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On a quasigroup prolongation and its recursive differentiability

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Abstract. We consider a method of quasigroups prolongation by adding two new elements and by using two transversals that intersect in exactly one cell. In a Latin square L of order q, along with usual transversals we call "free transversals" any set of q cells taken by one from each row and each column of L. It is shown that a Latin square of order q can have at most $q-2$ (ordinary) transversals that intersect exactly in one cell, which is also the only common cell of every two transversals. It is proved that a latin square of order 4 has exactly 24 free transversals, 576 (ordinary) transversals, 96 pairs of free transversals that intersect exactly in one cell and 13824 pairs of ordinary transversals that intersect exactly in one cell. We also investigate the recursive 1-differentiability of considered prolongations of Latin squares of order 5, one of the transversals being on the main diagonal. The research done in this paper is applicable in cryptography and is important for data integrity and fault tolerance.
2020 Mathematics Subject Classification: 20N05, 05B15.
Keywords: quasigroup, transversal, free transversal, prolongation, recursively differentiable quasigroup, recursive derivative, Latin square.

Despre prelungirea quasigrupului şi derivabilitatea sa recursivă

Rezumat. Este considerată o metodă de prelungire a quasigrupurilor prin adjuncţia a două elemente noi şi utilizarea a două transversale care se intersectează într-o singură celulă. Într-un pătrat latin L de ordinul q, împreună cu transversalele obişnuite, sunt utilizate "transversale libere" - seturi de q celule luate câte una din fiecare linie şi fiecare coloană a lui L. Se demonstrează că un pătrat latin de ordinul q poate avea cel mult $q-2$ transversale (ordinare) ce se intersectează într-o singură celulă, care este şi singura celulă comună a fiecăror două transversale. Se demonstrează că un pătrat latin de ordinul 4 are exact 24 de transversale libere, 576 de transversale (ordinare), 96 de perechi de transversale libere care se intersectează exact într-o celulă şi 13824 de perechi de transversale ordinare care se intersectează exact într-o celulă. De asemenea, investigăm 1-derivabilitatea recursivă a prelungirilor considerate ale pătratelor latine de ordinul 5, una dintre transversale fiind pe diagonala principală. Cercetarea efectuată în această lucrare are aplicări în criptografie şi este importantă pentru integritatea datelor şi toleranţa la erori.
Cuvinte-cheie: quasigrup, transversală, transversală liberă, prelungire, quasigrup recursiv derivabil, derivată recursivă, pătrat latin.

1. Introduction

The concept of extending quasigroups was first examined by Bruck in 1944 for finite idempotent quasigroups [1]. For analogous constructions the term "prolongation" was introduced by Belousov in 1967 [2]. By "prolongation" of a finite quasigroup we mean a process of extending the quasigroup by adding one or more new elements and redefining the operation to obtain a new quasigroup of higher order.

Other methods of prolongations or modifications of the known ones have been proposed by Osborn, Yamamoto, Denes and Pasztor, Belousov and Belyavskaya, Belyavskaya, Deriyenko and Dudek, and others [3].

A method of quasigroups prolongation by adding two new elements and by using two transversals that intersect exactly in one cell is considered in the present work.

Estimations of the number of prolongations, using the proposed method, are given. One of our purposes is to study the recursive differentiability of the considered prolongations. In a Latin square L of order q, along with usual transversals, we call a "free transversal" any set of q cells taken by one from each row and each column of L. We show that a Latin square of order q can have at most $q-2$ (ordinary) transversals that intersect exactly in one cell, which is also the only common cell of every two transversals. We use the considered method of quasigroups prolongation mainly for Latin squares of order 4 and 5 (there does not exist Latin squares of order ≤ 3 which have two transversals intersecting exactly in one cell). It is proved that a Latin square of order 4 has exactly 24 free transversals, 576 (ordinary) transversals, 96 pairs of free transversals that intersect exactly in one cell, and 13824 pairs of ordinary transversals that intersect exactly in one cell. It is known that there does not exist recursively differentiable quasigroups of order 6, due to orthogonality of the recursive derivatives and the non existence of orthogonal Latin squares of order 6. Hence, the prolongations of Latin squares of order 4, by adding two new elements, are not recursively 1-differentiable. In the case of Latin squares of order 5 we prove that:

1. There exist exactly 48 different Latin squares of order 5 with two transversals intersecting in one cell, where one transversal is on the main diagonal.

2. The prolongations by adding two new elements and using two new transversals, that intersect exactly in one cell, one of which is on the main diagonal, and have the order of elements $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 1\}$, are not recursively 1-differentiable.

The (ordinary) transversal of a Latin square of order n is a set of n cells taken by one from each row and each column, in which the elements are pairwise different. We call a free transversal of a Latin square L of order n any set of n cells taken exactly one from each row and each column of L.

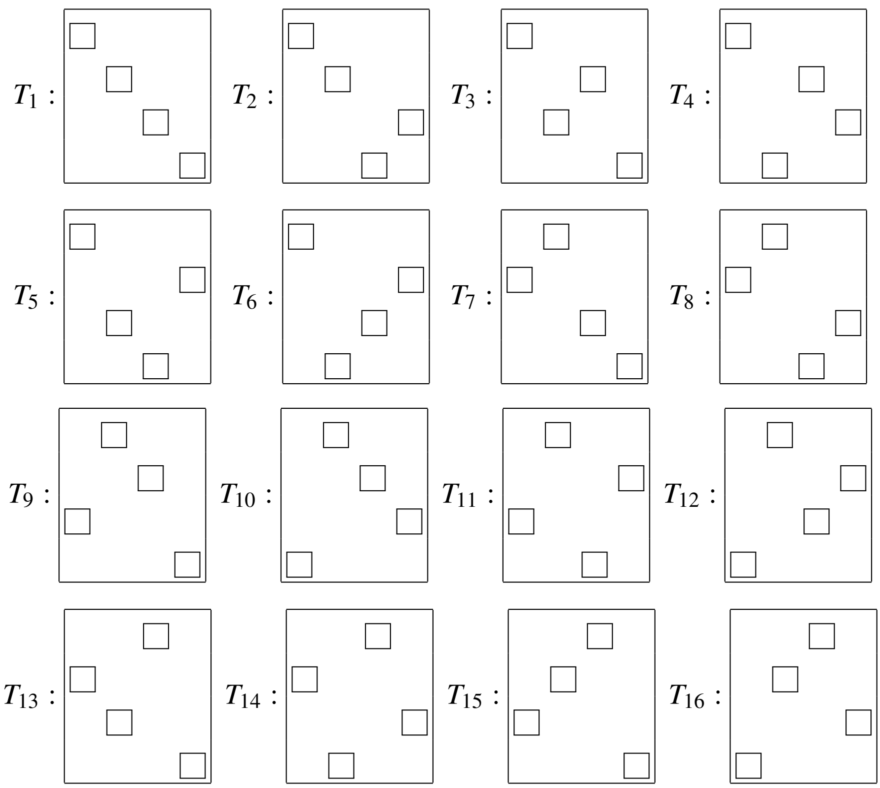
Proposition 1.1. *A Latin square of order n can have at most $n-2$ ordinary transversals that intersect exactly in one cell, if this is also the only common cell of every two transversals.*

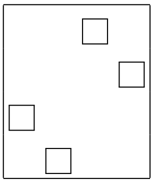
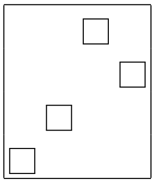
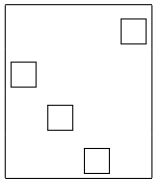
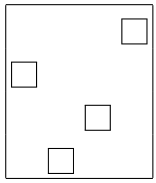
Proof. Let L be a Latin square of order n with the elements $1, 2, ..., n$ in its cells which has $n-2$ ordinary transversals that intersect exactly in one cell, which is also the only common cell of every two transversals. Let a be the element in the common cell of the $n-2$ transversals of the Latin square of order n. If L would have one more transversal with the same common cell, then this transversal must intersect a line which does not contain the common cell or in the cell with the element a or in the cell from the column (row) with the common cell, but both cases are impossible as the transversal already contains the element a in the common cell with the other $n-2$ transversals. $\square$

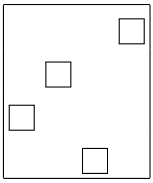
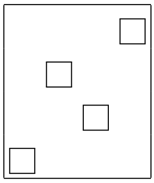
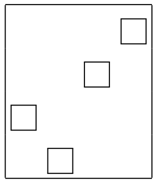
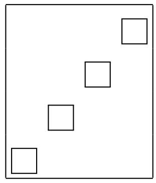
Corollary 1.1. *There does not exist Latin squares of order 3 with 2 transversals that intersect exactly in one cell.*

Proposition 1.2. *The Latin square of order four has exactly 24 free transversals.*

The proof follows immediately from the possibility of selecting cells from each row (column). These free transversals are given bellow:



T_{17} :  T_{18} :  T_{19} :  T_{20} : 

T_{21} :  T_{22} :  T_{23} :  T_{24} : 

Corollary 1.2. *Any Latin square of order four has exactly 576 (ordinary) transversals.*

Proposition 1.3. *The total number of pairs of free transversal that intersect exactly in one cell is 96.*

Proof. The proof follows from Proposition 1.2. The total number of pairs of free transversals that intersect exactly in one cell is 96 and these pairs are given bellow:

$$(T_1,T_4),(T_1,T_5),(T_2,T_3),(T_2,T_6),(T_3,T_6),(T_4,T_5),(T_7,T_{10}),(T_7,T_{11}),$$
$$(T_8,T_9),(T_8,T_{12}),(T_9,T_{12}),(T_{10},T_{11}),(T_{13},T_{16}),(T_{13},T_{17}),(T_{14},T_{15}),$$
$$(T_{14},T_{18}),(T_{15},T_{18}),(T_{16},T_{17}),(T_{19},T_{22}),(T_{19},T_{23}),(T_{20},T_{21}),$$
$$(T_{20},T_{24}),(T_{21},T_{24}),(T_{22},T_{23}),(T_7,T_{14}),(T_7,T_{19}),(T_8,T_{13}),(T_8,T_{20}),$$
$$(T_{13},T_{20}),(T_{14},T_{19}),(T_1,T_{16}),(T_1,T_{21}),(T_2,T_{15}),(T_2,T_{22}),(T_{15},T_{22}),$$
$$(T_{16},T_{21}),(T_3,T_{10}),(T_3,T_{23}),(T_4,T_9),(T_4,T_{24}),(T_9,T_{24}),(T_{10},T_{23}),$$
$$(T_5,T_{12}),(T_5,T_{17}),(T_6,T_{11}),(T_6,T_{18}),(T_{11},T_{18}),(T_{12},T_{17}),(T_9,T_{21}),$$
$$(T_9,T_{17}),(T_{11},T_{15}),(T_{11},T_{23}),(T_{15},T_{23}),(T_{17},T_{21}),(T_3,T_{18}),(T_3,T_{19}),$$
$$(T_5,T_{13}),(T_5,T_{24}),(T_{13},T_{24}),(T_{18},T_{19}),(T_1,T_{12}),(T_1,T_{20}),(T_6,T_7),$$
$$(T_6,T_{22}),(T_7,T_{22}),(T_{12},T_{20}),(T_2,T_{10}),(T_2,T_{14}),(T_4,T_8),(T_4,T_{16}),$$
$$(T_8,T_{16}),(T_{10},T_{14}),(T_{10},T_{18}),(T_{10},T_{22}),(T_{12},T_{16}),(T_{12},T_{24}),$$
$$(T_{16},T_{24}),(T_{18},T_{22}),(T_4,T_{17}),(T_4,T_{20}),(T_6,T_{14}),(T_6,T_{23}),(T_{14},T_{23}),$$
$$(T_{17},T_{20}),(T_2,T_{11}),(T_2,T_{19}),(T_5,T_8),(T_5,T_{21}),(T_8,T_{21}),(T_{11},T_{19}),$$
$$(T_1,T_9),(T_1,T_{13}),(T_3,T_7),(T_3,T_{15}),(T_7,T_{15}),(T_9,T_{13})$$

□

Corollary 1.3. *In any Latin square of order four there exist exactly 13824 pairs of ordinary transversals that intersect exactly in one cell.*

Proof. By fixing one of two transversals, we can fill its cells in 4! ways. Then the cells of the second transversal (which intersects the first in exactly one cell) can be filled in

3! ways. Therefore, there are a total of $96 \cdot 4! \cdot 3! = 13824$ pairs of regular transversals, which have only one cell in common. □

Let $(Q, \cdot)$ be a quasigroup. The recursive derivative of order k of the quasigroup $(Q, \cdot)$, denoted by $(\overset{k}{\cdot})$, $k \in N$, is defined as follows:

$$x \overset{0}{\cdot} y = x \cdot y; \; x \overset{1}{\cdot} y = y \cdot (x \cdot y);$$
$$x \overset{k}{\cdot} y = (x \overset{k-2}{\cdot} y) \cdot (x \overset{k-1}{\cdot} y), for\ all\ k \geq 2.$$

A binary quasigroup $(Q, \cdot)$ is recursively r-differentiable ($r \geq 0$) if each of its recursive derivatives $(\overset{k}{\cdot})$, where $k = 0, ..., r$, is a quasigroup operation.

Known estimations of the order of recursive differentiability of finite quasigroups are given in [4-7]. In particular, it is known that:

1. The maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q - 2$;
2. There exist finite binary quasigroups that are recursively $(q - 2)$ – differentiable for any primary order $q \geq 3$.

2. Methods

The method of construction of quasigroups prolongations by adding two new elements and by using two transversals that intersect exactly in one cell and the recursive differentiability of such prolongations for some particular transversals in Latin squares of order 5 were announced [8].

We examine the recursive derivability in a general case of such prolongations and study the recursive 1-differentiability of Latin squares of order 5 with a fixed transversal on the main diagonal bellow.

Let $(Q, \cdot)$ be a finite quasigroup of order q and let the Cayley table of the quasigroup $(Q, \cdot)$ contain 2 transversals T_1 and T_2, which have a single common cell, located at the intersection of the row of element x with the column of element y. We denote the element in the common cell of transversals T_1 and T_2 by a. We now consider the set $Q' = Q \cup \{\xi_1, \xi_2\}$, where $\xi_1, \xi_2 \notin Q$. On the set Q' we define the operation $\circ$ as follows: we complete the cells, different from the common cell, of one transversal with ξ_1, and of the other with ξ_2. We transfer the elements in the cells of the two transversals, except for the element in the common cell, preserving their order, into the cells of the row and, respectively, of the column of elements ξ_1 and ξ_2. The remaining cells in the row of element x, the column of element y, and at the intersection of the rows and columns of elements ξ_1 and ξ_2 can be filled in several possible ways, but following this method, there are exactly 12 possibilities for extending the quasigroup Q by adjointing two new elements and using 2 transversals that intersect exactly in a cell. These possibilities are illustrated in the diagrams below:

$$
I.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\
\xi_2 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a
\end{array}
\qquad
II.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\
\xi_2 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1
\end{array}
$$

$$
III.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\
\xi_2 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2
\end{array}
\qquad
IV.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \\
\xi_2 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1
\end{array}
$$

$$
V.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \\
\xi_2 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a
\end{array}
\qquad
VI.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\
\xi_2 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2
\end{array}
$$

$$
VII.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\
\xi_2 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1
\end{array}
\qquad
VIII.\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\
\xi_2 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2
\end{array}
$$

IX.

$\circ$	$\dots$	$\dots$	y_0	$\dots$	$\dots$	ξ_1	ξ_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_0	$\dots$	$\dots$	a	$\dots$	$\dots$	ξ_1	ξ_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
ξ_1	$\dots$	$\dots$	ξ_1	$\dots$	$\dots$	ξ_2	a
ξ_2	$\dots$	$\dots$	ξ_2	$\dots$	$\dots$	a	ξ_1

X.

$\circ$	$\dots$	$\dots$	y_0	$\dots$	$\dots$	ξ_1	ξ_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_0	$\dots$	$\dots$	a	$\dots$	$\dots$	ξ_1	ξ_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
ξ_1	$\dots$	$\dots$	ξ_2	$\dots$	$\dots$	a	ξ_1
ξ_2	$\dots$	$\dots$	ξ_1	$\dots$	$\dots$	ξ_2	a

XI.

$\circ$	$\dots$	$\dots$	y_0	$\dots$	$\dots$	ξ_1	ξ_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_0	$\dots$	$\dots$	a	$\dots$	$\dots$	ξ_2	ξ_1
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
ξ_1	$\dots$	$\dots$	ξ_2	$\dots$	$\dots$	ξ_1	a
ξ_2	$\dots$	$\dots$	ξ_1	$\dots$	$\dots$	a	ξ_2

XII.

$\circ$	$\dots$	$\dots$	y_0	$\dots$	$\dots$	ξ_1	ξ_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_0	$\dots$	$\dots$	a	$\dots$	$\dots$	ξ_2	ξ_1
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
ξ_1	$\dots$	$\dots$	ξ_1	$\dots$	$\dots$	a	ξ_2
ξ_2	$\dots$	$\dots$	ξ_2	$\dots$	$\dots$	ξ_1	a

As an example we consider a quasigroup of order 5 with two transversals – one marked in red and one marked in blue – that meet at position $(1,1)$. The prolongation by adding two new elements ξ_1 and ξ_2 and using the transversals $T_1 = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$ and $T_2 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$ is given in the second Cayley table bellow:

$\cdot$	1	2	3	4	5
1	2	4	1	3	5
2	1	3	5	2	4
3	5	2	4	1	3
4	4	1	3	5	2
5	3	5	2	4	1

$\longrightarrow$

$\circ$	1	2	3	4	5	ξ_1	ξ_2
1	ξ_1	4	1	3	5	2	ξ_2
2	1	ξ_1	ξ_2	2	4	3	5
3	5	2	ξ_1	1	ξ_2	4	3
4	4	ξ_2	3	ξ_1	2	5	1
5	3	5	2	ξ_2	ξ_1	1	4
ξ_1	ξ_2	1	5	4	3	ξ_1	2
ξ_2	2	3	4	5	1	ξ_2	ξ_1

Proposition 2.1. *Six extensions out of the 12 possible (Cases I, IV, VI, VII, X, XI), by the adjunction of two new elements and using two transversals that intersect exactly in a cell, are not recursively 1-differentiable.*

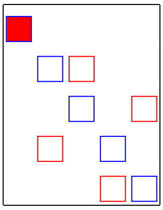
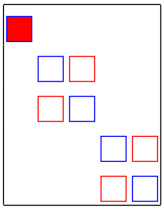
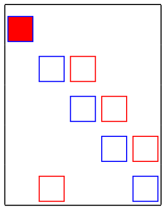
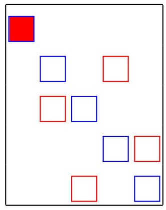
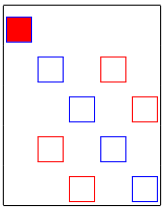
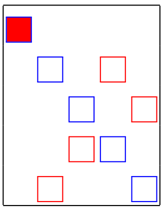
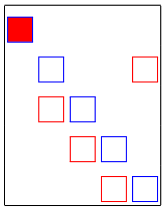
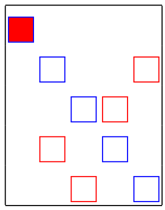
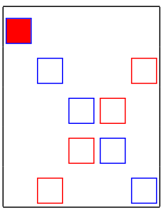
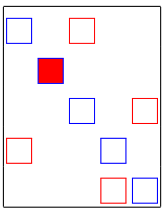
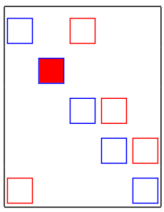
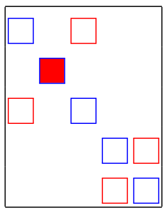
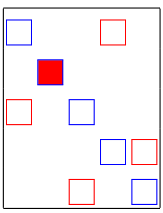
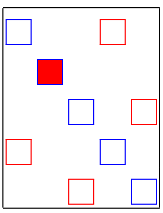
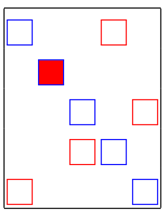
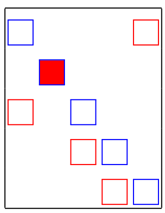
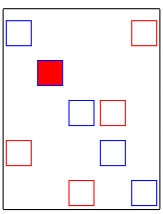
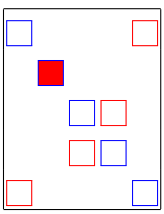
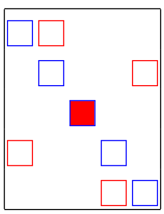
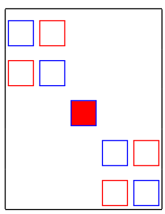
Proof. In the mentioned cases the recursive derivative $\overset{1}{\circ}$ is not a quasigroup operation as it is shown bellow:

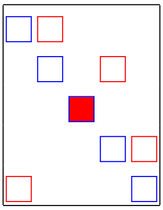
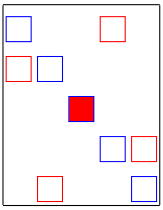
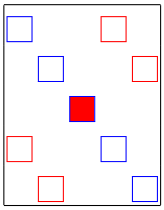
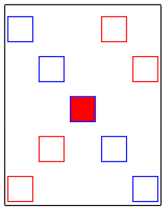
I. $\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_1$, $\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_1$.

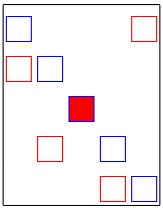
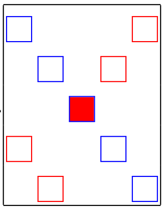
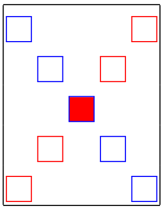
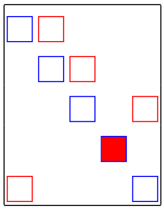
IV. $\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_1$, $\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_1$.
VI. $\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_2$, $\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_2$.
VII. $\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_2$, $\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_2$.
X. $x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_1 = a$, $x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_2 = a$.
XI. $x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_2 = a$, $x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_1 = a$. □

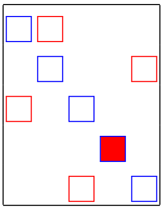
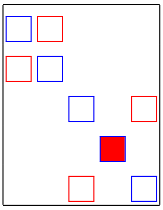
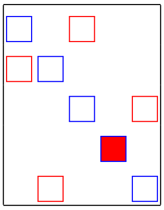
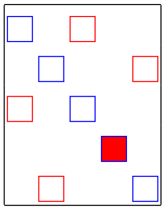
Proposition 2.2. *In a Latin square of order 5, the total number of pairs of free transversals, one of which is on the main diagonal that intersect exactly in one cell, is 45.*

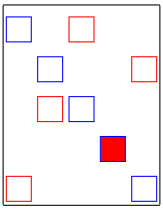
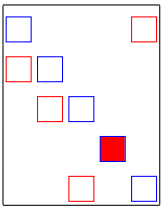
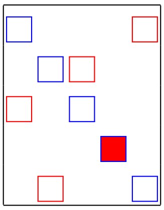
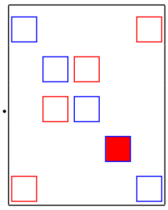
Proof. The 45 diagrams illustrating Latin squares with two free transversals, one of which is on the main diagonal, that intersect exactly in one cell are the following:

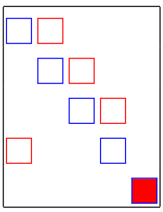
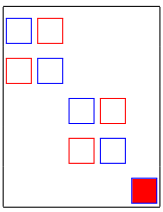
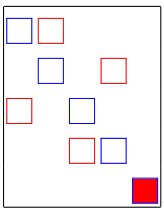
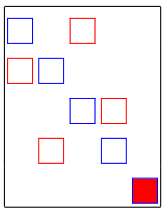
1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 
15. 
16. 
17. 
18. 
19. 
20. 

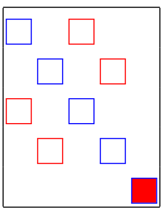
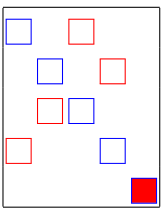
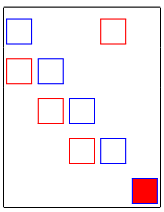
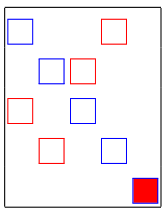
21. 
22. 
23. 
24. 

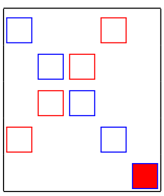
25. 
26. 
27. 
28. 

29. 
30. 
31. 
32. 

33. 
34. 
35. 
36. 

37. 
38. 
39. 
40. 

41. 
42. 
43. 
44. 

45. 

□

Proposition 2.3. *There exist exactly 48 different Latin squares of order 5 with two transversals intersecting in one cell, where one transversal is on the main diagonal with the fixed order of elements.*

Proof. The 48 Latin squares of order 5 with two transversals, one of which is on the main diagonal with the fixed order of elements $\{1,2,3,4,5\}$, that intersect exactly in one cell are:

$$L_1 = \begin{array}{|ccccc|} \hline 1&3&5&2&4\\ 3&2&4&5&1\\ 5&4&3&1&2\\ 2&5&1&4&3\\ 4&1&2&3&5\\ \hline \end{array} \quad L_2 = \begin{array}{|ccccc|} \hline 1&3&5&2&4\\ 5&2&4&1&3\\ 4&1&3&5&2\\ 3&5&2&4&1\\ 2&4&1&3&5\\ \hline \end{array} \quad L_3 = \begin{array}{|ccccc|} \hline 1&5&4&3&2\\ 4&2&5&1&3\\ 2&1&3&5&4\\ 5&3&2&4&1\\ 3&4&1&2&5\\ \hline \end{array} \quad L_4 = \begin{array}{|ccccc|} \hline 1&4&2&5&3\\ 4&2&5&3&1\\ 2&5&3&1&4\\ 5&3&1&4&2\\ 3&1&4&2&5\\ \hline \end{array}$$

$$L_5 = \begin{array}{|ccccc|} \hline 1&4&5&2&3\\ 3&2&4&5&1\\ 2&5&3&1&4\\ 5&3&1&4&2\\ 4&1&2&3&5\\ \hline \end{array} \quad L_6 = \begin{array}{|ccccc|} \hline 1&3&2&5&4\\ 5&2&4&1&3\\ 4&5&3&2&1\\ 3&1&5&4&2\\ 2&4&1&3&5\\ \hline \end{array} \quad L_7 = \begin{array}{|ccccc|} \hline 1&4&5&3&2\\ 3&2&4&5&1\\ 2&5&3&1&4\\ 5&1&2&4&3\\ 4&3&1&2&5\\ \hline \end{array} \quad L_8 = \begin{array}{|ccccc|} \hline 1&3&2&5&4\\ 5&2&4&3&1\\ 4&5&3&1&2\\ 2&1&5&4&3\\ 3&4&1&2&5\\ \hline \end{array}$$

$$L_9 = \begin{array}{|ccccc|} \hline 1&5&4&2&3\\ 3&2&5&1&4\\ 2&4&3&5&1\\ 5&3&1&4&2\\ 4&1&2&3&5\\ \hline \end{array} \quad L_{10} = \begin{array}{|ccccc|} \hline 1&3&2&5&4\\ 4&2&5&1&3\\ 5&4&3&2&1\\ 3&5&1&4&2\\ 2&1&4&3&5\\ \hline \end{array} \quad L_{11} = \begin{array}{|ccccc|} \hline 1&5&4&3&2\\ 3&2&5&1&4\\ 2&4&3&5&1\\ 5&1&2&4&3\\ 4&3&1&2&5\\ \hline \end{array} \quad L_{12} = \begin{array}{|ccccc|} \hline 1&3&2&5&4\\ 4&2&5&3&1\\ 5&4&3&1&2\\ 2&5&1&4&3\\ 3&1&4&2&5\\ \hline \end{array}$$

$$L_{13} = \begin{array}{|ccccc|} \hline 1&4&5&2&3\\ 5&2&4&3&1\\ 2&1&3&5&4\\ 3&5&1&4&2\\ 4&3&2&1&5\\ \hline \end{array} \quad L_{14} = \begin{array}{|ccccc|} \hline 1&5&2&3&4\\ 5&2&4&1&3\\ 2&4&3&5&1\\ 3&1&5&4&2\\ 4&3&1&2&5\\ \hline \end{array} \quad L_{15} = \begin{array}{|ccccc|} \hline 1&3&4&5&2\\ 4&2&5&3&1\\ 5&1&3&2&4\\ 2&5&1&4&3\\ 3&4&2&1&5\\ \hline \end{array} \quad L_{16} = \begin{array}{|ccccc|} \hline 1&3&4&5&2\\ 3&2&5&1&4\\ 4&5&3&2&1\\ 5&1&2&4&3\\ 2&4&1&3&5\\ \hline \end{array}$$

$$L_{17} = \begin{array}{|ccccc|} \hline 1&4&2&5&3\\ 5&2&1&3&4\\ 4&5&3&2&1\\ 3&1&5&4&2\\ 2&3&4&1&5\\ \hline \end{array} \quad L_{18} = \begin{array}{|ccccc|} \hline 1&5&4&3&2\\ 3&2&1&5&4\\ 5&4&3&2&1\\ 2&1&5&4&3\\ 4&3&2&1&5\\ \hline \end{array} \quad L_{19} = \begin{array}{|ccccc|} \hline 1&4&5&2&3\\ 5&2&1&3&4\\ 4&1&3&5&2\\ 3&5&2&4&1\\ 2&3&4&1&5\\ \hline \end{array} \quad L_{20} = \begin{array}{|ccccc|} \hline 1&4&5&2&3\\ 3&2&1&5&4\\ 4&5&3&1&2\\ 5&3&2&4&1\\ 2&1&4&3&5\\ \hline \end{array}$$

$$L_{21} = \begin{array}{|ccccc|} \hline 1&5&2&3&4\\ 4&2&1&5&3\\ 5&4&3&1&2\\ 2&3&5&4&1\\ 3&1&4&2&5\\ \hline \end{array} \quad L_{22} = \begin{array}{|ccccc|} \hline 1&5&4&3&2\\ 4&2&1&5&3\\ 5&1&3&2&4\\ 2&3&5&4&1\\ 3&4&2&1&5\\ \hline \end{array} \quad L_{23} = \begin{array}{|ccccc|} \hline 1&5&4&2&3\\ 5&2&1&3&4\\ 4&1&3&5&2\\ 2&3&5&4&1\\ 3&4&2&1&5\\ \hline \end{array} \quad L_{24} = \begin{array}{|ccccc|} \hline 1&4&5&3&2\\ 4&2&1&5&3\\ 5&1&3&2&4\\ 3&5&2&4&1\\ 2&3&4&1&5\\ \hline \end{array}$$

$$L_{25} = \boxed{\begin{matrix} 1&3&4&5&2\\ 4&2&5&1&3\\ 5&1&3&2&4\\ 3&5&2&4&1\\ 2&4&1&3&5 \end{matrix}}\ L_{26} = \boxed{\begin{matrix} 1&5&4&2&3\\ 3&2&1&5&4\\ 5&4&3&1&2\\ 2&3&5&4&1\\ 4&1&2&3&5 \end{matrix}}\ L_{27} = \boxed{\begin{matrix} 1&4&5&3&2\\ 3&2&1&5&4\\ 4&5&3&2&1\\ 5&1&2&4&3\\ 2&3&4&1&5 \end{matrix}}\ L_{28} = \boxed{\begin{matrix} 1&3&5&2&4\\ 5&2&4&3&1\\ 4&1&3&5&2\\ 2&5&1&4&3\\ 3&4&2&1&5 \end{matrix}}$$

$$L_{29} = \boxed{\begin{matrix} 1&3&2&5&4\\ 5&2&4&3&1\\ 4&5&3&1&2\\ 2&1&5&4&3\\ 3&4&1&2&5 \end{matrix}}\ L_{30} = \boxed{\begin{matrix} 1&4&2&5&3\\ 5&2&4&3&1\\ 2&5&3&1&4\\ 3&1&5&4&2\\ 4&3&1&2&5 \end{matrix}}\ L_{31} = \boxed{\begin{matrix} 1&4&5&3&2\\ 5&2&4&1&3\\ 2&1&3&5&4\\ 3&5&2&4&1\\ 4&3&1&2&5 \end{matrix}}\ L_{32} = \boxed{\begin{matrix} 1&4&2&5&3\\ 5&2&1&3&4\\ 4&5&3&1&2\\ 2&3&5&4&1\\ 3&1&4&2&5 \end{matrix}}$$

$$L_{33} = \boxed{\begin{matrix} 1&5&4&2&3\\ 4&2&5&3&1\\ 2&1&3&5&4\\ 5&3&1&4&2\\ 3&4&2&1&5 \end{matrix}}\ L_{34} = \boxed{\begin{matrix} 1&5&2&3&4\\ 4&2&1&5&3\\ 5&4&3&2&1\\ 3&1&5&4&2\\ 2&3&4&1&5 \end{matrix}}\ L_{35} = \boxed{\begin{matrix} 1&3&5&2&4\\ 3&2&4&5&1\\ 4&5&3&1&2\\ 5&1&2&4&3\\ 2&4&1&3&5 \end{matrix}}\ L_{36} = \boxed{\begin{matrix} 1&3&4&5&2\\ 3&2&5&1&4\\ 5&4&3&2&1\\ 2&5&1&4&3\\ 4&1&2&3&5 \end{matrix}}$$

$$L_{37} = \boxed{\begin{matrix} 1&3&4&5&2\\ 5&2&1&3&4\\ 4&5&3&2&1\\ 2&1&5&4&3\\ 3&4&2&1&5 \end{matrix}}\ L_{38} = \boxed{\begin{matrix} 1&3&5&2&4\\ 5&2&4&1&3\\ 2&4&3&5&1\\ 3&5&1&4&2\\ 4&1&2&3&5 \end{matrix}}\ L_{39} = \boxed{\begin{matrix} 1&5&4&2&3\\ 3&2&5&1&4\\ 4&1&3&5&2\\ 5&3&2&4&1\\ 2&4&1&3&5 \end{matrix}}\ L_{40} = \boxed{\begin{matrix} 1&5&2&3&4\\ 3&2&4&5&1\\ 5&4&3&1&2\\ 2&1&5&4&3\\ 4&3&1&2&5 \end{matrix}}$$

$$L_{41} = \boxed{\begin{matrix} 1&4&2&5&3\\ 4&2&5&3&1\\ 5&1&3&2&4\\ 3&5&1&4&2\\ 2&3&4&1&5 \end{matrix}}\ L_{42} = \boxed{\begin{matrix} 1&4&5&3&2\\ 4&2&1&5&3\\ 2&5&3&1&4\\ 5&3&2&4&1\\ 3&1&4&2&5 \end{matrix}}\ L_{43} = \boxed{\begin{matrix} 1&3&4&5&2\\ 4&2&5&3&1\\ 2&5&3&1&4\\ 5&1&2&4&3\\ 3&4&1&2&5 \end{matrix}}\ L_{44} = \boxed{\begin{matrix} 1&3&5&2&4\\ 4&2&1&5&3\\ 5&4&3&1&2\\ 3&5&2&4&1\\ 2&1&4&3&5 \end{matrix}}$$

$$L_{45} = \boxed{\begin{matrix} 1&4&5&3&2\\ 3&2&4&5&1\\ 5&1&3&2&4\\ 2&5&1&4&3\\ 4&3&2&1&5 \end{matrix}}\ L_{46} = \boxed{\begin{matrix} 1&4&2&5&3\\ 3&2&5&1&4\\ 4&5&3&2&1\\ 5&3&1&4&2\\ 2&1&4&3&5 \end{matrix}}\ L_{47} = \boxed{\begin{matrix} 1&5&2&3&4\\ 5&2&4&1&3\\ 4&1&3&5&2\\ 2&3&5&4&1\\ 3&4&1&2&5 \end{matrix}}\ L_{48} = \boxed{\begin{matrix} 1&5&4&2&3\\ 5&2&1&3&4\\ 2&4&3&5&1\\ 3&1&5&4&2\\ 4&3&2&1&5 \end{matrix}}$$

The corresponding 240 second transversals for the transversal given on the main diagonal of the Latin squares $L_1 - L_{48}$, are the following:

L_1: $T_1 = \{(1,1), (2,3), (3,5), (4,2), (5,4)\}$, $T_2 = \{(1,1), (2,4), (3,2), (4,5), (5,3)\}$,
$T_3 = \{(1,3), (2,2), (3,4), (4,5), (5,1)\}$, $T_4 = \{(1,5), (2,2), (3,1), (4,3), (5,4)\}$,
$T_5 = \{(1,4), (2,5), (3,3), (4,2), (5,1)\}$, $T_6 = \{(1,5), (2,4), (3,3), (4,1), (5,2)\}$,
$T_7 = \{(1,2), (2,5), (3,1), (4,4), (5,3)\}$, $T_8 = \{(1,3), (2,1), (3,5), (4,4), (5,2)\}$,
$T_9 = \{(1,2), (2,3), (3,4), (4,1), (5,5)\}$, $T_{10} = \{(1,4), (2,1), (3,2), (4,3), (5,5)\}$.

L_2: $T_{11} = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_{12} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{13} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{14} = \{(1,5),(2,2),(3,4),(4,1),(5,3)\}$,
$T_{15} = \{(1,4),(2,1),(3,3),(4,5),(5,2)\}$, $T_{16} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{17} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{18} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$,
$T_{19} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{20} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$.

L_3: $T_{21} = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_{22} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{23} = \{(1,3),(2,2),(3,4),(4,5),(5,1)\}$, $T_{24} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,
$T_{25} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{26} = \{(1,5),(2,4),(3,3),(4,1),(5,2)\}$,
$T_{27} = \{(1,2),(2,5),(3,1),(4,4),(5,3)\}$, $T_{28} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,
$T_{29} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{30} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_4: $T_{31} = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_{32} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$,
$T_{33} = \{(1,4),(2,2),(3,5),(4,3),(5,1)\}$, $T_{34} = \{(1,5),(2,2),(3,4),(4,1),(5,3)\}$,
$T_{35} = \{(1,2),(2,5),(3,3),(4,1),(5,4)\}$, $T_{36} = \{(1,4),(2,2),(3,3),(4,5),(5,2)\}$,
$T_{37} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$, $T_{38} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{39} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{40} = \{(1,3),(2,1),(3,4),(4,2),(5,5)\}$.

L_5: $T_{41} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{42} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,

L_6: $T_{43} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{44} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_7: $T_{45} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{46} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$,
$T_{47} = \{(1,3),(2,2),(3,4),(4,5),(5,1)\}$, $T_{48} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,
$T_{49} = \{(1,2),(2,5),(3,3),(4,1),(5,4)\}$, $T_{50} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{51} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{52} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{53} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{54} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_8: $T_{55} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{56} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$,
$T_{57} = \{(1,4),(2,2),(3,1),(4,5),(5,3)\}$, $T_{58} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{59} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{60} = \{(1,5),(2,1),(3,3),(4,2),(5,4)\}$,
$T_{61} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{62} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$,
$T_{63} = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{64} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$.

L_9: $T_{65} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{66} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{67} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{68} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{69} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{70} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,
$T_{71} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{72} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{73} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{74} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_{10}: $T_{75} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{76} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{77} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$, $T_{78} = \{(1,5),(2,2),(3,1),(4,3),(5,4)\}$,
$T_{79} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{80} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{81} = \{(1,2),(2,3),(3,5),(4,4),(5,1)\}$, $T_{82} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,
$T_{83} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{84} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$.

L_{11}: $T_{85} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{86} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_{12}: $T_{87} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{88} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,

L_{13}: $T_{89} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{90} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,
$T_{91} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{92} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{93} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{94} = \{(1,4),(2,5),(3,3),(4,2),(5,1)\}$,
$T_{95} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{96} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{97} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{98} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$.

L_{14}: $T_{99} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{100} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
$T_{101} = \{(1,4),(2,2),(3,5),(4,3),(5,1)\}$, $T_{102} = \{(1,5),(2,2),(3,4),(4,1),(5,3)\}$,
$T_{103} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{104} = \{(1,5),(2,1),(3,3),(4,2),(5,4)\}$,
$T_{105} = \{(1,2),(2,5),(3,1),(4,4),(5,3)\}$, $T_{106} = \{(1,3),(2,1),(3,5),(4,4),(5,2)\}$,
$T_{107} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$, $T_{108} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_{15}: $T_{109} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{110} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,
$T_{111} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{112} = \{(1,4),(2,2),(3,5),(4,3),(5,1)\}$,
$T_{113} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{114} = \{(1,5),(2,1),(3,3),(4,2),(5,4)\}$,
$T_{115} = \{(1,2),(2,5),(3,1),(4,4),(5,3)\}$, $T_{116} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,
$T_{117} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{118} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$.

L_{16}: $T_{119} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{120} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
$T_{121} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{122} = \{(1,4),(2,2),(3,1),(4,5),(5,3)\}$,
$T_{123} = \{(1,4),(2,5),(3,3),(4,2),(5,1)\}$, $T_{124} = \{(1,5),(2,4),(3,3),(4,1),(5,2)\}$,
$T_{125} = \{(1,2),(2,3),(3,5),(4,4),(5,1)\}$, $T_{126} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{127} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{128} = \{(1,3),(2,1),(3,4),(4,2),(5,5)\}$.

L_{17}: $T_{129} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$, $T_{130} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{131} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$, $T_{132} = \{(1,5),(2,2),(3,1),(4,3),(5,4)\}$,
$T_{133} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{134} = \{(1,4),(2,5),(3,3),(4,2),(5,1)\}$,
$T_{135} = \{(1,3),(2,1),(3,5),(4,4),(5,2)\}$, $T_{136} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,
$T_{137} = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{138} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$.

L_{18}: $T_{139} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$, $T_{140} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{141} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{142} = \{(1,4),(2,2),(3,5),(4,3),(5,1)\}$,
$T_{143} = \{(1,2),(2,5),(3,3),(4,1),(5,4)\}$, $T_{144} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{145} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{146} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{147} = \{(1,3),(2,1),(3,4),(4,2),(5,5)\}$, $T_{148} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$.

L_{19}: $T_{149} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$, $T_{150} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_{20}: $T_{151} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$, $T_{152} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
$T_{153} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{154} = \{(1,5),(2,2),(3,4),(4,1),(5,3)\}$,
$T_{155} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{156} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,
$T_{157} = \{(1,3),(2,1),(3,5),(4,4),(5,2)\}$, $T_{158} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,
$T_{159} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{160} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_{21}: $T_{161} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$, $T_{162} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
$T_{163} = \{(1,4),(2,2),(3,1),(4,5),(5,3)\}$, $T_{164} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{165} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{166} = \{(1,5),(2,4),(3,3),(4,1),(5,2)\}$,

$T_{167} = \{(1,2),(2,3),(3,5),(4,4),(5,1)\}$, $T_{168} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,
$T_{169} = \{(1,3),(2,1),(3,4),(4,2),(5,5)\}$, $T_{170} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$.

L_{22}: $T_{171} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$, $T_{172} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_{23}: $T_{173} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$, $T_{174} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{175} = \{(1,3),(2,2),(3,4),(4,5),(5,1)\}$, $T_{176} = \{(1,5),(2,2),(3,1),(4,3),(5,4)\}$,
$T_{177} = \{(1,2),(2,5),(3,3),(4,1),(5,4)\}$, $T_{178} = \{(1,4),(2,1),(3,3),(4,5),(5,2)\}$,
$T_{179} = \{(1,2),(2,3),(3,5),(4,4),(5,1)\}$, $T_{180} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{181} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$, $T_{182} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_{24}: $T_{183} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$, $T_{184} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{185} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{186} = \{(1,4),(2,2),(3,1),(4,5),(5,3)\}$,
$T_{187} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{188} = \{(1,5),(2,1),(3,3),(4,2),(5,4)\}$,
$T_{189} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$, $T_{190} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{191} = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{192} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_{25}: $T_{193} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{194} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,

L_{26}: $T_{195} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{196} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,

L_{27}: $T_{197} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{198} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,

L_{28}: $T_{199} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{200} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,

L_{29}: $T_{201} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$, $T_{202} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,

L_{30}: $T_{203} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$, $T_{204} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,

L_{31}: $T_{205} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{206} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{32}: $T_{207} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{208} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,

L_{33}: $T_{209} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{210} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,

L_{34}: $T_{211} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{212} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{35}: $T_{213} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{214} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{36}: $T_{215} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{216} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{37}: $T_{217} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{218} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{38}: $T_{219} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{220} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,

L_{39}: $T_{221} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{222} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{40}: $T_{223} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{224} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,

L_{41}: $T_{225} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{226} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{42}: $T_{227} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{228} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{43}: $T_{229} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{230} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$,

L_{44}: $T_{231} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{232} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$,

L_{45}: $T_{233} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{234} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$,

L_{46}: $T_{235} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{236} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$,

L_{47}: $T_{237} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{238} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$,

L_{48}: $T_{239} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{240} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$. $\square$

Proposition 2.4. *There does not exist recursively 1-differentiable prolongations of Latin squares of order 5, obtained by adding two new elements and two transversals, one of which is on the main diagonal with the fixed order of elements* $\{1, 2, 3, 4, 5\}$ *or* $\{2, 3, 4, 5, 1\}$*, that intersect exactly in one cell.*

Proof. The proof is obtained by direct calculation. For Latin squares with each of the 240 pairs of transversals, 6 types of prolongations according to extension schemes II, III, V, VIII, IX, XII were considered. □

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References

[1] Bruck, R.H. Some results in the theory of quasigroups. *Trans. Amer. Math. Soc.*, 1944, vol. 55, pp. 19–52.

[2] Belousov, V. Foundations of the Theory of Quasigroups and Loops. Moscow, Nauka, 1967 (in Russian).

[3] Keedwell, A.A., Denes, J. Latin Squares and Their Applications. *Second edition. North Holland*, 2015. DOI: https://doi.org/10.1016/C2014-0-03412-0.

[4] Gonsales, S., Couselo, E., Markov, V.T., Nechaev A.A. Recursive MDS codes and recursively differentiable quasigroups. *Diskret. Mat.*, 1998, vol. 10, no. 2 pp. 3–29.

[5] Syrbu, P. On the order of recursive differentiability of finite binary quasigroups. *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2023, vol. 103, no.3, pp. 103–106.

[6] Syrbu, P., Cuzneţov, E. On recursively differentiable k-quasigroups. *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2022, vol. 99, no. 2, pp. 68–75.

[7] Syrbu, P., Cuzneţov, E. On recursive 1-differentiability of quasigroup prolongations. *Bul. Acad. Ştiinţe Repub. Mold., Mat.*, 2023, vol. 101, no. 2, pp. 102–109.

[8] Cuzneţov, E. On a method of prolongation of quasigroups. *Proceedings of the International Conference dedicated to the 60th anniversary of the foundation of Vladimir Andrunachievici Institute of Mathematics and Computer Science*, October 10-13, 2024, Chisinau, Republic of Moldova, pp. 55–60.

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