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Center conditions for a cubic differential system with one invariant straight line and one invariant conic

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Abstract. In this work we find the center conditions for a cubic system of differential equations with a critical point of a center or a focus type having one invariant straight line and one invariant conic. The center-focus problem is studied by using the Darboux integrability and the rational reversibility methods.

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Keywords: cubic system of differential equations, the center-focus problem, invariant algebraic curve, Darboux integrability, rational reversibility.

Condiţii de existenţă a centrului pentru un sistem diferenţial cubic cu o dreaptă invariantă şi o conică invariantă

Rezumat. În lucrare se determină condiţii de existenţă a centrului pentru un sistem cubic de ecuaţii diferenţiale, cu punct critic de tip centru sau focar, care posedă o dreaptă invariantă şi o conică invariantă. Problema deosebirii centrului de focar se studiază aplicând integrabilitatea Darboux şi reversibilitatea raţională.

Cuvinte-cheie: sistem cubic de ecuaţii diferenţiale, problema centrului şi focarului, curbă algebrică invariantă, integrabilitatea Darboux, reversibilitate raţională.

1. Introduction

We consider the cubic system of differential equations

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are relatively prime polynomials in the ring of real polynomials in the variables x, y and $\dot{x} = dx/dt, \dot{y} = dy/dt$. The origin of coordinates $O(0,0)$ is a critical point which is a center or focus (a fine focus) for (1).

The problem of distinguishing between a center and a focus (the center-focus problem) is open for cubic systems (1). It is completely solved for: quadratic systems, cubic symmetric systems, the Kukles system, and some families of polynomial differential systems of higher degree.

The center-focus problem was solved for some subclasses of cubic differential systems (1) with algebraic solutions: two parallel invariant straight lines [5], [25]; three invariant straight lines [8], [26], [27]; four invariant straight lines [8], [19], [22]; two invariant straight lines and one invariant cubic [13], [14]; two invariant straight lines and one invariant conic [10], [11], [12].

An approach to the center-focus problem is based on the theory of integrability. It means investigating the integrability of (1) in some neighborhood of the critical point $O(0,0)$. The integrability conditions were found for some subclasses of cubic differential systems (1) with invariant algebraic curves in [4], [6], [7], [16], [21]. It was found that every center in a cubic differential system (1) is provided by the Darboux integrability if the system has four invariant straight lines [19] or the system has two invariant straight lines and one invariant conic [8].

The Darboux integrability conditions were determined for: cubic systems (1) with two parallel invariant straight lines [5], a class of reversible cubic systems [1] and some complex cubic systems [20].

The purpose of this work is to find the center conditions for a cubic system (1) that has two invariant algebraic curves. The paper is structured as follows. In Section 2, we review established results related to the existence of invariant algebraic curves and the Darboux integrability. Section 3 examines the existence of Darboux first integrals that consist of an invariant straight line and an irreducible invariant conic. In Section 4, we apply the method of rational reversibility to determine the center conditions for a cubic system (1) that contains an invariant straight line and an invariant conic.

2. Invariant algebraic curves and Darboux integrability

Invariant algebraic curves play a crucial role in the study of the integrability of polynomial differential systems. They provide significant insights into the qualitative behavior of solutions and help in identifying the first integrals.

Definition 2.1. *An algebraic curve $\Phi(x,y)=0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x,y]$ is an invariant algebraic curve of a differential system (1) if there exists a polynomial $K(x,y) \in \mathbb{C}[x,y]$ such that*

$$\frac{\partial \Phi}{\partial x}P(x,y)+\frac{\partial \Phi}{\partial y}Q(x,y)=\Phi(x,y)K(x,y). \tag{2}$$

The polynomial $K(x,y)$ is called the cofactor of the invariant algebraic curve $\Phi(x,y)=0$.

It is a very hard problem to find invariant algebraic curves for a given system (1) because, in general, we do not have any evidence about the degree of a curve [24]. Not all polynomial differential systems admit invariant algebraic curves.

We analyze the center-focus problem for the cubic system (1) under the assumption that it possesses irreducible invariant algebraic curves in $\mathbb{C}[x, y]$. The notation $\mathbb{C}[x, y]$ denotes the ring of polynomials in two variables with complex coefficients [13].

Definition 2.2 ([8])**.** *The invariant algebraic curve $\Phi(x, y) = 0$ is said to be an algebraic solution of system (1) if and only if $\Phi(x, y)$ is an irreducible element of $\mathbb{C}[x, y]$.*

Knowledge of invariant algebraic curves is fundamental in the study of polynomial differential systems. They provide key information about integrability, phase portraits, stability, and global dynamics. The necessary and sufficient conditions for the existence of invariant algebraic curves in a cubic system (1) were determined when the curves are: straight lines [8], [18], [19], [27]; straight lines and conics [10], [11], [9], [8]; straight lines and cubics [13], [14]; conics [15]; cubics [17].

According to [7], [8], system (1) is considered integrable on an open set D of $\mathbb{R}^2$ if there exists a nonconstant analytic function $F : D \longrightarrow \mathbb{R}$ that remains constant along all solution curves $(x(t), y(t))$ within D, meaning that $F(x(t), y(t)) = C$ for all t where the solution is defined. This function F is called a *first integral* of the system on D.

Suppose that the function F exists in D. Then all the solutions of the cubic system (1) in D are known [24] and $F(x, y) = C$ gives every solution of (1) for some $C \in \mathbb{R}$. Clearly F is a first integral if and only if F solves the partial differential equation

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} \equiv 0. \tag{3}$$

For cubic system (1), we study the algebraic integrability which is called *the Darboux integrability* [2], [24]. Darboux's method provides a systematic way to construct a first integral or an integrating factor. Suppose that the curves $\Phi_j = 0,\ j = \overline{1, k}$ are invariant algebraic curves of (1) and $\alpha_j \in \mathbb{C}$. A first integral of the form

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_k^{\alpha_k}, \tag{4}$$

is called a *Darboux first integral*.

We mention that for cubic systems (1), the conditions for the existence of integrating factors of the form $\mu = \Phi^{\beta}$ were obtained in [17] when $\Phi = 0$ is an invariant cubic and in [15] when $\Phi = 0$ is an invariant conic. First integrals and integrating factors of the form $l_1^{\alpha_1}\Phi^{\alpha_2}$, composed of one invariant straight line $l_1 = 0$ and one invariant cubic $\Phi = 0$, were determined in [7], [16]. In this paper, we study for cubic system (1) the problem of the existence of first integrals of the form

$$l_1^{\alpha}\Phi^{\beta} = C, \tag{5}$$

where $l_1 = 0$ is an invariant straight line and $\Phi = 0$ is an invariant conic.

It is known [24] that the origin will be a center for system (1) if and only if there exists a nonconstant analytic first integral

$$x^2 + y^2 + F_3(x, y) + \cdots + F_m(x, y) + \cdots = C$$

in some neighborhood of $O(0,0)$, where F_m are homogeneous polynomials of degree m.

3. Cubic systems with two invariant algebraic curves

Assume that $Ax + By + 1 = 0$ is a real invariant straight line of the cubic differential system (1). Then, by a transformation of the form $x \to \omega(x\cos\alpha - y\sin\alpha)$, $y \to \omega(x\sin\alpha + y\cos\alpha)$, we can bring this line to the form $x = 1$. In [16] the following lemma was proved.

Lemma 3.1. *A straight line $x = 1$ is an invariant straight line for cubic system* (1) *if and only if the following set of conditions is satisfied*

$$r = 0,\ p = -f,\ m = -c - 1,\ k = -a. \tag{6}$$

When conditions (6) are satisfied, we obtain a cubic system of the form

$$\begin{cases} \dot{x} = (1 - x)(y + xy + ax^2 + cxy + fy^2) \equiv P(x, y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y). \end{cases} \tag{7}$$

Let us assume that the cubic differential system (7) has an irreducible invariant conic

$$\Phi(x, y) \equiv a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0, \tag{8}$$

where $a_{01}, a_{10}, a_{02}, a_{11}, a_{20}$ are real parameters and $(a_{02}, a_{11}, a_{20}) \neq 0$. For every conic curve (8) the following quantities [8] are invariants

$$\begin{aligned} &I_1 = a_{02} + a_{20},\ I_2 = (4a_{20}a_{02} - a_{11}^2)/4, \\ &I_3 = (4a_{20}a_{02} - a_{20}a_{01}^2 + a_{11}a_{01}a_{10} - a_{10}^2a_{02} - a_{11}^2)/4 \end{aligned} \tag{9}$$

with respect to the rotation of axes. The conic (8) is: a parabola when $I_2 = 0$, an ellipse when $I_2 > 0$ and a hyperbola when $I_2 < 0$. If $I_3 = 0$, then the conic (8) is reducible into two straight lines.

By Definition 2.1, the curve (8) is an invariant conic for cubic system (7) if and only if there exists a cofactor $K(x, y) = c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2$ such that

$$P(x, y)\frac{\partial \Phi}{\partial x} + Q(x, y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x, y)K(x, y), \tag{10}$$

where $c_{10}, c_{01}, c_{20}, c_{11}, c_{02} \in \mathbb{R}$.

The identity (10) yields a system $\{F_{ij}=0,\ i+j=1,2,3,4\}$ of fourteen equations for the unknowns $c_{kl}, a_{kl},\ k+l=1,2$. When $i+j=1,2$, we find from (10) that

$$c_{10}=-a_{01},\ c_{01}=a_{10},\ c_{11}=a_{01}^2-da_{01}-a_{10}^2+ca_{10}-2a_{02}+2a_{20},$$
$$c_{20}=aa_{10}+a_{01}a_{10}-ga_{01}-a_{11},\ c_{02}=a_{11}-ba_{01}+fa_{10}-a_{01}a_{10}$$

and when $i+j=3,4$, we obtain from (10) the system of algebraic equations

$$\begin{aligned}
F_{40} &\equiv (a_{20}-s)a_{11}+(ga_{01}-aa_{10}-2a-a_{01}a_{10})a_{20}=0,\\
F_{31} &\equiv (2a_{20}-2s)a_{02}+(a_{10}^2-a_{01}^2+da_{01}-ca_{10}-2c-2)a_{20}+\\
&+(a_{11}+ga_{01}-aa_{10}-a-a_{01}a_{10}-q)a_{11}-2a_{20}^2=0,\\
F_{22} &\equiv (3a_{11}-aa_{10}-a_{01}a_{10}+ga_{01}-2q)a_{02}-3a_{11}a_{20}+\\
&+a_{11}(a_{10}^2-a_{01}^2+da_{01}-ca_{10}-c-n-1)+\\
&+((a_{10}+b)a_{01}-f(a_{10}+2))a_{20}=0,\\
F_{13} &\equiv (2a_{02}+a_{10}^2-a_{01}^2-2a_{20}+da_{01}-ca_{10}-2n)a_{02}-a_{11}^2+\\
&+((a_{10}+b)a_{01}-f(a_{10}+1)-l)a_{11}=0,\\
F_{04} &\equiv ((a_{10}+b)a_{01}-2l-fa_{10}-a_{11})a_{02}=0,\\
F_{30} &\equiv (a_{11}-a)a_{10}+(a_{01}+2a)a_{20}-ga_{11}+\\
&+ga_{01}a_{10}-aa_{10}^2-a_{01}a_{10}^2-sa_{01}=0,\\
F_{21} &\equiv a_{10}^3+(a-d+2a_{01})a_{11}-aa_{10}a_{01}+(2c-3a_{10})a_{20}+\\
&+2(a_{10}-g)a_{02}-ca_{10}^2-ca_{10}-a_{10}+da_{01}a_{10}+\\
&+ga_{01}^2-2a_{01}^2a_{10}-qa_{01}=0,\\
F_{12} &\equiv (3a_{01}-2d)a_{02}+(c-b-2a_{10})a_{11}+2(f-a_{01})a_{20}-a_{01}^3+\\
&+da_{01}^2+(2a_{10}+b-c)a_{01}a_{10}-na_{01}-fa_{10}^2-fa_{10}=0,\\
F_{03} &\equiv (f-a_{01})a_{11}-(a_{10}+2b)a_{02}+\\
&+(a_{01}a_{10}+ba_{01}-fa_{10}-l)a_{01}=0.
\end{aligned}\tag{11}$$

We shall study the consistency of (11) in a_{10}, a_{01}, a_{20}, a_{11}, a_{02} and establish the conditions under which the system has one solution.

4. Cubic systems and first integrals

In this section, we study for cubic system (1) the problem of the existence of first integrals of the form

$$F(x,y)\equiv(x-1)^{\alpha}(a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1)^{\beta}=C,\tag{12}$$

where the invariant conic is irreducible and α,β are nonzero real exponents.

According to [8], the relation (12) is a first integral for the system (1) if and only if the identity (3) holds. We will use this identity to find the first integrals (12) of system (1).

Theorem 4.1. *The cubic differential system (1), where $P(x, y)$ and $Q(x, y)$ are relatively prime polynomials, does not possess Darboux first integrals in the form of* (12).

Proof. The identity (3) being applied to (12) yields the following system of equations

$$\{H_{ij} = 0, \ \ i + j = 1, 2, 3, 4\} \tag{13}$$

in the coefficients of (1) and the parameters $a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \alpha, \beta, \ \alpha\beta \neq 0$.

From equations $H_{01} = 0$ and $H_{10} = 0$ of the system (13) we obtain $\alpha = a_{10}\beta$ and $a_{01} = 0$. Then the equations $H_{02} = 0$, $H_{11} = 0$, and $H_{20} = 0$ of (13) yield the following

$$a_{11} = 0, a_{20} = (a_{10} + 2a_{02} + a_{10}^2)/2.$$

From equations $H_{ij} = 0, \ i + j = 3$ of the system (13) we find

$$a = 0, d = f, a_{10} = -2b, g = [(](b + c)a_{02} + b(2b^2 - 3b + 1)]/a_{02}.$$

Then the equations $H_{ij} = 0, \ i + j = 4$ of (13) imply

$$l = bf, a_{10} = -2b, q = [f(ba_{02} - a_{02} + 2b^3 - 3b^2 + b)]/a_{02},$$
$$n = bc + b, s = [(a_{02} - b + 2b^2)(b - 1)(c + 1)]/a_{02}.$$

We find that right-hand sides of (1) have a common factor $1 + (c + 1)x + fy = 0$ in contradictions to the assumption of Theorem. ■

Remark 4.1. *There exists quadratic differential systems with first integrals containing one invariant straight line and one invariant conic. For example, in* [3] *it was shown that for quadratic system*

$$\dot{x} = -y - x^2 - y^2, \ \dot{y} = x(1 + y)$$

the straight line $y + 1 = 0$ *and the conic* $6x^2 + 3y^2 + 2y - 1 = 0$ *are invariants. This system has a first integral* $(y + 1)^2(6x^2 + 3y^2 + 2y - 1) = C$.

5. Cubic systems and rational reversibility

As established in [28], if the differential system (1) has a critical point $O(0, 0)$ of center or focus type and remains invariant under reflection with respect to the axis $X = 0$ and reversion of time, then $O(0, 0)$ is a center for system (1).

It is evident that the critical point $(O(0, 0)$ is a center for the system (1) if a diffeomorphism exists $H : U \longrightarrow V, \ H = \{X = g(x, y), \ Y = h(x, y)\}, \ H(0, 0) = (0, 0)$, which brings the system (1) to a system that has an axis of symmetry [28].

In this paper, we obtain centers by rational reversibility. We seek a rational transformation, invertible in a neighborhood of $O(0, 0)$, of the form [6], [23]

$$x = \frac{a_1X + b_1Y}{a_3X + b_3Y - 1}, \ y = \frac{a_2X + b_2Y}{a_3X + b_3Y - 1} \tag{14}$$

with $a_j, \ b_j \in \mathbb{R}, \ j = 1, 2, 3$, which maps the critical point $O(0, 0)$ to $X = Y = 0$.

Applying this transformation to system (7) we get a quartic system

$$\dot{X} = \sum_{i+j=0}^{4} A_{ij} X^i Y^j, \quad Y = \sum_{i+j=0}^{4} B_{ij} X^i Y^j, \tag{15}$$

where A_{ij}, B_{ij} are polynomials that depend on both the coefficients of system (1) and the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ from the mapping (14).

We will show that the parameters in (14) can be found such that the system (15) is equivalent, in some neighborhood of $O(0,0)$, with a polynomial system [6]

$$\dot{X} = Y + M(X^2, Y), \quad \dot{Y} = -X(1 + N(X^2, Y)). \tag{16}$$

This system is symmetric with respect to the axis $X = 0$ and the critical point $O(0,0)$ is a center. The systems (15) and (16) are equivalent if the following conditions are fulfilled:

$$B_{40} = 0,\ A_{13} \equiv B_{04} = 0,\ A_{31} \equiv B_{22} = 0,\ A_{10} \equiv B_{01} = 0,\ A_{00} = B_{00} = 0,$$

and

$$\begin{aligned}
A_{30} \equiv\ & 2aa_3b_2a_1^2 + [2a_3(c-g) - (c+s+1)a_1 - a_2(q+f)]b_2a_2a_1 + \\
& + a_2^3(lb_1 - nb_2) - ab_2a_1^3 + 2a_2^2a_3(bb_1 + (f-d)b_2) = 0, \\
A_{12} \equiv\ & b_1^3(qa_2 + 2a_3g) + [2(d+a)a_3 + a_2(2n - c - 3s - 1)]b_2b_1^2 + \\
& + [a_2(3l - 2f + 3a - 2q) + 2a_3(c+b)]b_2^2b_1 + \\
& + [2fa_3 - fa_1 - (n - 2c - 2)a_2]b_2^3 = 0, \\
A_{11} \equiv\ & [db_1 + b_2(c + 2b - 2g)]b_1a_2 + \\
& + 3a_3 + b_2^2[ca_1 - a_2(d - 2f + 2a)] = 0, \\
A_{01} \equiv\ & b_2^2 + b_1^2 - 1 = 0,\ A_{10} \equiv b_2a_2 + b_1a_1 = 0, \\
B_{04} \equiv\ & [sb_1^4 + b_2b_1(b_1^2(q-a) + b_2b_1(n-c-1) + b_2^2(l-f))]a_3 = 0, \\
B_{22} \equiv\ & [a_2b_2^2a_1(-2f + 3a - q) + da_2b_1^2a_3 + na_2^2b_1^2 + ca_1b_2^2a_3 + \\
& + (3l - f - 2q)a_2^2b_1b_2 + (3s - 2(n - c - 1))a_2^2b_2^2 - (c+1)b_2^2a_1^2 + \\
& + (c + 2b - 2g)a_2a_3b_1b_2 - (d + 2a - 2f)a_2b_2^2a_3 + a_3^2]a_3 = 0, \\
B_{03} \equiv\ & (-aa_2 - ga_3)b_1^3 - [(c+s+1)a_2 + (d+a)a_3]b_1^2b_2 + \\
& + [(-f-q)a_2 - (c+b)a_3]b_1b_2^2 + [la_1 - na_2 - fa_3]b_2^3 = 0, \\
B_{21} \equiv\ & -fa_1^3b_2 + (2n - c - 3s - 1)a_1^2a_2b_2 + (d-a)a_1^2a_3b_2 - \\
& - (2q - 3a - 3l + 2f)b_2a_2^2a_1 + [-fb_1 - (n + 2c + 2)b_2]a_2^3 + \\
& + b_2a_3a_2a_1(2b - g) + [b_2(f - 2a) + b_1(c - b)]a_3a_2^2 = 0, \\
B_{02} \equiv\ & ab_1^2a_2 - (g-c)b_2a_2b_1 - b_2^2(da_2 - ba_1 - fa_2) - a_3 = 0, \\
B_{20} \equiv\ & 2a_3 + ga_1^3 + fa_2^3 + (d+a)a_2a_1^2 + (c+b)a_2^2a_1 = 0, \\
B_{10} \equiv\ & a_2^2 + a_1^2 - 1 = 0.
\end{aligned} \tag{17}$$

Theorem 5.1. *The cubic differential system* (1) *with two algebraic solutions* $x-1=0$, $a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1=0$ *is rationally reversible if one of the following conditions* (i), (ii), (iii) *holds:*

(i) $c=-3/2,\ b=m=1/2,\ g=-1,\ f=l=p=r=0,\ k=-a,\ q=(a-d)/2$, $s=-(2a+a_{01})(a_{01}^2-da_{01}+n)/(2a_{01})$, $2na_{01}^2-(a+2dn)a_{01}+2n^2+n)=0$;

(ii) $c=-3/2,\ b=m=1/2,\ g=-1,\ l=f/2,\ k=-a,\ p=-f,\ r=0$, $d=[(8a_{02}+1)a_{01}]/(8a_{02})$, $q=(a-d)/2$, $n=(fa_{01}+8a_{02}^2-2a_{02})/(8a_{02})$, $s=[(2a+a_{01})a_{01}]/(16a_{02})$;

(iii) $c=-3/2,\ b=m=1/2,\ g=-1,\ l=f/2,\ k=-a,\ p=-f,\ q=(a-d)/2,\ a=[(a_{01}^2-da_{01}+2a_{20})(4a_{20}-1)a_{01}]/(a_{01}^2-16a_{02}a_{20}),\ f=[4(d-a_{01})a_{01}^2a_{02}+2a_{01}a_{02}(8a_{02}-4a_{20}+1)-16da_{02}^2]/(a_{01}^2-16a_{02}a_{20}),\ n=[(d-a_{01}^3)a_{01}+a_{01}^2(8a_{02}a_{20}+3a_{02}-2a_{20})-2da_{01}a_{02}(4a_{20}+1)+4a_{02}a_{20}(4a_{20}-4a_{02}+1)]/(a_{01}^2-16a_{02}a_{20}),\ s=[a_{01}^2a_{20}(8a_{20}-1)+2da_{01}a_{20}(1-4a_{20})+4a_{20}^2(4a_{20}-4a_{02}-1)]/(a_{01}^2-16a_{02}a_{20}),\ r=0$.

Proof. We study the consistency of systems $\{(17), (11)\}$ considering two cases: $a_3=0$ and $a_3\neq 0$. According to [6], the equations $A_{01}=0$ and $B_{10}=0$ from (17) can be parametrized as follows:

$$a_1=\frac{2u}{u^2+1},\ a_2=\frac{u^2-1}{u^2+1},\ b_1=\frac{2v}{v^2+1},\ b_2=\frac{v^2-1}{v^2+1}, \tag{18}$$

where u and v are real parameters. Then $A_{10}=0$ becomes $A_{10}\equiv e_1e_2=0$, where

$$e_1=u-v+uv+1,\ e_2=v-u+uv+1.$$

Assume that $e_1=0$. Then the equation $e_1=0$ yields $v=(1+u)/(1-u)$ and $A_{10}\equiv 0$.

1. Let $a_3=0$. Then $B_{04}\equiv 0$ and $B_{22}\equiv 0$. When $u=0$, the equations of (17) yield

$$r=q=p=l=k=f=d=a=0,\ m=-1-c.$$

In this case, the cubic system has two parallel invariant straight lines $1-x=0$, $1+(c+1)x=0$ and the center-focus problem was solved in [5], [25].

When $u=-1$, the equations of (17) imply

$$r=q=p=l=k=g=f=c=b=a=0,\ m=-1.$$

The cubic system has the invariant straight lines $1-x=0$, $1+x=0$ and center-focus problem was solved in [5], [25].

If $u(u+1)\neq 0$, then the equations of (17) yield

$r=0, p=-f, m=-c-1, k=-a, a=[bu(20u^2-6u^4-6)+(f-d)(u^2(u^4-7u^2+7)-1)]/[2(1-u^2)^3], c=[2b(6u^3-u^5-u)-f(u^6-1)+(4d-7f)(u^2-u^4)]/[2u(u^2-1)^2], g=[(f+d)(1-u^6)+(f-7d)(u^2-u^4)+b(2u^5+2u-$

$12u^3)]/[4u(1-u^2)^2], n = [f(1-u^{14})-4(b-1)(u+u^{13})-(15f+4d)(u^2-u^{12})+8(4b-11)(u^3+u^{11})-3(15f-4d)(u^4-u^{10})+4(95-7b)(u^5+u^9)+(61f-8d)(u^6-u^8)-16u^7(37+8b)]/[2u(1+u^2)^4(u^2-1)^2], l = [f(u^8-10u^4-4u^2-4u^6+1)+(7+b)(4u^5-4u^3)+(1-b)(4u-4u^7)]/[(1+u^2)^4], q = [6(124-15b)(u^5+u^9)-6(b-4)(u^{13}+u-10u^3-10u^{11})-24(44+13b)u^7-(31f+9d)(u^2-u^{12})+3(23f-15d)(u^4-u^{10})+(3d+5f)(1-u^{14})+3(35f-11d)(u^6-u^8)]/[2(u^2+1)^4(u^2-1)^3],$ $s = [(f+d)(9u^2+u^{18}-9u^{16}-1)-2(2+b)(u+u^{17})+64(u^3+u^{15})+4(9f+d)(u^{14}-u^4)+8(13b-46)(u^5+u^{13})+4(21f-19d)(u^6-u^{12})-64(2b-15)(u^7+u^{11})+2(65f-31d)(u^8-u^{10})-4u^9(326+115b)]/[4u(1-u^4)^4].$

In this case the cubic system possesses two invariant straight lines $1-x=0$, $(1+u^2)^2-(1+u^4-6u^2)x+4(u^3-u)y=0$ and center-focus problem was solved in [6].

2. Let $a_3 \neq 0$. Then from the equation $B_{20}=0$ of (17) we get

$$a_3 = [u^2(3f-4d-4a)(u^2-1)-2u(c+b)(u^4+1)-f(u^6-1)+ \\ +4(c+b-2g)u^3]/[2(1+u^2)^3].$$

Assume that $u=0$. If $a=0$, then the equations of (17) yield

$$s=r=q=p=n=l=k=d=a=0,\ m=-c-1.$$

The cubic system has the invariant straight lines $1-x=0$, $1+(c+1)x=0$ and center-focus problem was solved in [5], [25].

Assume that $u=0$ and let $a\neq 0$. Then the equations of (17) yield

$$d=-3a,\ f=-2a,\ c=b-2,\ g=-1,\ l=-2ab,\ k=-a,\ m=1-b,\ n=2a^2, \\ p=q=2a,\ r=s=0.$$

The cubic system has three invariant straight lines $1-x=0$, $1-2ay=0$, $1-x-2ay=0$ and center-focus problem was solved in [26].

Assume that $u=-1$. If $g=-1$, then from the equations of (17) we get

$$c=-3/2,\ b=m=1/2,\ g=-1,\ l=f/2,\ k=-a,\ p=-f,\ q=(a-d)/2,\ r=0.$$

The equation $F_{04}=0$ of (11) implies two cases to be investigated: $a_{02}=0$ and $a_{02}\neq 0$.

Let $a_{02}=0$. If $a_{11}=0$, then $a_{01}a_{20}\neq 0$ and $F_{03}\equiv(a_{01}-f)(2a_{10}+1)=0$. When $a_{01}=f$, the equations $F_{22}\neq 0$ and when $a_{10}=(-1)/2$, we obtain that $F_{21}\neq 0$.

Assume that $a_{11}\neq 0$. We express s, n and a_{11} from the equations $F_{40}=0$, $F_{22}=0$ and $F_{13}=0$, respectively. In this case we have $F_{03}\equiv i_1i_2i_3=0$, where

$$i_1=a_{01}-f,\ i_2=2a_{10}+3,\ i_3=f.$$

If $i_1=0$, then $a_{01}=f$ and $F_{12}=0$ yields $a_{10}=-1-a_{20}$. In this case the conic is reducible.

If $i_1 \neq 0$ and $i_2 = 0$, then $a_{10} = (-3)/2$ and $F_{21} = 0$ implies $a_{20} = 1/2$. In this case the conic is also reducible.

If $i_1 i_2 \neq 0$ and $i_3 = 0$, then $f = 0$ and $F_{12} = 0$ implies $a_{10} = -1$. In this case we obtain the set of conditions (i) for the existence of an invariant conic

$$(a_{01}^2 - da_{01} + n)x^2 + a_{01}xy + 2(x - a_{01}y - 1) = 0.$$

Let now $a_{02} \neq 0$. We express a_{11}, s and n from equations $F_{04} = 0$, $F_{31} = 0$ and $F_{13} = 0$ of (11), respectively. Then $F_{03} \equiv j_1 j_2 = 0$, where $j_1 = a_{10} + 1$, $j_2 = fa_{01} - f^2 - a_{02}$.

Assume that $j_1 = 0$, then $a_{10} = -1$. If $a_{20} = a_{01}^2/(16a_{02})$, then $F_{22} = F_{40} = 0$ yields

$$d = [(8a_{02} + 1)a_{01}]/(8a_{02}).$$

In this case we get the set of conditions (ii) for the existence of an invariant conic

$$(a_{01}x - 4a_{02}y)^2 - 16a_{02}x + 16a_{01}a_{02}y + 16a_{02} = 0.$$

If $a_{20} \neq a_{01}^2/(16a_{02})$, then we express a and f from the equations $F_{40} = F_{30} = 0$ and $F_{12} = F_{22} = 0$ of (11). In this case we have the set of conditions (iii) for the existence of an invariant conic

$$2a_{20}x^2 - a_{01}xy + 2a_{02}y^2 - 2x + 2a_{01}y + 2 = 0.$$

Assume that $j_1 \neq 0$ and let $j_2 = 0$. Then $a_{10} = fa_{01} - f^2$ and $F_{21} = 0$ yields

$$d = (6a_{20} - 2a_{10}^2 - a_{10} + 4f^2 + 2af)/(2f).$$

If $a_{01} = 2f$, then $F_{22} = 0$ implies $a_{20} = a_{10}^2/4$ and the conic is reducible.

Let $a_{01} \neq 2f$. Then we express a from $F_{12} = 0$ and $F_{22} = 0$ yields $a_{20} = (-2a_{10} - 1)/4$. In this case the conic is also reducible.

Assume that $u = 0$ and let $g \neq -1$. Then the equations of (17) yield

$$r = p = l = k = f = a = 0, \; q = -d, \; m = 2, \; g = -2, \; c = -3, \; b = 1.$$

In this case the cubic system has the invariant straight lines $1 - x = 0$, $1 - 2x = 0$ and center-focus problem was solved in [5], [25].

Assume now $u(u+1) \neq 0$. We express a, s, l, g, n, q, from the equations $A_{11} = 0$, $A_{12} = 0$, $A_{30} = 0$, $B_{02} = 0$, $B_{04} = 0$, $B_{21} = 0$ of (17), respectively. In this case we obtain that $B_{03} \equiv hf_1 = 0$, $B_{22} \equiv hf_2 = 0$, where

$$\begin{aligned}
h &= (7f - 4d)(u^2 - u^4) + f(u^6 - 1) + 2(c + b + 2)(u + u^5) - 4(c + 3b)u^3,\\
f_1 &= (2d - 3f)(1 + 14u^4 + u^8) - 2(11c + 25b + 4)(u^3 - u^5) +\\
&\quad + 2(5c + 7b + 4)(u - u^7) - 8(2d - 5f)(u^2 + u^6),\\
f_2 &= 2(b - 2 - c)(1 + u^8) + (2d - 15f)(u - u^7) - 8(6c + 16b + 1)u^4 +\\
&\quad + 2(11c + 17b + 8)(u^2 + u^6) + (81f - 46d)(u^3 - u^5).
\end{aligned}$$

If $h = 0$, then the equation of (17) imply

$r = 0$, $p = -f$, $m = -c - 1$, $k = -a$, $a = [f(1 + 7u^4 - 7u^2 - u^6) - 2(c + b + 2)(u+u^5)+4(3c-b+10)u^3]/[8(u^2-1)u^2]$, $d = 2a+[2(c+5-2b)u]/(1-u^2)$, $g = [f(15u^2-15u^4+u^6-1)+2u(2+c+b)(1+u^4)-4(3c+3b+14)u^3]/(16u^3)$, $l = [4bu(1 - u^6) + 8f(u^2 + 2u^4 + u^6) + 4(b - 8)(u^3 - u^5)]/(1 + u^2)^4$, $n = [f(u^{12} - 8u^{10} + 32u^6 + 7u^8 + 7u^4 + 1 - 8u^2) - 2(c + b + 2)u - u^{11}) - 6(c - 3b - 2)(u^3-u^9)-4(c-5b+44)(u^5-u^7)]/[u(u^2-1)(1+u^2)^4]$, $q = [f(1+19u^{12}-19u^2+33u^4-33u^{10}+53u^6-53u^8-u^{14})+2(33c-46-47b)(u^5+u^9)+4(3c+5b+16)(u^3+u^{11})-2(c+b+2)(u+u^{13})+8u^7(13c+152-29b)]/[4(u^2-1)(1+u^2)^4u^2]$, $s = [f(1+u^4-19u^2+21u^6-u^{10}-21u^8+19u^{12}-u^{14})+2(17c-62-15b)(u^5+u^9)+4(c+5b+14)(u^3+u^{11})-2(c+b+2)(u+u^{13})+8u^7(7c+82-13b)]/[4u(1-u^2)^2(1+u^2)^4]$.

We have two invariant straight lines $1 - x = 0$, $(1 + u^2)^2 - 8u^2x - 4(u^2 - 1)uy = 0$ and center-focus problem was solved in [6].

Assume that $h \neq 0$. We find the resultant of the polynomials f_1, f_2 with respect to d and obtain that $Res(f_1, f_2, d) = 0$, if

$b = [(c+2)(u^6-1)-f(6u^5-52u^3+6u)+(15c+22)u^2(1-u^2)]/[(1+u^2)^2(u^2-1)]$.

In this case we express d from the equations $B_{03} \equiv B_{22} = 0$ and the equations of (17) yield

$r = 0$, $p = -f$, $m = -c - 1$,, $k = -a$, $a = [(6u^2 - u^4 - 1)f]/[2(1 - u^2)^2]$, $b = [(2+c)(u^6-1)-f(6u-52u^3+6u^5)+(22+15c)(u^2-u^4)]/[(1+u^2)^2(u^2-1)]$, $d = [f(3 + 3u^8 - 100u^2 - 100u^6 + 306u^4) + (3 + 2c)(12u^7 + 52u^3 - 52u^5 - 12u)]/[2(1-u^4)^2]$, $g = [1-u^6+2f(u-30u^3+u^5)+(23+16c)(u^4-u^2)]/[(1+u^2)^2(u^2-1)]$, $l = [(2+c)(u^6+7u^2-7u^4-1)+fu(20u^2-6u^4-6)][f(1-14u^2+u^4)+4(1+c)(u^3-u)]/[(1+u^2)^4(u^2-1)]$, $n = [4f(9+5c)(u^{11}-u)+2(32c^2+104c+72-49f^2)(u^2+u^{10})+92f(11+7c)(u^3-u^9)+(1391f^2-1152c-384c^2-768)(u^4+u^8)+8f(445+301c)(u^7-u^5)+4(160c^2+472c-791f^2+312)u^6+f^2(u^{12}+1)]/[2(1-u^2)^2(1+u^2)^4]$, $q = [4f(567+394c)u^6+2(87+127c+40c^2-61f^2)(u^9-u^3)-f(1103+720c)(u^4+u^8)+4(353f^2-223c-76c^2-147)(u^7-u^5)+2(f^2-3c-3)(u^{11}-u)-f(1+u^{12})+2f(33+14c)(u^2+u^{10})]/[(1-u^2)^2(1+u^2)^4]$, $s = u[f(1 - 36u^2 + 54u^4 - 36u^6 + u^8) + 8(1 + c)(u^7 + 3u^3 - 3u^5 - u)][f(2u^5 + 2u - 28u^3) - (15 + 8c)(u^2 - u^4) - u^6 + 1)]/[(1 - u^4)^4]$.

The cubic system has three invariant straight lines $1 - x = 0$, $(fu^4 + 4cu^3 - 14fu^2 - 4cu + f)(2ux + u^2y - y) + (u^4x - 14u^2x + x - 8u^3y + 8uy - u^4 - 2u^2 - 1)(1 - u^2) = 0$, $(fu^4-14fu^2+4cu^3-4cu+f)(2ux+u^2y-y)+(8u^2x+4u^3y-4uy+(1+u^2)^2)(u^2-1) = 0$ and center-focus problem was solved in [26].

Assume that $e_2 = 0$. It is easy to see that $e_2(u, v) = e_1(-u, -v)$ and the case $e_2 = 0$ is equivalent to the case $e_1 = 0$. ∎

Theorem 5.2. *The critical point $O(0, 0)$ is a center for a cubic differential system* (1)*, with two algebraic solutions $x - 1 = 0$, $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$, if one of the conditions* (i), (ii), (iii) *is satisfied.*

The proof of Theorem 5.2 follows directly from Theorem 5.1, if the cubic system (1) is rationally reversible, then the critical point $O(0, 0)$ is a center [23], [28].

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