{(1)})^2+60(S_{18},C_0)^{(1)}(S_{14},C_1)^{(2)}+28D_1^3(S_{18},D_2)^{(1)}.
\end{aligned}
$$

Here the polynomials

$$A_1 = S_{24}/288, \quad A_2 = S_{27}/72,$$
$$A_5 = (S_{23}, C_3)^{(4)}/2^7/3^5, \quad A_6 = (S_{26}, D_3)^{(2)}/2^5/3^3$$

are affine invariants and

$$T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72,$$

$$\begin{aligned} T_6 =&(3C_1D_3^2 - 27C_1T_3 + 54C_1T_4 + 4C_3D_2^2 - 2C_3S_{14}+ \\ &+ 16C_3S_{14} - 4C_2D_2D_3 + 2C_2S_{17} + 12C_2S_{21} - 4C_2S_{19})/2^4/3^2, \end{aligned}$$

$$\begin{aligned} T_{11} =&(D_3^2, C_2)^{(2)} - 9(T_3, C_2)^{(2)} + 18(T_4, C_2)^{(2)} - 6(D_3^2, D_2)^{(1)}+ \\ &+ 54(T_3, D_2)^{(1)} - 108(T_4, D_2)^{(1)} + 12D_2S_{26} - 12(S_{26}, C_2)^{(1)}+ \\ &+ 432C_2A_1 - 2160C_2A_2)/2^7/3^4, \end{aligned}$$

$$T_{16} =(S_{23}, D_3)^{(2)}/2^6 3^3, T_{17} = (S_{26}, D_3)^{(1)}/2^5/3^3,$$

$$\begin{aligned} T_{74} =&(2187T_3^2C_0 + 8748T_4^2C_0 + 20736T_{11}C_2^2 - 62208T_{11}C_1C_3+ \\ &+ 108C_3D_1D_2D_3^2 - 8C_2D_2^2D_3^2 - 54C_2D_1D_3^3 + 6C_1D_2D_3^3+ \\ &+ 27C_0D_3^4 - 54C_3D_3^2S_8 + 108C_3D_3^2S_9 + 27C_2D_3^2S_{11} - 27C_2D_3^2S_{12}+ \\ &+ 4C_2D_3^2S_{14} - 32C_2D_3^2S_{15} + 54D_1D_3^2S_{16} - 3C_1D_3^2S_{17} + 6C_1D_3^2S_{19}- \\ &- 9T_3(54C_0(18T_4 + D_3^2) + 54C_3(2D_1D_2 - S_8 + 2S_9) - C_2(8D_2^2+ \\ &+ 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) + 54D_1S_{16} + 3C_1(2D_2D_3- \\ &-S_{17}+2S_{19}-6S_{21}))-576T_6(2D_2D_3-S_{17}+2S_{19}-6S_{21})-18C_1D_3^2S_{21}+ \\ &+ 18T_4(6C_1D_2D_3 + 54C_0D_3^2 + 54C_3(2D_1D_2 - S_8 + 2S_9) - C_2(8D_2^2+ \\ &+ 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) + 54D_1S_{16} - 3C_1S_{17}+ \\ &+ 6C_1S_{19} - 18C_1S_{21}))/2^8/3^4, \end{aligned}$$

$$T_{44} =((S_{23}, C_3)^{(1)}, D_3)^{(2)}, \ T_{133} = \big(T_{74},C_3\big)^{(1)}, \ T_{137} = \big(T_{74},D_3\big)^{(1)}/6,$$

$$T_{136} =\big(T_{74}, C_3\big)^{(2)}/24, \ T_{140} = \big(T_{74}, D_3\big)^{(2)}/12,$$

$$T_{141} =\big(T_{74}, C_3\big)^{(3)}/36, \ T_{144} = \big(T_{133}, C_3\big)^{(4)}, \ T_{145} = \big(T_{137}, C_3\big)^{(3)}$$

are *T-comitants* of cubic systems (1) (see [15] for the definition of a $T$-comitant). We note that for the above invariant polynomials, we preserve the notations introduced in [8].

Using a different notation for the coefficients, we rewrite the cubic systems (1) as:

$$\begin{aligned} \dot{x} &= a+cx+dy+gx^2+2hxy+ky^2+px^3+3qx^2y+3rxy^2+sy^3 \equiv P(x,y), \\ \dot{y} &= b+ex+fy+lx^2+2mxy+ny^2+tx^3+3ux^2y+3vxy^2+wy^3 \equiv Q(x,y). \end{aligned} \tag{3}$$

Let $L(x,y) = W + Ux + Vy = 0$ be an invariant straight line of this family of cubic systems. Then, we get

$$UP(x,y) + VQ(x,y) = (W + Ux + Vy)(F + Dx + Ey + Ax^2 + 2Bxy + Cy^2),$$

and this identity yields the following equations:

$$\begin{aligned} Eq_1 &= tV + (p-A)U = 0, \\ Eq_2 &= (3u-A)V + (3q-2B)U = 0, \\ Eq_3 &= (3v-2B)V + (3r-C)U = 0, \\ Eq_4 &= (s-C)U + Vw = 0, \\ Eq_5 &= lV + (g-D)U - AW = 0, \\ Eq_6 &= (2m-D)V + (2h-E)U - 2BW = 0, \\ Eq_7 &= (n-E)V + kU - CW = 0, \\ Eq_8 &= eV + (c-F)U - DW = 0, \\ Eq_9 &= (f-F)V + dU - EW = 0, \\ Eq_{10} &= bV + aU - FW = 0. \end{aligned} \tag{4}$$

The infinite singularities (real or complex) of systems (3) are determined by the linear factors in the factorization over $\mathbb{C}$ of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

All possible configurations of invariant lines, in the case, when the total multiplicity of these lines (including the line at infinity) equals seven possessing at infinity four distinct infinite singularities (all real, or two real and two complex), are determined in [5, 4, 3, 6]. In these papers, the author studied the above-mentioned systems according to the type of configurations of invariant straight lines. Additionally, the affine invariant conditions for the class of cubic systems possessing two real and two complex singularities at infinity was constructed.

In this paper, the class of cubic systems with four real distinct infinite singularities and invariant straight lines in the configuration of the type $(3,3)$ is considered. All possible configurations of invariant straight lines for this class were constructed in [5] (see *Figure 1*). Our goal is to determine the affine invariant conditions for the realization of each one of these 14 configurations.

According to [17] (see also [19]) we have the following results ( Lemma 1.1, Lemma 1.2 and Theorem 1.1).

**Lemma 1.1.** *A cubic system $S \in \mathbb{CS}$ has 4 real distinct infinite singularities if and only if*

$$\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0.$$

**Lemma 1.2.** *If a cubic system $S \in \mathbb{CS}$ has 4 real distinct infinite singularities, then this system could be brought via a linear transformation to the canonical form*

$$\begin{cases} x' = p_0 + p_1(x,y) + p_2(x,y) + (p+r)x^3 + (s+v)x^2y + qxy^2, \\ y' = q_0 + q_1(x,y) + q_2(x,y) + px^2y + (r+v)xy^2 + (q+s)y^3, \end{cases} \tag{5}$$

*with $rs(r+s) \neq 0$ and $C_3 = xy(x-y)(rx+sy)$.*

**Theorem 1.1** ([3]). *Assume that a cubic system $S \in \mathbb{CS}$ possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex. Then the following conditions are satisfied, respectively:*

- *(i)* *two triplets* $\Rightarrow$ $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
- *(ii)* *one triplet and one couple* $\Rightarrow$ $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
- *(iii)* *one triplet* $\Rightarrow$ $\mathcal{V}_4 = \mathcal{U}_2 = 0$;
- *(iv)* *3 couples* $\Rightarrow$ $\mathcal{V}_3 = 0$;
- *(v)* *2 couples* $\Rightarrow$ $\mathcal{V}_5 = 0$.

According to [5] the following lemma is valid:

**Lemma 1.3.** *Assume the family of cubic system possessing 4 real distinct infinite singularities, i.e. the conditions $\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0$ hold. We additionally consider that for this family the condition $\mathcal{V}_1 = \mathcal{V}_2 = 0$ is satisfied. Then:*

*(A)* *this family of cubic systems could be brought via an affine transformation and time rescaling to the systems*

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + y^3; \end{aligned} \tag{6}$$

*(B)* *a cubic system* (6) *has invariant straight lines of total multiplicity 7 (including the line at infinity) in the configuration of the type* $(3,3)$ *if and only if the following conditions hold:*

$$k = d = h = e = l = m = 0,\ (c-f)^2 + (a^2-b^2)^2 \neq 0. \tag{7}$$

So, according to (7) systems (6) became of the form

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + y^3. \tag{8}$$

We denote

$$\xi_1 = -(27a^2 + 4c^3), \;\; \xi_2 = -(27b^2 + 4f^3), \;\; \nu_1 = a^2 + c^2, \;\; \nu_2 = b^2 + f^2.$$

According to [5, Theorem 3.2, Subsection 3.1] we have the following lemma:

**Lemma 1.4.** *Assume that for a system* (8) *the conditions given below in terms of the polynomials* $\xi_1$, $\xi_2$, $\nu_1$ *and* $\nu_2$ *are satisfied. Then this system could be brought via an affine transformation and time rescaling to one of the presented below canonical systems* (9)–(17)*. Moreover, this system possesses one of the configurations Config. 7.1a – 7.14a (see Figure 1) if and only if the conditions under the parameters a and b of the corresponding canonical system (when these conditions exist) are satisfied, respectively:*

$\xi_1\xi_2 > 0,\ \xi_1 + \xi_2 > 0 \quad \Rightarrow (9) \quad \Leftrightarrow$ *Config. 7.1a;*

$$\xi_1\xi_2 > 0,\ \xi_1 + \xi_2 < 0 \quad \Rightarrow (10), \left[\begin{array}{ll} ab \neq 0 & \Leftrightarrow \textit{Config. 7.2a}; \\ ab = 0, a+b \neq 0 & \Leftrightarrow \textit{Config. 7. 3a}; \\ a = b = 0 & \Leftrightarrow \textit{Config. 7.4a}; \end{array}\right.$$

$$\xi_1\xi_2 < 0 \quad \Rightarrow (11)\ \textit{with} \left[\begin{array}{ll} b \neq 0 & \Leftrightarrow \textit{Config. 7.5a}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.6a}; \end{array}\right.$$

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 > 0,\ \nu_1\nu_2 \neq 0 \quad \Leftrightarrow (12) \quad \Rightarrow$ *Config. 7.7a;*

$$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 < 0,\ \nu_1\nu_2 \neq 0 \quad \Rightarrow (13)\ \textit{with} \left[\begin{array}{ll} b \neq 0 & \Leftrightarrow \textit{Config. 7.8a}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.9a}; \end{array}\right.$$

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 > 0,\ \nu_1\nu_2 = 0 \quad \Rightarrow (14) \quad \Leftrightarrow$ *Config. 7.10a;*

$$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 < 0,\ \nu_1\nu_2 = 0 \quad \Rightarrow (15)\ \textit{with} \left[\begin{array}{ll} b \neq 0 & \Leftrightarrow \textit{Config. 7.11a}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.12a}; \end{array}\right.$$

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 = 0,\ \nu_1\nu_2 \neq 0 \quad \Rightarrow (16) \quad \Leftrightarrow$ *Config. 7.13a;*

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 = 0,\ \nu_1\nu_2 = 0 \quad \Rightarrow (17) \quad \Leftrightarrow$ *Config. 7.14a.*

The canonical systems indicated in the statement of the above lemma are the following ones:

$$\dot{x} = x(x-1)(x-a), \quad \dot{y} = y(y-b)(y-c), \quad a(a+1)bc(b-c) \neq 0, \tag{9}$$

$$\dot{x} = x\big[(x+a)^2 + c\big], \quad \dot{y} = y\big[(y+b)^2 + f\big], \quad c > 0,\ f > 0. \tag{10}$$

$$\dot{x} = x(x-1)(x-a), \quad \dot{y} = y\big[(y+b)^2 + c\big], \quad a(a-1) \neq 0\ c > 0. \tag{11}$$

$$\dot{x} = x^2(x-1), \quad \dot{y} = y(y-b)(y-c), \quad bc(b-c) \neq 0. \tag{12}$$

$$\dot{x} = x^2(x-1), \quad \dot{y} = y\big[(y+b)^2 + c\big], \quad c > 0. \tag{13}$$

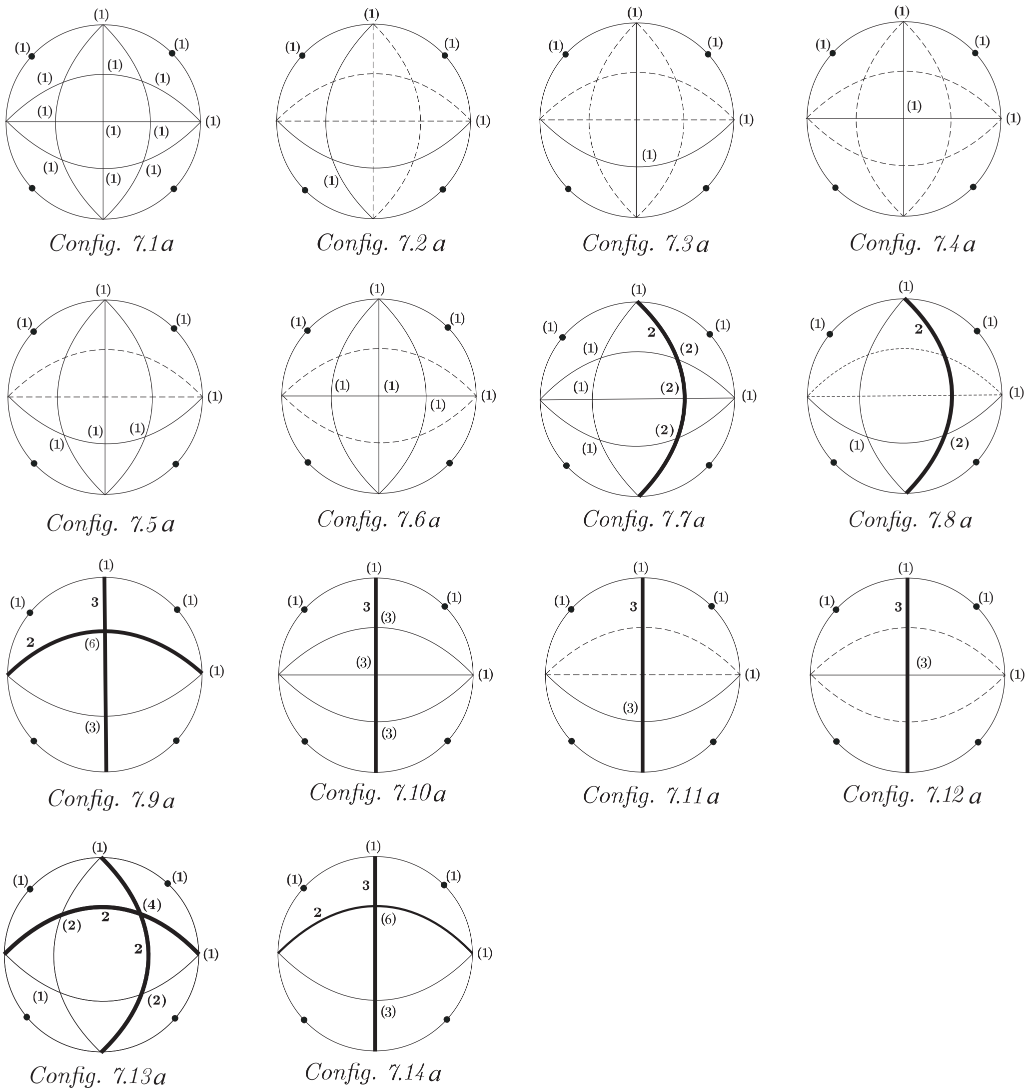


**Figure 1.** Configurations of invariant straght lines for systems in $CSL_{(3,3)}^{4r\infty}$

$$\dot{x} = x^3, \quad \dot{y} = y(y-1)(y-b), \;\; b(b-1) \neq 0. \tag{14}$$

$$\dot{x} = x^3, \quad \dot{y} = y\left[1 + (y+b)^2\right]. \tag{15}$$

$$\dot{x} = x^2(x-1), \quad \dot{y} = y^2(y-b), \;\; b \neq 0. \tag{16}$$

$$\dot{x} = x^3, \quad \dot{y} = y^2(y-1). \tag{17}$$

## 2. Invariant criteria for the realization of the configurations *Config. 7.1a* - *Config. 7.14a* of systems belonging to $CLS^{4r\infty}_{(3,3)}$

First we prove the following lemma.

**Lemma 2.1.** *An arbitrary non-degenerate cubic system belongs to the class $CLS^{4r\infty}_{(3,3)}$ if and only if the following conditions hold:*

$$\mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \mathcal{L}_1 = \mathcal{L}_8 = 0, \mathcal{L}_2^2 + \mathcal{K}_1^2 \neq 0. \tag{18}$$

*Proof.* According to Lemma 1.3 systems (6) could have two triplets of invariant straight line if and only if the conditions (7) are satisfied.

For systems (6) we calculate:

$$\mathcal{L}_1 = -20736(lx^3 + 2mx^2y - 2hxy^2 - ky^3).$$

Evidently $\mathcal{L}_1 = 0$ is equivalent to $l = m = h = k = 0$.

We define the new invariant polynomial

$$\mathcal{L}_8 = T_{15} - 2T_{14}$$

and evaluate it for systems (6) in the case $l = m = h = k = 0$:

$$\mathcal{L}_8 = 3ex^2 - 3dy^2.$$

It is evident that the condition $\mathcal{L}_8 = 0$ is equivalent to $d = e = 0$. Therefore, we have found out the invariant conditions which are equivalent with the first part of the conditions (7). So, applying $\mathcal{L}_1 = \mathcal{L}_8 = 0$ to systems (6) we arrive at systems (8) for which we calculate

$$\mathcal{L}_2 = -186624(c - f)xy, \quad \mathcal{K}_1 = 2^{17}3^{15}5^4 7^4 \cdot 817(a^2 - b^2)(x^2 - y^2).$$

Therefore, we deduce that the condition $\mathcal{L}_2^2 + \mathcal{K}_1^2 \neq 0$ is equivalent to

$$(c - f)^2 + (a^2 - b^2)^2 \neq 0.$$

The proof is complete. □

Next, we prove our main result.

**Theorem 2.1.** *Assume that for a generic cubic system* (3) *the conditions* (18) *are satisfied, i.e. this system belongs to the class $CLS^{4r\infty}_{(3,3)}$. Then this system has one of the configurations Config. 7.1 – 7.14 if and only if one of the following sets of conditions is satisfied, correspondingly:*

$(A_1)$ $\mathcal{H}_1 > 0$, $\mathcal{H}_2 > 0$ $\Leftrightarrow$ *Config. 7.1a;*

$(A_2)$ $\mathcal{H}_1 > 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_3 \neq 0$ $\Leftrightarrow$ *Config. 7.2a;*

$(A_3)$ $\mathcal{H}_1 > 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_3 = 0$, $\mathcal{H}_4 \neq 0$ $\Leftrightarrow$ *Config. 7.3a;*

$(A_4)$ $\mathcal{H}_1 > 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_3 = \mathcal{H}_4 = 0$ $\Leftrightarrow$ *Config. 7.4a;*

$(A_5)$ $\mathcal{H}_1 < 0$, $\mathcal{H}_3 \neq 0$ $\Leftrightarrow$ *Config. 7.5a;*

$(A_6)$ $\mathcal{H}_1 < 0$, $\mathcal{H}_3 = 0$ $\Rightarrow$ *Config. 7.5a or 7.6a;*

$(A_7)$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 > 0$, $\mathcal{H}_5 \neq 0$ $\Leftrightarrow$ *Config. 7.7a;*

$(A_8)$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_5 \neq 0$, $\mathcal{H}_3 \neq 0$ $\Leftrightarrow$ *Config. 7.8a;*

$(A_9)$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_5 \neq 0$, $\mathcal{H}_3 = 0$ $\Leftrightarrow$ *Config. 7.9a;*

$(A_{10})$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 > 0$, $\mathcal{H}_5 = 0$ $\Leftrightarrow$ *Config. 7.10a;*

$(A_{11})$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_5 = 0$, $\mathcal{K}_2 \neq 0$ $\Leftrightarrow$ *Config. 7.11a;*

$(A_{12})$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$, $\mathcal{H}_5 = 0$, $\mathcal{K}_2 = 0$ $\Leftrightarrow$ *Config. 7.12a;*

$(A_{13})$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 = 0$, $\mathcal{H}_3 \neq 0$ $\Leftrightarrow$ *Config. 7.13a;*

$(A_{14})$ $\mathcal{H}_1 = 0$, $\mathcal{H}_2 = 0$, $\mathcal{H}_3 = 0$ $\Leftrightarrow$ *Config. 7.14a.*

*Proof.* Consider a cubic system belonging to $CLS^{4r\infty}_{(3,3)}$. As it was prove earlier, such a system via an affine transformation an time rescaling could be brought to the canonical form (8). For these systems we calculate

$$\begin{aligned}\mathcal{H}_1 =& 2^{12}3^3(27a^2+4c^3)(27b^2+4f^3) = 2^{12}3^3\xi_1\xi_2;\\ \mathcal{H}_2 =& 2^7 3^4(27a^2+27b^2+4c^3+4f^3)(x^2+y^2) = 2^7 3^4(\xi_1+\xi_2)(x^2+y^2).\end{aligned} \tag{19}$$

**The statement** $(A_1)$. According to Theorem 2.1, we have $\mathcal{H}_1 > 0, \mathcal{H}_2 > 0$ and by (19) these conditions are equivalent to $\xi_1\xi_2 > 0$ and $\xi_1+\xi_2 > 0$, respectively. As a result, according to Lemma 1.4, we get systems (9) for which we calculate:

$$\mathcal{H}_1 = 2^{12}3^3a^2b^2c^2(a-1)^2(b-c)^2.$$

Therefore, the condition $\mathcal{H}_1 > 0$ imply $a(a+1)bc(b-c) \neq 0$. Thus, according to Lemma 1.4 we arrive at configuration *Config. 7.1a*. This completes the proof of the statemant $(A_1)$ of the theorem.

**The statements** $(A_2)-(A_4)$. We observe that the conditions $\mathcal{H}_1 > 0$, $\mathcal{H}_2 < 0$ are common for all these three statements. On the other hand these conditions are equivalent to $\xi_1\xi_2 > 0$ and $\xi_1+\xi_2 < 0$. By [5] via an affine transformation and time rescaling systems

(8) could be brought to the form (10) for which we calculate

$$\begin{aligned}\mathcal{H}_1 =& 2^{16}3^3cf(a^2+c)^2(b^2+f)^2;\\ \mathcal{H}_2 =& -2^93^4[c(a^2+c)^2+f(b^2+f)^2)](x^2+y^2).\end{aligned}$$

It is evident that $\mathcal{H}_1 > 0$ implies $cf > 0$ and due to $\mathcal{H}_2 < 0$ we get $c > 0$ and $f > 0$.

For systems (10) we calculate:

$$\begin{aligned}\mathcal{H}_3 =& 2^{10}15^217ab(a^2+9c)(b^2+9f)x^6y^6(x^4-y^4);\\ \mathcal{H}_4 =& 2x^4y^4\big[b(b^2+9f)x-a(a^2+9c)y\big].\end{aligned}$$

Assume first that the condition $\mathcal{H}_3 \neq 0$ is satisfied. Since $c > 0$ and $f > 0$ we conclude that this condition is equivalent to $ab \neq 0$. So, according to Lemma 1.4 in this case we get the configuration *Config. 7.2a* and the statement $(A_2)$ of our theorem is proved.

If $\mathcal{H}_3 = 0$ we get $ab = 0$ and we investigate two cases: $\mathcal{H}_4 \neq 0$ and $\mathcal{H}_4 = 0$.

The condition $\mathcal{H}_4 \neq 0$ implies $a^2 + b^2 \neq 0$. So, according to Lemma 1.4 in this case we get configurations *Config. 7.3a* and hence, the statement $(A_3)$ is proved.

Assume finally $\mathcal{H}_3 = \mathcal{H}_4 = 0$. This implies $a = b = 0$ and by Lemma 1.4 we get *Config. 7.4a*. Thus, we proved the statement $(A_4)$ of the theorem.

**The statements** $(A_5)$, $(A_6)$. In this case we have $\xi_1\xi_2 < 0$ and according to [5] via an affine transformation and time rescaling systems (8) could be brought to the form (11) for which we calculate

$$\begin{aligned}\mathcal{H}_1 =& -2^{14}3^3(a-1)^2a^2c(b^2+c)^2,\\ \mathcal{H}_3 =& 2^93^25^217b(a-2)(a+1)(2a-1)(b^2+9c)x^6y^6(x^4-y^4).\end{aligned}$$

We observe that the condition $\mathcal{H}_1 < 0$ guarantees $a(a-1) \neq 0$ and $c > 0$, i.e. the conditions mentioned in (11) hold. At the same time due to $c > 0$ the condition $\mathcal{H}_3 \neq 0$ imply $b \neq 0$. So according to Lemma (1.4) the conditions $\mathcal{H}_1 < 0$ and $\mathcal{H}_3 \neq 0$ give us *Config. 7.5a*

Assume now $\mathcal{H}_3 = 0$. In this case, we get two possibilities: $b = 0$ or $b \neq 0$ and $(a-2)(a+1)(2a-1)$. In the first case, by Lemma 1.4 we have the configuration *Config. 7.6a*, whereas in the second case we arrive at the configuration *Config. 7.5a*. So we conclude that the statements $(A_5)$ and $(A_6)$ of Theorem 2.1 are valid.

We point out that the problem of determining of an invariant polynomial which gouverns the condition $b = 0$ remains open.

**The statement** $(A_7)$**.** In this case for systems (8) the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 > 0$ and $\mathcal{H}_5 \neq 0$ are satisfied. The first two conditions give us $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 > 0$ for systems

(8) for which we have

$$\mathcal{H}_5 = -2^8 3^3 (c\xi_2 + f\xi_1). \tag{20}$$

We claim that the condition $\mathcal{H}_5 \neq 0$ implies $\nu_1\nu_2 \neq 0$. Indeed, supposing the contrary that $\nu_1 = 0$ (respectively $\nu_2 = 0$) we get $a = c = 0$ (respectively $b = f = 0$) and this leads to the condition $\mathcal{H}_5 = 0$. This proves our claim and hence, we have the condition $\nu_1\nu_2 \neq 0$. Then according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (13) for which we calculate:

$$\mathcal{H}_5 = 2^8 3^2 b^2 (b-c)^2 c^2.$$

Evidently the condition $\mathcal{H}_5 \neq 0$ implies $bc(b-c) \neq 0$ and we get the condition required for systems (13).

So, accoridng to Lemma 1.4 in this case we get configuration *Config. 7.7a* and hence, the statement $(A_7)$ is proved.

**The statements** $(A_8)$, $(A_9)$**.** In both cases the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$ and $\mathcal{H}_5 \neq 0$ are satisfied. Then for systems (8) the first two conditions give us $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 < 0$. Moreover, as it was shown in the case of the statement $(A_7)$ the condition $\mathcal{H}_5 \neq 0$ implies $\nu_1\nu_2 \neq 0$. So according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (14) for which we calculate:

$$\mathcal{H}_2 = -\,2^9 3^4 c(b^2+c)^2(x^2+y^2).$$

Clearly the condition $\mathcal{H}_2 < 0$ yields $c > 0$, i. e. we get the condition required for systems (14).

In order to distinguish the conditions $b \neq 0$ (the statement $(A_8)$) and $b = 0$ (the statement $(A_9)$) for systems (14) we evaluate the invariant polynomial $\mathcal{H}_3$:

$$\mathcal{H}_3 = 2^{10} 3^8 5^2 17 b(b^2+9c)x^6(x-y)y^6(x+y)(x^2+y^2).$$

Since $c > 0$, we obtain that the conditions $b = 0$ is equivalent to $\mathcal{H}_3 = 0$. Therefore by Lemma 1.4 we arrive at the configuration *Config. 7.8a* if $b \neq 0$ (i. e. $\mathcal{H}_3 \neq 0$) and *Config. 7.9a* if $b = 0$ (i. e. $\mathcal{H}_3 = 0$).

**The statement** $(A_{10})$**.** As earlier we determine that for systems (8) the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 > 0$ imply $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 > 0$.

We claim that the condition $\mathcal{H}_5 = 0$ implies $\nu_1\nu_2 = 0$. Indeed, since $\xi_1\xi_2 = 0$, we may assume that $\xi_1 = 0$ due to the change $(x, y, a, b, c, f) \mapsto (y, x, b, a, f, c)$. Then we have $\xi_1 = -(25a^2 + 4c^3) = 0$. According to (20) the condition $\xi_1 = 0$ and $\xi_2 \neq 0$ implies $c = 0$ and then, we have $\xi_1 = 27a^2 = 0$, i. e. we get $a = 0$. Evidently we arrive at the codnition

$\nu_1 = 0$ and this complete the prove of our claim. Then according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (14) for which we calculate:

$$\mathcal{H}_2 = 2^7 3^4 b^2 (b-1)^2 (x^2 + y^2).$$

Evidently the condition $\mathcal{H}_2 > 0$ implies $b(b-1) \neq 0$, i. e we get the condition required for systems (14).

So, accoridng to Lemma 1.4 in this case we get configuration *Config. 7.10a* and hence, the statement $(A_{10})$ is proved.

**The statements** $(A_{11})$, $(A_{12})$**.** In both cases the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$ and $\mathcal{H}_5 = 0$ are satisfied. Simillary as in the case of statement $(A_{10})$ it can be proved that the condition $\mathcal{H}_5 = 0$ implies $\nu_1\nu_2 = 0$. So according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (15) for which we calculate:

$$\mathcal{H}_1 = \mathcal{H}_5 = 0,\ \mathcal{H}_2 = -2^9 3^4 (b^2+1)^2 (x^2 + y^2).$$

In order to distinguish the conditions $b \neq 0$ (the statement $(A_{11})$) and $b = 0$ (the statement $(A_{12})$) for systems (15) we evaluate the invariant polynomial $\mathcal{K}_2$:

$$\mathcal{K}_2 = 2b(9 + b^2) x^5 y^4.$$

We get that the conditions $b = 0$ is equivalent to $\mathcal{K}_2 = 0$. Therefore by Lemma (1.4) we arrive at the configuration *Config. 7.11a* if $b \neq 0$ (i. e. $\mathcal{K}_2 \neq 0$) and *Config. 7.12a* if $b = 0$ (i. e. $\mathcal{K}_2 = 0$).

**The statements** $(A_{13})$, $(A_{14})$**.** We observe that in both cases the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 = 0$ are satified and this is equivalent with $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 = 0$. For systems (8) with $\xi_1 = \xi_2 = 0$, we set two new parameters $u$ and $v$ as follows: $a = 2u^3, b = 2v^3$ and then we get $c = -3u^2, f = -3v^2$. In this case we calculate:

$$\begin{aligned} \mathcal{H}_3 &= 2^{10} 3^2 5^2 17 u^3 v^3 x^6 y^6 (x^4 - y^4), \\ \mathcal{H}_4 &= -54 x^4 y^4 (v^3 x - u^3 y), \end{aligned} \tag{21}$$

and $\nu_1 = u^4(9 + 4u^2)$, $\nu_2 = v^4(9 + 4v^2)$. It is clear that $\nu_1\nu_2 \neq 0$ is equivalent to $\mathcal{H}_3 \neq 0$.

Thus by Lemma 1.4 via an affine transformation of coordinates and time rescaling systems (8) could be brought to the form (16) for which we calculate:

$$\mathcal{H}_3 = 2^{10} 3^2 5^2 17 b^3 x^6 y^6 (x^4 - y^4) \neq 0 \quad \Rightarrow \quad b \neq 0.$$

Therefore by Lemma 1.4 we arrive at the configuration *Config. 7.13a*.

Let now $\mathcal{H}_3 = 0$, i. e. $\nu_1\nu_2 = 0$. This implies $uv = 0$ and we claim that $u^2 + v^2 \neq 0$ due to the condition $\mathcal{K}_1^2 + \mathcal{L}_2^2 \neq 0$. Indeed, for systems (8) with the parameters $a, b, c, f$ given above, we calculate $\mathcal{L}_2 = 559872(u^2 - v^2)xy \neq 0$ and this proves our claim.

Thus we have $\nu_1\nu_2 = 0$ and $\nu_1^2 + \nu_2^2 \neq 0$ and according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (17) and consequently we obtain the configuration *Config. 7.14a*. □

**Acknowledgments.** *This work is supported by the Program SATGED 011303, Moldova State University.*

## References


[1] Artes, J., Grunbaum, B. and Llibre, J. On the number of invariant straight lines for polynomial differential system. *Pac. J. Appl. Math.*, 1998, vol. 184, 317–327.

[2] Artés, J. C., Llibre, J., Schlomiuk, D. and Vulpe, N. *Geometric configurations of singularities of planar polynomial differential systems* [ *A global classification in the quadratic case*]. Birkhäuser/Springer, Cham, [2021]. xii+699 pp. ISBN: 978-3-030-50569-1; 978-3-030-50570-7

[3] Bujac, C., Vulpe, N. The family of cubic differential systems with two real and two complex distinct infinite singularities and invariant straight lines of the type (2, 2, 2). *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2024, vol. 104-105, no. 1-2, 84–99.

[4] Bujac, C., Schlomiuk D., Vulpe N. The family of cubic differential systems with two real and two complex distinct infinite singularities and invariant straight lines of the type (3,1,1,1). *Electronic Journal of Qualitative Theory of Differential Equations*, 2023, no. 40, 1–94.

[5] Bujac, C., Schlomiuk, D., Vulpe, N. Cubic differential systems with invariant straight lines of total multiplicity seven and four real distinct infinite singularities. *Electronic Journal of Differential Equations*, 2021, vol. 2021, no. 83, 1–110. ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

[6] Bujac, C. The classification of a family of cubic differential systems in terms of configurations of invariant lines of the type (3, 3). *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2019, vol. 90, no. 2, 79–98.

[7] Bujac, C. One subfamily of cubic systems with invariant lines of total multiplicity eight and with two distinct real infinite singularities. *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2015, vol. 77, no. 1, 48–86.

[8] Bujac, C., Vulpe, N. Cubic differential systems with invariant lines of total multiplicity eight and with four distinct infinite singularities. *J. Math. Anal. Appl.*, 2015, vol. 423, 1025–1080.

[9] Bujac, C., Vulpe, N. Cubic systems with invariant straight lines of total multiplicity eight and with three distinct infinite singularities. In: *Qual. Theory Dyn. Syst.*, 2015, vol. 14, no. 1, 109–137.

[10] Bujac, C., Vulpe, N. Classification of cubic differential systems with invariant straight lines of total multiplicity eight and two distinct infinite singularities. *Electron. J. Qual. Theory Differ. Equ.*, 2015, no. 60, 1–38.

[11] Bujac, C., Vulpe, N. Cubic differential systems with invariant straight lines of total multiplicity eight possessing one infinite singularity. *Qual. Theory Dyn. Syst.*, 2017, vol. 16, no. 1, 1–30.

[12] Grace, J. H., Young, A. *The algebra of invariants*. New York: Stechert, 1941.

[13] Schlomiuk, D., Vulpe, N. Planar quadratic differential system with invariant straight lines of at least five total multiplicity. *Qual. Theory Dyn. Syst.*, 2024, no. 5, 135–194.

[14] Schlomiuk, D., Vulpe, N. Global classification of the planar Lotka–Volterra differential systems according to their configurations of invariant straight line. *Journal of Fixed Point Theory and Applications*, 2010, vol. 8, no.1, 69 p.

[15] Schlomiuk, D., Vulpe, N. Planar quadratic differential systems with invariant straight lines of at least five total multiplicity. *Qualitative Theory of Dynamical Systems*, 2004, no. 5, 135–194.

[16] Sibirskii, K.S. *Introduction to the algebraic theory of invariants of differential equations.* Translated from Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[17] Llibre, J. and Vulpe, N. I. Planar cubic polynomial differential systems with the maximum number of invariant straight lines. *Rocky Mountain J. Math.*, 2008, vol. 38, 1301–1373.

[18] Olver, P.J. *Classical Invariant Theory* (London Mathematical Society student texts: 44), Cambridge University Press, 1999.

[19] Popa, M.N. The number of comitants that are involved in determining the number of integral lines of a cubic differential system. *Izv. Akad. Nauk Moldav. SSSR Mat.*, 1990, no. 1, 67–69 (in Russian).

*Received: October 25, 2024* *Accepted: December 30, 2024*

(Cristina Bujac) "Vladimir Andrunachievici" Institute of Mathematics and Computer Sciences, Moldova State University, 5 Academiei St., Chişinău, Republic of Moldova
*E-mail address*: `cristina@bujac.eu`