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On some Noetherian criteria for singular integral operators with bounded coefficients

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Abstract. In this paper, Noetherian conditions are established for singular integral operators with measurable and bounded coefficients. The study is carried out in spaces of integrable functions with certain weights, and the integration contour is assumed to be piecewise smooth. The established Noetherian conditions depend on the space in which the research is carried out, on the coefficients of the operators, as well as on the sizes of the angles formed by the integration contour at its corner points. In the case of the Lyapunov-type contour, the presented Noetherian conditions are necessary and sufficient in which the operators studied are Noetherian.
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Asupra unor criterii noetheriene pentru operatorii integrali singulari cu coeficienți mărginiți

Rezumat. În această lucrare sunt stabilite condiții noetheriene pentru operatorii integrali singulari cu coeficienți măsurabili și mărginiți. Studiul este realizat în spații de funcții integrabile cu anumite ponderi, conturul de integrare se presupune neted pe porțiuni. Condițiile noetheriene stabilite, depind de spațiu în care se desfășoară cercetarea, de coeficienții operatorilor, precum și de mărimile unghiurilor formate de conturul de integrare în punctele sale unghiulare. În cazul conturului de tip Lyapunov condițiile noetheriene prezentate sunt necesare și suficiente în care operatorii studiați sunt Noetherieni.
Cuvinte-cheie: operator singular, factorizarea funcțiilor, norma esențială, condiții noetheriene, spații cu ponderi.

1. Introduction

The previous theory of one-dimensional singular integral equations with continuous coefficients was systematically developed by N. Muskhelishvili, F. Gakhov, I. Vekua and others. Based on these classical developments, numerous studies were carried out in the case of singular integral equations with discontinuous coefficients. In particular,

I. Simonenko, I. Gohberg, and N. Krupnik established various local theories for singular integral equations, which reduce global problems to local ones. In an attempt to use these local theories, K. Clancey and I. Gohberg [1] investigated the concept of local factorization of functions. They showed that the existence of a global factorization is equivalent to the existence of a local factorization at each point of the integration contour, and the Noetherian conditions of singular integral operators are determined by the existence of factorization of the coefficients of the respective operators.

In this paper, the results of research in the theory of operators with local properties [2-4] are applied to obtain some sufficient conditions under which singular operators with scalar or matrix coefficients are Noetherian. These conditions are established for singular integral operators acting in the spaces $L_p(\Gamma, \rho)$, where Γ is a piecewise Lyapunov contour and ρ is the weight introduced by the mathematician B. Khvedelidze [5].

It is shown that the Noetherian conditions depend on the coefficients of the operator, on the space $L_p(\Gamma, \rho)$, but also on the sizes of the angles formed by the contour at its corner points. Thus, the Noetherian conditions at different points of the contour are different, and especially, at the self-intersection points of the contour. In the case, where the integration contour Γ does not contain corner points, the results presented in this paper also become necessary in which the operators under study are Noetherian.

2. Invertibility and local factorization

Let γ be a rectifiable arc in the complex plane $\mathbb{C}$. The notation $L_p(\gamma)$ $(1 \le p \le \infty)$ will be used for the usual Lebesgue spaces with respect to arc length measure on γ. In case γ is a piecewise Lyapunov arc and $1 < p < \infty$, we define the singular integral operator S on $L_p(\gamma)$ by

$$(S_\gamma \varphi)(t) = \frac{1}{\pi i} \int_\gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau.$$

The fact that S_γ defines a bounded operator on $L_p(\gamma)$ $(1 < p < \infty)$, even if the arc γ has cusps, is established in [5], [6]. When the arc γ is a piecewise Lyapunov contour Γ, one defines the usual analytic projectors

$$(P_\gamma \varphi)(t) = \frac{\varphi(t)}{2} + \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau, \qquad (Q_\gamma \varphi)(t) = \varphi(t) - (P_\gamma \varphi)(t),$$

bounded in $L_p(\Gamma)$ $(1 < p < \infty)$. The range of the projector P_γ on $L_p(\Gamma)$ is denoted by $L_p^+(\Gamma)$, while $L_p^-(\Gamma)$ denotes the span of the range of Q_γ on $L_p(\Gamma)$ and the constant functions $(1 < p < \infty)$. Further, $L_\infty^\pm(\Gamma) = L_\infty(\Gamma) \cap L_2^\pm(\Gamma)$. Let a be a function in $L_\infty(\Gamma)$. We denote by T_a the singular integral operator $T_a = aP_\gamma + Q_\gamma$ acting on $L_p(\Gamma)$

$(1 < p < \infty)$. A general reference to the theory of operators of the form T_a is the monograph [7].

Let t_0 be a fixed point on the Lyapunov contour Γ and let φ belong to $L_p(\Gamma)$. In this section we will discuss the equivalence of various interpretations of the phrase *"the operator T_a on $L_p(\Gamma)$ is locally invertible at t_0"*.

Definition 2.1. *Let $a \in L_\infty(\Gamma)$, where Γ is a Lyapunov contour. The operator $T_a = aP+Q$ in $L_p(\Gamma)$ is called left-Fredholm, right-Fredholm or Fredholm operator at a point $t_0 \in \Gamma$ if there is a function $b \in L_\infty(\Gamma)$ and a neighborhood U of the point t_0 such that $a(t) = b(t)$ for $t \in U$, and the operator $T_b = bP + Q$ is left-Fredholm, respectively right-Fredholm or Fredholm.*

The next definition of *"local invertibility"* is due to Simonenko [2], see also the references in [3].

Definition 2.2. *We say that the operator $T_a = aP + Q$ in $L_p(\Gamma)$ admits a local right (left) regulator at a point $t_0 \in \Gamma$ if there exist an operator R, a compact operator K in $L_p(\Gamma)$, and a function u continuous on Γ and equal to one in a neighborhood of t_0, such that*

$$RT_aM_u = M_u + K; \qquad M_uT_aR = M_u + K,$$

where M_u denotes the operator in $L_p(\Gamma)$ of multiplication by the function u.

We say that the operator T_a admits a local regulator at t_0 if it admits both a right and a left local regularizer at t_0.

Let Γ be a Lyapunov contour bounding the domain G in the complex plane. It will be assumed that the point zero belongs to G. Let a be an element of $GL_\infty(\Gamma)$; here $GL_\infty(\Gamma)$ denotes the group of invertible elements in $L_\infty(\Gamma)$. The following is the standard definition of factorization of the function a relative to L_p.

Definition 2.3. *We say that a function a admits a factorization in the space L_p $(1 < p < \infty)$ if it can be represented in the form $a(t) = a_-(t)\, t^\kappa\, a_+(t),\ \ t \in \Gamma$, where κ is an integer and the following conditions hold:*

1) *the pair (a_+^{-1}, a_+) belongs to $L_p^+(\Gamma_+) \times L_q^+(\Gamma_+)$ and the pair (a_-, a^{-1}) belongs to $L_p^-(\Gamma) \times L_q^-(\Gamma)$ $(p^-1 + q^-1 = 1)$;*

2) *the operator $a_+^{-1}Sa_-^{-1}I$ is bounded in $L_p(\Gamma)$.*

The following definition of local factorization is similar to the previous definition and is contained in [2] (see also [1]).

Definition 2.4. *A function a admits a local factorization in L_p at a point $t_0 \in \Gamma$ if there exists a neighborhood γ of the point t_0 such that $a(t) = a_-(t)a_+(t)$, where*
(i) the pair of functions(a_+^{-1}, a_+) belongs to $L_p^+(\Gamma_+) \times L_q^+(\Gamma_+)$ and (a_-, a^{-1}) belongs to $L_p^+(\Gamma_-) \times L_q^+(\Gamma_-)$; here $\Gamma_\pm \subset G_+ \cup \Gamma$ are piecewise Lyapunov contours such that $\Gamma_\pm \cap \Gamma = \gamma$.
(ii) the operator $a_+^{-1}Sa_-^{-1}I$ is bounded in $L_p(\gamma)$.

In [1] it was established that a function a which admits a local factorization in L_p at every point $t_0 \in \Gamma$ also admits a (global) factorization in L_p. In the present work, alongside the norms and essential norms of the operator S_Γ, we will use the equivalence between the notions of local factorization of a function in $L_p(\Gamma)$ and local invertibility of the singular integral operator $T_a = aP + Q$. The main result that will be used is the following:

Theorem 2.1. *Let Γ be a Lyapunov contour. Assume $a \in GL_\infty(\Gamma)$, and let $t_0 \in \Gamma$.*

1) *Assume that $1 < p < \infty$. If a function a admits a local factorizationin L_p at a point t_0, then the operator $T_a = aP + Q$ in $L_p(\Gamma)$ has a local regulator at t_0.*

2) *If the operator $T_a = aP + Q$ in L_2 is locally invertible at t_0, then a admits a local factorization in L_p at t_0.*

3. Noetherian conditions for the operator $A = aP_\Gamma + bQ_\Gamma$ with scalar coefficients

Let a and b be two measurable and bounded functions on the contour Γ. Here we will establish sufficient conditions under which the operator

$$A = aP_\Gamma + bQ_\Gamma \tag{3.1}$$

is Noetherian in the space $L_2(\Gamma, \rho)$. If the contour Γ is of Lyapunov type and $\rho \equiv 1$, these conditions coincide with the conditions established by I. Simonenko [2] and, in addition, are also necessary under which the operator (3.1) is Noetherian in the space $L_2(\Gamma, \rho)$.

Henceforth, we will assume that the contour Γ satisfies the following conditions:

a) Γ has a finite number of self-intersection points $\tau_1, \tau_2, \ldots, \tau_m$ and a finite number of angular points τ_k $(k = r + 1, \ldots, m)$;
b) at the points τ_k $(k = r + 1, \ldots, m)$, the tangents to Γ are perpendicular;
c) the contour Γ can be completed up to a closed composite contour $\widetilde{\Gamma}$ of Lyapunov type by pieces, which divides the complex plane into two sets $F^+_{\widetilde{\Gamma}}$ and $F^-_{\widetilde{\Gamma}}$ and $\widetilde{\Gamma}$ is the boundary for each of these sets $F^\pm_{\widetilde{\Gamma}}$. Obviously, the contour $\widetilde{\Gamma}$ is determined unambiguously.

We denote by $\pi\alpha_0(t)$ $(0 < \alpha_0(t) < 2)$ the angle formed by the contour Γ at the point $\tau \neq \tau_k$ $(k = r+1, \dots, m)$. We will also need the functions $\alpha(t) = \min\{\alpha_0(t),\ 2-\alpha_0(t)\}$ $(0 < \alpha(t) < 1)$ and $\theta(t)$ $(t \in \Gamma \setminus \{\tau_1, \tau_2, \dots, \tau_m\})$ defined by the relation

$$ctg\theta(t) = \frac{1}{2}\max_{-1\leq z\leq 1}\left|(1+z)\left(\frac{1-z}{1+z}\right)^{\frac{\alpha(t)}{2}} - (1-z)\left(\frac{1+z}{1-z}\right)^{\frac{\alpha(t)}{2}}\right|. \tag{3.2}$$

We note that the set of values of the function $\theta(t)$ is contained on the interval $]\frac{\pi}{4}, \frac{\pi}{2}]$ and $\theta(t) = \frac{\pi}{2}$ at all points t of the contour Γ where the Lyapunov conditions are verified. In particular, if the contour Γ is simple and of Lyapunov type, then the set of values of the function $\theta(t)$ is contained on the interval $]\frac{\pi}{4}, \frac{\pi}{2}]$ and $\theta(t) = \frac{\pi}{2}$.

Let τ_0 be an arbitrary point of the contour Γ different from the points $\tau_1, \tau_2, \dots \tau_r$. We denote by $\Lambda(\tau_0)$ an angle with the vertex at the point $z = 0$ of size $2\theta(\tau_0)$, where $\theta(t)$ is defined in (3.2). By $\Lambda(\tau_k)$ $(k = 1, ..., r)$ we will denote an angle with the vertex at the point $z = 0$ of size $\theta = \frac{\pi}{2}$.

Let $\widetilde{\Gamma}(\supset \Gamma)$ be a completion of the contour Γ up to a closed composite contour, which satisfies condition c). If $a \in \Gamma_\infty(\Gamma)$, then we denote by $\widetilde{a}$ the function

$$\widetilde{a}(\tau) = \begin{cases} a(t), & \text{for} \quad t \in \Gamma, \\ 1, & \text{for} \quad t \in \widetilde{\Gamma} \setminus \Gamma. \end{cases}$$

Let us also denote by $M(\Gamma)$ the set of measurable and bounded functions on Γ that satisfy the conditions:

(i) $essinf[a(t)] > 0$;

(ii) for any point $\tau \in \widetilde{\Gamma}$ there exists a neighborhood $u(\tau) \subset \widetilde{\Gamma}$ of the point τ and two functions g_τ^+, g_τ^- which belong respectively to the sets[1] $L_\infty^+(\widetilde{\Gamma})$, $L_\infty^-(\widetilde{\Gamma})$ together with $(g_t^+)^{-1}$ and $(g_t^-)^{-1}$, such that for $t \in u(\tau)$ the set of values of the function $g_\tau^+(t)\, a(t)\, g_\tau^-(t)$ is contained in the angle $\Lambda(\tau_0)$.

It can be easily shown that the set $M(\Gamma)$ does not depend on the choice of the composite contour $\widetilde{\Gamma}$, if Γ satisfies conditions a) – c).

Theorem 3.1. *Let $a \in M(\Gamma)$. Then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2(\Gamma)$.*

To facilitate the proof of the stated theorem, we will need the following three auxiliary propositions. Let Γ_0 be a simple closed Lyapunov contour on portions with a single corner point t_0.

[1]By $L_\infty^+(\widetilde{\Gamma})$ $(\subset L_\infty\widetilde{\Gamma})$ we denote the set of bounded analytic functions in the domain $F_{\widetilde{\Gamma}}^+(F_{\widetilde{\Gamma}}^-)$.

Lemma 3.1. *If the real and measurable function $\varphi(t)$ $(t \in \Gamma_0)$ satisfies the condition*

$$\operatorname*{ess\,sup}_{t\in\Gamma_0} |\varphi(t)| < \theta(\tau_0),$$

then the operator $A_\varphi = e^{i\varphi} P_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\Gamma_0)$.

Proof. Based on the results from [8], [9] we have

$$\inf_{T\in\mathfrak{T}} \|S_{\Gamma_0} + T\|_{L_2(\Gamma_0)} = ctg\frac{\theta(\tau_0)}{2},$$

where $\mathfrak{T}(= \mathfrak{T}(L_2(\Gamma_0)))$ is the set of compact operators in the space $L_2(\Gamma_0)$. The operator A_0 can be written in the form

$$A_0 = \frac{e^{i\varphi}+1}{2}\left(I + i\,tg\frac{\varphi}{2}\,S_{\Gamma_0}\right).$$

As

$$\inf_{T\in\mathfrak{T}} \left\|i\,tg\frac{\varphi}{2}\,S_{\Gamma_0} + T\right\|_{L_2(\Gamma_0)} \le \sup_{t\in\Gamma_0}\left|tg\frac{\varphi(t)}{2}\right| ctg\frac{\theta(\tau_0)}{2} < 1,$$

it follows that the operator A_0 is Noetherian (invertible) in the space $L_2(\Gamma_0)$. Lemma 3.1 is proved. □

Lemma 3.2. *Let $a \in L_\infty(\Gamma_0)$. If $\operatorname{ess\,inf}_{t\in\Gamma_0}|a(t)| > 0$ and its value set belongs to the angle $\Lambda(\tau_0)$, then the operator $B_a = aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\Gamma_0)$.*

Proof. The function a can be represented in the form $a(t) = \gamma|a|\cdot e^{i\varphi}$, where $\varphi(t)$ satisfies the conditions of Lemma 3.1 and γ is a complex number $(|\gamma| = 1)$. The function $b(t) = |a(t)|$ admits factorization (see [7]), $b = b_+ \cdot b_-$, where $b_+^{\pm 1} \in L_\infty^+(\Gamma_0)$ and $b_-^{\pm 1} \in L_\infty^-(\Gamma_0)$ and $b_+^{\pm 1} \in L_\infty^-(\Gamma_0)$. Then the operator B_0 can be represented in the form

$$B_0 = b_-\left(e^{i\varphi}P_{\Gamma_0} + Q_{\Gamma_0}\right)(\gamma b_+ P_{\Gamma_0} + b_-^{-1}Q_{\Gamma_0}). \tag{3.3}$$

By Lemma 3.1, the operator $e^{i\varphi}P_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\Gamma_0)$. As the operators b_-I and $\gamma b_+ P_{\Gamma_0} + b_-^{-1}Q_{\Gamma_0}$ are invertible in $L_2(\Gamma_0)$, from (3.3) it follows that B_0 is Noetherian in $L_2(\Gamma_0)$. The lemma is proved. □

Let $\widetilde{\Gamma}_0$ be a closed Lyapunov contour on portions with a single self-intersection point τ_0 and the tangents to $\widetilde{\Gamma}_0$ in τ_0 are perpendicular. As in the proof of Lemma 3.2, taking into account that in this case (see [8], [9])

$$\inf_{T\in\mathfrak{T}} \|S_{\Gamma_0} + T\|_{L_2(\Gamma_0)} = 1 + \sqrt{2},$$

the following lemma is also proved.

Lemma 3.3. *Let $a \in L_\infty(\tilde{\Gamma}_0)$. If $\operatorname{ess\,inf}_{t\in\Gamma_0}|a(t)| > 0$ and the set of values of the function a is in the angle with the vertex at the point $z = 0$ and of size $\frac{\pi}{2}$, then the operator $B_0 = aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\tilde{\Gamma}_0)$.*

Proof. We give the proof of the Theorem 3.1. Let $\tau_{r+1}, \dots, \tau_m$ denote all the angular points of the contour Γ. Let τ be an arbitrary point in $\widetilde{\Gamma}$, $g_\tau^\pm \in L_\infty^\pm(\tilde{\Gamma})$ and $u(\tau)$ be a neighborhood of the point τ, such that the set of values of the function $g_\tau^+(t)\widetilde{a}(t)g_\tau^-(t)$ lies in $\Lambda(\tau)$ for $t \in u(\tau)$. We can assume that $u(\tau)$ does not contain points in τ_k $(k = 1, 2, \dots, m)$ different from τ. Let

$$\tilde{a}_\tau(t) = \begin{cases} \widetilde{a}(t), & \text{for } t \in u(\tau), \\ \delta\left(g_\tau^+(t)g_\tau^-(t)\right)^{-1}, & \text{for } t \in \widetilde{\Gamma} \setminus u(\tau), \end{cases}$$

where δ is a complex number in $\Lambda(\tau)$. The set of values of the function $\widetilde{b}_\tau(t) = g_\tau^+(t)\tilde{a}_\tau(t)g_\tau^-(t)$ lies in the angle $\Lambda(\tau)$. We will show that the operator $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian in the space $L_2(\Gamma)$. Indeed, let $\widetilde{\Gamma}_\tau(\supset u(\tau))$ be a piecewise Lyapunov contour that has a single (angular) self-intersection point for $\tau = \tau_k$ $(k = 1, 2, \dots, r)$ (for $\tau = \tau_k$ $(k = r + 1, \dots, m)$) and that has no such points for τ other than the points $\tau = \tau_k$ $(k = 1, 2, \dots, m)$. We denote by $c_\tau(t)$ the function defined by the equality

$$c_\tau(t) = \begin{cases} \widetilde{b}_\tau(t), & \text{for } t \in u(\tau), \\ 1, & \text{for } t \in \widetilde{\Gamma}_\tau \setminus u(\tau). \end{cases}$$

As $\widetilde{b}_\tau(t) = 1$ for $t \in \widetilde{\Gamma} \setminus u(\tau)$, then the set of values of the function $c_\tau(t)$ $(t \in \widetilde{\Gamma}_\tau)$ also lies in the angle $\Lambda(\tau)$. From Theorem 1.1 Chapter VII of the work [7], we deduce that the operator $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian in the space $L_2(\widetilde{\Gamma})$ if and only if the operator $C_\tau = c_\tau P_{\widetilde{\Gamma}_\tau} + Q_{\widetilde{\Gamma}_\tau}$ is Noetherian in the space $L_2(\widetilde{\Gamma}_\tau)$. If $\tau = \tau_k$ $(k = r + 1, \dots, m)$ or $\tau = \tau_k$ $(k = 1, \dots, r)$, then, by virtue of Lemmas 3.2 and 3.3, respectively, it follows that the operator C_τ is Noetherian in the space $L_2(\widetilde{\Gamma}_\tau)$. However, if $\tau \in \widetilde{\Gamma} \setminus \{\tau_1, \dots, \tau_m\}$, then from Theorem 5 of [10], it follows that C_τ is Noetherian in $L_2(\widetilde{\Gamma}_\tau)$.

Therefore, the operator $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian for any $\tau \in \widetilde{\Gamma}$. We consider the operator $\widetilde{A}_\tau = \tilde{a}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ and write it in the form

$$\widetilde{A}_\tau = (g_\tau^-)^{-1}\delta\big(\widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}\big)\big((g_\tau^+)^{-1}P_{\widetilde{\Gamma}} + \delta g_\tau^- Q_{\widetilde{\Gamma}}\big) \tag{3.4}$$

The operators $(g_\tau^-)^{-1}\delta I$ and $(g_\tau^-)^{-1}P_{\widetilde{\Gamma}} + \delta^{-1}g_\tau^- Q_{\widetilde{\Gamma}}$ are invertible in the space $L_2(\widetilde{\Gamma})$, and $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian in $L_2(\widetilde{\Gamma})$, so from (3.4) it follows that $\widetilde{A}_\tau$ is Noetherian.

Thus, the operator $\widetilde{A} = \tilde{a}P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ at any point $\tau \in \widetilde{\Gamma}$ is equivalent to the Noetherian operator $\widetilde{A}_\tau$. Based on Theorem 1.6 of [2], it follows that $\widetilde{A}$ is Noetherian in $L_2(\widetilde{\Gamma})$.

Then from Theorem 1.1 of Chapter VII of the paper [7], it follows that the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in $L_2(\Gamma)$. The theorem is proved. □

Let

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \quad (-1 < \beta_k < p - 1,\ t_k = \tau_k,\ k = 1, \dots, m,\ n \geq m)$$

and the function h defined in the paper [10]. From Theorem 3.1 and the results of the paper [11] it follows

Corollary 3.1. *Let the functions $a \in L_\infty(\Gamma)$. If $ah \in M(\Gamma)$, then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2(\Gamma, p)$.*

Corollary 3.2. *Let $a, b \in L_\infty(\Gamma)$ and $\operatorname{ess\,inf}_{t\in\Gamma} |b(t)| > 0$. If $ab^{-1}h \in M(\Gamma)$, then the operator $A = aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2(\Gamma, p)$.*

We note that the set $M(\Gamma)$ is quite large. Thus, if the contour Γ is of a simple closed Lyapunov type, then the class of functions $M(\Gamma)$ coincides with the class $\widetilde{A}(2, \Gamma)$, defined by I. Simonenko in the paper [12] and, in addition, the operator $aP_\Gamma + bQ_\Gamma$ is Noetherian in $L_2(\Gamma, p)$ if and only if $ab^{-1}h \in M(\Gamma)$.

Let the contour Γ be composite, closed and satisfy the conditions of Theorem 2.1. We denote by $\widetilde{M}(\Gamma) \subset L_\infty(\Gamma)$ the set of functions satisfying the conditions:

(i) $\operatorname{ess\,inf}_{t\in\Gamma} |a(t)| > 0$;

(ii) for any $\tau \in \Gamma \setminus \{\tau_1, \dots, \tau_m\}$ there exists a neighbourhood $u(\tau) \subset \Gamma$ of the point τ and two functions g_τ^+, g_τ^- which belong respectively to the sets $L_\infty^+(\Gamma)$, $L_\infty^-(\Gamma)$ together with $(g_\tau^+)^{-1}$ and $(g_\tau^-)^{-1}$ such that for any $t \in u(\tau)$ the set of values of the function $g_\tau^+(t)a(t)g_\tau^-(t)$ lies in an angle with the vertex at $z = 0$ of measure π;

(iii) at the points τ_k $(k = 1, \dots, r)$ there exists $a(\tau_k \pm 0)$, where the quantities $a(\tau_k \pm 0)$ are defined in the works [7], [8].

From Theorem 3.1 and the results of paper [11] it follows

Theorem 3.2. *For the operator $A = aP_\Gamma + bQ_\Gamma$ $(a, b \in L_\infty(\Gamma))$ to be Noetherian in the space $L_2(\Gamma)$ it is necessary and sufficient that the function ab^{-1} belongs to the set $\widetilde{M}(\Gamma)$.*

From Theorem 3.2 and Corollary 3.2, it also follows

Corollary 3.3. *The operator $A = aP_\tau + bQ_\tau$ is Noetherian in the space $L_2(\Gamma, p)$ if and only if the function $ab^{-1}h$ belongs to the set $\widetilde{M}(\Gamma)$.*

4. Noetherian conditions for systems of singular integral operators

We will study the operator $A = aP_\tau + bQ_\tau$ in the case where the coefficients of the operator are matrices of order m with elements in $L_\infty(\Gamma)$.

Let $\mathfrak{A}$ be a Banach space of functions, defined on the contour Γ. By $\mathfrak{A}^{m\times m}$ we denote the Banach algebra of matrix functions of order m with elements in $\mathfrak{A}$.

Definition 4.1. *Let the contour Γ be closed and $a \in GL^{l\times l}(\Gamma)$. We will say that the matrix a admits a factorization [13] in the space $L^l_p(\Gamma, p)$, if it can be represented in the form*

$$a = a_- \, d \, a_+, \tag{4.1}$$

where $d = \left(\delta_{jk}(t - z_0)^{k_j}\right)^l_{j,k=1}$, $z_0 \in F^+_\Gamma$ and k_l are integers ($k_1 \geq k_2 \geq \cdots \geq k_j$), and the matrices $a_\pm$ satisfy the conditions:

(i) $a_- \in^- L^{l\times l}_p(\Gamma, p)$, $\quad a_+ \in^+ L^{l\times l}_q(\Gamma, p^{1-q})$ and $a_-^{-1} \in^- L^{l\times l}_q(\Gamma, p^{1-q})$, $\quad a_+^{-1} \in^+ L^{l\times l}_p(\Gamma, p)$, $(p^{-1} + q^{-1} = 1)$;

(ii) the operator $a_+^{-1} P_\Gamma a_-^{-1} I$ is bounded in the space $L^l_p(\Gamma, p)$, where by P_Γ is denoted the operator $(\delta_{jk} P_\Gamma)^l_{j,k=1}$.

Theorem 4.1. *The matrix $a \in GL^{l\times l}_\infty(\Gamma)$ admits a factorization in the space $L^l_p(\Gamma, p)$ if and only if the operator*

$$A = aP_\tau + Q_\tau \qquad (Q_\tau = I - P_\tau)$$

is Noetherian in the space $L^l_p(\Gamma, p)$.

Theorem 4.2. *Let the matrix $a \in GL^{l\times l}_\infty(\Gamma)$ admits a factorization in the space $L^l_p(\Gamma, p)$, where*

$$k_1 \geq k_2 \geq \cdots \geq k_s \geq 0, \quad k_{s+1} \leq k_{s+2} \leq \cdots \leq k_j < 0 \quad (0 \leq s \leq l),$$

then

$$\dim\ker A = -\sum_{i=s+1}^{l} k_i, \qquad \dim\operatorname{coker} A = \sum_{i=1}^{s} k_i.$$

The proof of these theorems is done in a similar way to the one in Theorem 3.1 of [13]. Let the function h be defined in [11]. From Theorems 3.1 and 3.2 and from the above, it follows

Corollary 4.1. *The matrix $a \in GL^{l\times l}_\infty(\Gamma)$ admits a factorization in $L^l_p(\Gamma, p)$, if and only if the matrix $b = ha$ admits a factorization in the space $L^l_p(\Gamma)$. If $b = b_+ d b_-$, then $a = h_+^{-1} b_+ d b_- h_-^{-1}$ represents the factorization of the matrix a in $L^l_p(\Gamma, p)$.*

Corollary 4.2. *The operator $aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, p)$, if and only if the operator $A_h = haP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma)$. Furthermore,*

$$\operatorname{Ind} A\big|_{L_p^l(\Gamma,p)} = \operatorname{Ind} A_h\big|_{L_p^l(\Gamma)}.$$

As we mentioned, if $a, b \in L_\infty(\Gamma)$ and $b(t) \geq \delta > 0$, then the operators $A = aP_\Gamma + Q_\Gamma$ and $B = baP_\Gamma + Q_\Gamma$ are or are not Noetherian in the space $L_p(\Gamma, p)$ and $\operatorname{Ind} A = \operatorname{Ind} B$.

We will show that this property is not possessed by operators of the form A and B in the space $L_p^l(\Gamma, p)$. Moreover, we will show that for any p $(1 \leq p < \infty)$ and any natural l $(l \geq 2)$ there exist two matrices $a, b \in L_\infty^{l\times l}(\Gamma)$, such that sets[2] $b(t) \geq \delta > 0$, $baP_\Gamma + Q_\Gamma$ are Noetherian in $L_p^l(\Gamma, p)$, but the operator $aP_\Gamma + Q_\Gamma$ is not Noetherian in this space.

Based on Corollary 4.2, we can consider $p(t) = 1$. Obviously, it is enough to consider $l = 2$ and $\Gamma = \Gamma_0$, where Γ_0 is the unit circle. As the matrices a and b will have elements from $CP(\Gamma)$, then we can also consider $p = 2$. So, let $t = e^{i\theta}$ $(0 \leq \theta < 2\pi)$,

$$a(t) = \begin{pmatrix} 2 - \cos\frac{\theta}{2} & -1 \\ -4 + \sin\frac{\theta}{2} & 2 - \cos\frac{\theta}{2} \end{pmatrix}, \qquad b(t) = \begin{pmatrix} 3 + 2\cos\frac{\theta}{2} & 1 \\ 1 & 2 - \cos\frac{\theta}{2} \end{pmatrix}.$$

The matrices a and b are continuous at any point $t \in \Gamma_0 \setminus \{1\}$. Under these conditions the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian (see [7]) in the space $L_2^2(\Gamma_0)$, if and only if

$$\det\big[\mu a(t+0) + (1-\mu)a(t-0)\big] \neq 0 \quad (t \in \Gamma_0,\ 0 \leq \mu \leq 1).$$

It is directly verified that for $\mu_0 = \frac{1}{2}$ and $t_0 = 1$ we have

$$\det\big[\mu_0 a(t_0+0) + (1-\mu_0)a(t_0-0)\big] = 0.$$

Therefore, the operator $A = aP_{\Gamma_0} + Q_{\Gamma_0}$ is not Noetherian in the space $L_2^2(\Gamma_0)$. Let $c = ab$.

It is easy to show that

$$\det\big[\mu c(t+0) + (1-\mu)c(t-0)\big] \neq 0 \quad (t \in \Gamma_0,\ 0 \leq \mu \leq 1),$$

i.e. the operator $B = abP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in $L_2^2(\Gamma_0)$.

Theorem 4.3. *Let the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ satisfy the following conditions:*

(i) for any $\tau \in \Gamma \setminus \{\tau_1, \ldots, \tau_m\}$ there exists a neighborhood $u(\tau)$ and a pair of matrices g_τ^+ and g_τ^-, $(g_\tau^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma)$, such that

$$\operatorname{Re}^3\big(g_\tau^+(t)a(t)g_\tau^-(t)\big) \geq \delta_\tau > 0 \quad (t \in u(\tau));$$

[2]Recall that $H \geq M$, where H and M are Hermitian matrices, means that $\sum_{i=1}^l \sum_{j=1}^l (c_{ij} - d_{ij})(x_i - \bar{x}_j) \geq 0$, where c_{ij} and d_{ij} are the elements of the matrices H și M, respectively.

[3]Recall that $\operatorname{Re} B = \frac{1}{2}(B + B^*)$.

(ii) there exist the limits $a_j(\tau_k \pm 0)$;

(iii) the spectrum of matrices $a^{-1}(\tau_k+0)\cdot a(\tau_k-0)$ $(k=1,\dots,m)$ have no common points with the semiaxis $\mathbb{R}^-$.

Then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2^l(\Gamma)$.

The proof can be done using the results of the paper [13] and the local principle from [2] and [10].

If the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ is piecewise continuous, then the conditions of Theorem 3.3 are also necessary for the operator $A = aP_\Gamma + Q_\Gamma$ to be Noetherian in the space $L_2^l(\Gamma)$. Indeed, let A be Noetherian in the space $L_2^l(\Gamma)$. Then from [13] we deduce that the spectrum of the matrix $a^{-1}(\tau+0)\cdot a(\tau-0)$ $(\tau\in\Gamma)$ has no common points with the semiaxis R^-. It remains to use the following lemma.

Lemma 4.1. *Let B and C be two non-degenerate numerical matrices. For there to exist numerical matrices M and N such that*

$$\operatorname{Re}(MBN) > 0 \quad \textit{and} \quad \operatorname{Re}(MCN) > 0 \tag{4.2}$$

it is necessary and sufficient that the spectrum of the matrix $B^{-1}C$ has no points in common with R^-.

Proof. Let X be the matrix that reduces the matrix $B^{-1}C$ to the Jordan form,

$$XB^{-1}CX^{-1} = \operatorname{diag}(G_1,\dots,G_s),$$

where

$$G_k = \begin{pmatrix} \lambda_k & \cdot & \cdot & 0 \\ 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 1 & \lambda_k \end{pmatrix},$$

and $\operatorname{diag}(G_1,\dots,G_s)$ is the diagonal matrix with the elements G_k $(k=1,\dots,s)$ on the diagonal.

As $\lambda_k \notin R^-$, there exists a number $\gamma_k \in \mathbb{C}$, such that $\operatorname{Re}\gamma_k > 0$ and $\delta_k = \operatorname{Re}(\gamma_k\lambda_k) > 0$. Let

$$F_k = \begin{pmatrix} \varepsilon_k^{l_k-1}\gamma_k & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_k^{l_k-2}\gamma_k & 0 & \cdots & 0 \\ 0 & 0 & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \gamma_k \end{pmatrix},$$

where l_k is the order of the matrix G_k, ε_k is a positive number. Direct calculations show that the principal minor of order r of the matrix $\mathrm{Re}(G_kF_k)$ has the form $\varepsilon_k^{\omega_k} g(\varepsilon_k)$, where $\omega_k = (2l - r - 1)\frac{r}{2}$, and $g(\varepsilon_k) \to \delta_k^r$ for $\varepsilon_k \to 0$. It follows that we can choose $\varepsilon_k > 0$, such that $\mathrm{Re}(G_kF_k) > 0$. We denote $N = X^{-1}F$, where $F = \mathrm{diag}(F_1, \dots, F_s)$, and $M = XB^{-1}$. Based on what has been demonstrated, the matrices $MBN(= F)$ and $MCN = XB^{-1}CX^{-1}F$ satisfy conditions (4.2). The sufficiency is proved.

The necessity of conditions (4.2) can be demonstrated as follows. Consider on the unit circle Γ_0 a nondegenerate matrix a of continuous functions at any point $z \neq 1$ and $a(1 + 0) = B$, $a(1 - 0) = C$. If there are two matrices M and N such that $\mathrm{Re}(MBN) > 0$ and $\mathrm{Re}(MCN) > 0$, then in the space $L_2^l(\Gamma_0)$ the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is locally Noetherian at the point $z = 1$. Since $\det a(z) \neq 0$ for $z \neq 1$ and a is continuous at any $z \in \Gamma_0 \setminus \{1\}$, it follows that the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is locally Noetherian also at these points of the contour Γ_0. Therefore, the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in $L_2^l(\Gamma_0)$. From Theorem 2 of [13], it follows that the spectrum of the matrix $B^{-1}C$ has no points in common with R^-. The lemma is proved. □

Next, to make the exposition easier, we will assume that the contour Γ is made up of two closed curves γ_1 and γ_2 of Liapunov type on portions with a single common point t_0. At the point t_0 the tangents to γ_1 and γ_2 form an angle of measure $\frac{\pi}{2}$.

In the following we will establish a class of matrices $L^{l\times l}(\Gamma)$ (which also contains the unitary matrices) such that the operator $aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, \rho)$, where

$$\rho(t) = \prod_0^n |t - t_k|^{\beta_k},$$

t_0, t_1, $\dots$, t_n are distinct points on Γ in which all the angular points of the contour Γ are included and $-1 < \beta_k < p - 1$ $(k = 0, \dots, n)$.

To begin with, we will consider $p = 2$. Let $\tau \in \gamma_1 \setminus \{t_0\}$ $(\tau \in \gamma_2 \setminus \{t_0\})$ and z_1 (z_2) be fixed points of $F_{\gamma_1}^+$ $(F_{\gamma_2}^+)$. We denote by $h_\tau(t)$ the function $(t - z_1)^{-\beta(\tau)/2}((t - z_2)^{-\beta(\tau)/2})$, where $\beta(\tau) = \beta_k$ for $\tau = t_k$ $(k = 0, \dots, n)$ and $\beta(\tau) = 0$ at the other points of the contour Γ, and β_k and t_k respectively are the numbers and points that define the weight $\rho(t)$.

For $\tau = t_0$ we define the function $h_\tau(t)$ as follows:

$$h_{t_0}(t) = (t - z_1)^{\sigma_1}(t - z_2)^{\sigma_2},$$

where σ_1 and σ_2 are real numbers satisfying the condition $\sigma_1 + \sigma_2 = -\dfrac{\beta_0}{2}$. The functions $(t - z_j)^{-\beta(\tau)/2}$ $((t - z_j)^{\sigma_j})$ $(j = 1, 2)$ are defined so that they are continuous on the complex plane with the cut that joins the point z_j with ∞ and intersects the contour Γ at a single point τ (t_0).

Theorem 4.4. *The operator*

$$A = aP_\Gamma + bQ_\Gamma \quad (a, b \in L_\infty^{l\times l}(\Gamma))$$

in the space $L_2^l(\Gamma,\rho)$ is locally Noetherian at the point $\tau \in \Gamma$ if and only if this property is possessed by the operator

$$A_\tau = h_\tau a P_\Gamma + bQ_\Gamma$$

in the space $L_2^l(\Gamma)$.

The proof is based on the factorization of the function h_τ, applying Theorem 2.2, therefore we do not dwell on it.

Consider the following function (see [8])

$$\theta(t) = \begin{cases} \operatorname{arctg}\left[\dfrac{1}{2}\max\limits_{-1\le z\le 1}\left|(1+z)\left(\dfrac{1-z}{1+z}\right)^{\delta(t)} - (1-z)\left(\dfrac{1+z}{1-z}\right)^{\delta(t)}\right|\right], & t \in \Gamma\setminus\{t_0\}, \\ \dfrac{\pi}{4}, & t = t_0, \end{cases} \tag{4.3}$$

where $\delta(t) = \dfrac{\alpha(t)}{2\pi}$, $\alpha(t) = \min\bigl(\alpha_0(t),\, 2\pi - \alpha_0(t)\bigr)$, and $\alpha_0(t)$ is the angle formed by the contour Γ at the point $t \ne t_0$.

We denote by $M_p^l(\Gamma)$ the set of function matrices $a(t) = \bigl(a_{jk}(t)\bigr)_{j,k=1}^l$ $(t \in \Gamma)$ of order l with elements $a_{jk}(t) \in L_\infty(\Gamma)$ that satisfy the following conditions:

(i) $\operatorname{Vraiinf}\limits_{t\in\Gamma} \det|a(t)| > 0$;

(ii) for any point $\tau \in \Gamma$ except perhaps for a finite number of points τ_k $(k = 1, 2, \dots, s)$, there exists a neighborhood $u(\tau) \subset \Gamma$ of the point τ and two function matrices $g_\tau^\pm$, $(g_\tau^+)^{\pm 1} \in^+ L_\infty^{l\times l}(g_\tau^-)^{\pm1} \in^- L_\infty^{l\times l}$ such that for any $t \in u(\tau)$ the relation

$$\operatorname{Re}\bigl(g_\tau^+(t) h_\tau a(t) g_\tau^-(t)\bigr) > c(\tau)\cos\theta(\tau), \tag{4.4}$$

where $\theta(\tau)$ is the function defined by relation (3.3) and $c(\tau)$ is the norm of the operator $h_\tau a$ in the space $L_2^l(u(\tau))$. As, it is known, $c(\tau) = \sup_{t\in u(\tau)} s_1\bigl(h_\tau(t)a(t)\bigr)$, where $s_1(h_\tau a)$ is the largest eigenvalue of the matrix $(h_\tau a a^* h_\tau^*)$;

(iii) at the points τ_k $(k = 1, 2, \dots, s)$ there exist the finite limits $a(\tau_k - 0)$ and $a(\tau_k + 0)$ of the matrix $a(t)$ and the spectrum of the matrix $e^{-\pi i\beta(\tau_k)} a^{-1}(\tau_k + 0)a(\tau_k - 0)$ does not intersect the negative semi-axis $\mathbb{R}^-$.

It is easy to see that if all points τ_k $(k = 1, 2, \dots, s)$ where the limits $a(\tau \pm 0)$ exist are regular, then conditions (iii) are equivalent to conditions (ii). This can be deduced using Lemma 3.1.

Theorem 4.5. *Let $a \in M_p^l(\Gamma)$. Then the operator*

$$A = aP_\Gamma + Q_\Gamma$$

is Noetherian in the space $L_2^l(\Gamma, \rho)$.

We note that if Γ is a closed contour of Lyapunov type and $\rho(t) = 1$, then this theorem was proved by I. Simonenko (see [10], Theorem 8), and the set $M_1^l(\Gamma)$ coincides with the class $\widetilde{M}(2, \Gamma)$ defined in [10]. Given that Γ is a simple Lyapunov-type contour, Theorem 4.3 is contained in the work [14].

We will prove two lemmas from which we will immediately deduce Theorem 4.5 and which, at the same time, are also of interest in themselves.

Lemma 4.2. *Let the contour Γ_0 have a single corner point t_0 and $\rho_0(t) = |t - \tau_0|^\beta$ $(-1 < \beta < 1)$. If the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ satisfies the condition*

$$\operatorname{Re}\big(h_{\tau_0}(t)a(t)\big) \geq \sigma > \|h_{\tau_0}a\| \cos\theta(\tau_0),$$

then the operator

$$A_0 = aP_{\Gamma_0} + Q_{\Gamma_0}$$

is Noetherian in the space $L_2^l(\Gamma, \rho)$ and its index are equal to zero.

Proof. Based on Corollary 3.3, it suffices to prove that the operator

$$h_0 a P_{\Gamma_0} + Q_{\Gamma_0} \qquad (h_0(t) = h_{\tau_0}(t))$$

is Noetherian in the space $L_2^l(\Gamma)$. Let $\varphi \in L_2^l(\Gamma_0, \rho)$ and $\|\varphi\| = 1$, then

$$\left\| \left(I - \frac{\sigma_{h_0} a}{\|h_0 a I\|^2}\right) \varphi \right\|^2 \leq 1 - \frac{2\sigma \operatorname{Re}(h_0 a)}{\|h_0 a I\|^2} + \frac{\sigma^2}{\|h_0 a I\|^2} \leq 1 - \frac{\sigma^2}{\|h_0 a I\|^2}.$$

Consequently,

$$\left\| I - \frac{\sigma h_0 a}{\|h_0 a I\|^2} \right\| \leq \sin\theta(\tau_0).$$

As (see [9]),

$$\inf_{T\in\mathfrak{T}} \|P_{\Gamma_0} + T\| = \frac{1}{\sin\theta(\tau_0)},$$

where $\mathfrak{T} = \mathfrak{T}(L_2^l(\Gamma_0))$ is the set of all compact operators in the space $L_2^l(\Gamma_0)$, then

$$\inf_{T\in\mathfrak{T}} \left\| \left(I - \frac{\sigma h_0 a}{\|h_0 a I\|^2}\right) P_{\Gamma_0} + T \right\| < 1.$$

It follows that the operator

$$I - \left(I - \frac{\sigma h_0 a}{\|h_0 a I\|^2}\right) P_{\Gamma_0}$$

is Noetherian in the space $L_2^l(\Gamma_0)$ and its index is equal to zero. We write the operator $h_0aP_{\Gamma_0} + Q_{\Gamma_0}$ in the form

$$h_0aP_{\Gamma_0} + Q_{\Gamma_0} = \Big(I - (I - \omega h_0 a)(\omega^{-1}P_{\Gamma_0} + Q_{\Gamma_0})\Big),$$

where $\omega = \sigma\|h_0aI\|^{-2}$. The operator $\omega^{-1}P_{\Gamma_0} + Q_{\Gamma_0}$ is invertible, therefore the operator $h_0aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian and its indices are equal to zero. The lemma is proved. □

Lemma 4.3. *Let the matrix of functions $a \in GL^{l\times l}(\Gamma_0)$ satisfy the following conditions:*

(i) there are two matrices $g_\tau^\pm$ such that $(g^+)^{\pm1} \in^+ L_\infty^{l\times l}(\Gamma_0)$, $(g^-)^{\pm1} \in^- L_\infty^{l\times l}(\Gamma_0)$;

(ii) $\mathrm{Re}\big(g^+(t)h_0(t)a(t)g^-(t)\big) \geq \sigma \geq \|h_0a\|\cos\theta(\tau_0)$,

then the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2^l(\Gamma_0,\rho)$.

The proof of the lemma follows from Lemma 4.2, from the equality

$$h_0aP_{\Gamma_0} + Q_{\Gamma_0} = (g^-)^{-1}\big(g^+h_0ag^-P_{\Gamma_0} + Q_{\Gamma_0}\big)\big((g^+)^{-1}P_{\Gamma_0} + g^-Q_{\Gamma_0}\big),$$

and from the invertibility of the operators $(g^-)^{-1}I$ and $g^+P_{\Gamma_0} + g^-Q_{\Gamma_0}$.

Proof. We give the proof of Theorem 4.5. From conditions (i) and (iii) of the definition of the set $M_\rho^l(\Gamma)$, it follows that the operator

$$A = aP_\Gamma + Q_\Gamma$$

is locally Noetherian at the points τ_k $(k = 1,2,\ldots,s)$. From conditions (ii) in the usual way, with the help of Lemma 4.3 and the local principle, it is established that the operator $aP_\Gamma + Q_\Gamma$ is locally Noetherian also at the points $\tau \in \Gamma\setminus\{\tau_1,\ldots,\tau_s\}$. Consequently the operator $aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2^l(\Gamma,\rho)$. The theorem is proved. □

Corollary 4.3. *Let $a, b \in L_\infty^{l\times l}(\Gamma)$. If*

$$\mathrm{Vraiinf}_{t\in\Gamma}\det|b(t)| > 0$$

and $b^{-1}a \in M_\rho^l(\Gamma)$, then the operator $aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2^l(\Gamma,\rho)$.

We note that the set $M_\rho^l(\Gamma)$ is quite large. This follows from the following corollary.

Corollary 4.4. *Let the functions a and $b \in CP_l(\Gamma)$. Then the operator $aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2^l(\Gamma,\rho)$ if and only if $b^{-1}a \in M_\rho^l(\Gamma)$.*

An important role in establishing Theorems 4.2 and 4.3 was played by the knowledge of the essential norms in the space $L_2^l(\Gamma, \rho)$ of the operators S_Γ, P_Γ and of the multiplication operator on the matrix a. If in the space $L_p^l(\Gamma, \rho)$ the norm is defined by the equality

$$\|\varphi\|^p = \sum_{j=1}^{l} \|\varphi_j\|^p,$$

then for $p = 2$, the norm of the operator A_0 is calculated from the relation

$$\|A_0\|_{L_2^l(\Gamma,\rho)} = \operatorname{esssup}_{t\in\Gamma} s(t), \tag{4.5}$$

where $s(t)$ is the largest eigenvalue of the matrix $\left(a^*(t)a(t)\right)^{1/2}$. Relation (4.5) has an important role in determining the Noetherian conditions for singular operators. Simple examples show us, however, that for $p \neq 2$ relation (4.5) is false. In order to use the essential norm of the operators S_Γ and P_Γ also for $p \neq 2$, we will modify the norm in the space $L_p^l(\Gamma, \rho)$ in such a way that (4.5) holds.

Lemma 4.4. *If in the space $L_p^l(\Gamma, \rho)$ the norm is defined by the equality*

$$\|\varphi\|^p = \int_\Gamma \left(\sum_{j=1}^{l} |\varphi_j(t)|^2 \right)^{p/2} \rho(t)\,|dt|, \tag{4.6}$$

then the relation (4.5) holds.

Proof. Obviously, for any $t_0 \in \Gamma$, the norm of the operator $a(t_0)I$ in the Euclidean space $\mathbb{C}^l$ is equal to $s(t_0)$. Therefore the norm of A_0 is equal to $\operatorname{esssup}_{t\in\Gamma} s(t)$ and the lemma is proven. □

We will assume that the norm in $L_p^l(\Gamma, \rho)$ is defined by (4.6). In this case (see [15]), the equalities

$$\|S_{\Gamma_0}\|_{L_p^l(\Gamma_0,|t-1|^{p-2-\beta}|t+1|^\beta)} = \|S_{\Gamma_0}\|_{L_p(\Gamma_0,|t-1|^{p-2-\beta}|t+1|^\beta)}$$

where Γ_0 is the unit circle and $\beta \in (-1, p-1)$. From this, in particular, it follows that

$$\|S_{\Gamma_0}\|_{L_p^l(\Gamma_0)} = \begin{cases} ctg\dfrac{\pi}{2p}, & \text{for } p \geq 2, \\ tg\dfrac{\pi}{2p}, & \text{for } 1 < p \leq 2, \end{cases}$$

and

$$\|S_{\Gamma_0}\|_{L_2^l(\Gamma_0,|\frac{t+1}{t-1}|^\beta)} = ctg\pi\frac{(1-|\beta|)}{4} \qquad (|\beta| < 1).$$

Let Γ be a closed Lyapunov contour. We denote by G_δ the set of all function matrices a in $L_\infty^{l\times l}(\Gamma)$ that satisfy the following conditions:

(i) for any point $\tau \in \Gamma$ there exists a neighborhood $u(\tau)$ of this point and two matrices g_τ^+ and g_τ^-, such that

$$(g_\tau^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma), \qquad (g_\tau^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma);$$

(ii) for $t \in u(\tau)$, the matrix $g_\tau^-(t)a(t)g_\tau^+(t)$ is unitary and $\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)\big)$ lies in an angle $\lambda_\tau(\delta)$ with the vertex at the point $z = 0$ and of measure δ.

Condition (ii) is equivalent to the fact that the set of values of the form

$$\big(g_\tau^-(t)a(t)g_\tau^+(t)\eta, \eta\big) \qquad (\eta \in \mathbb{C}^l,\ \|\eta\| = 1,\ t \in u(\tau))$$

lies inside the angle $\lambda_\tau(\delta)$.

Theorem 4.6. *Let $a \in G_\delta$, where*

$$tg\frac{\delta}{2} = \min\left(l^{\frac{2-p}{2p}}\, tg\frac{\pi}{2p}, l^{\frac{2-p}{2p}}\, ctg\frac{\pi}{2p}\right).$$

Then the operator $A = aP_+ + Q_-$ is Noetherian in the space $L_p^l(\Gamma)$.

The *proof* of this theorem is done in the usual way, using the local principle and the following lemma.

Lemma 4.5. *Let Γ_0 be the unit circle and b be a unitary matrix in $L_\infty^{l\times l}(\Gamma_0)$, and for any $t \in \Gamma_0$ its spectrum lies inside the angle $|\arg z| \le \alpha$, where*

$$tg\frac{\alpha}{2} = \min\left(l^{\frac{2-p}{2p}}\, tg\frac{\pi}{2p}, l^{\frac{2-p}{2p}}\, ctg\frac{\pi}{2p}\right).$$

Then the operator $B = bP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_p^l(\Gamma_0)$.

Proof. The operator B can be written in the form

$$B = \frac{1}{2}(b+e)\big(I + (b+e)^{-1}(b-e)S_{\Gamma_0}\big),$$

where e is the unit matrix of order l. For $l = 1$ there exists [14] an operator K_0 of rank one, such that in the real space $L_p(\Gamma_0)$, we have

$$\|S_{\Gamma_0} - K_0\|_{L_p(\Gamma_0)} = \begin{cases} ctg\dfrac{\pi}{2p}, & \text{for } p \ge 2,\\ tg\dfrac{\pi}{2p}, & \text{for } 1 < p \le 2.\end{cases}$$

From [5] we deduce that this relation is also true in the complex space $L_p(\Gamma_0)$. As the norm in $L_p^l(\Gamma_0)$ is defined by (3.6), then it is easy to deduce the equality

$$\big\|S_{\Gamma_0} - K_0\big\|_{L_p^l(\Gamma_0)} = \begin{cases} l^{\frac{2-p}{2p}}\, ctg\dfrac{\pi}{2p}, & \text{for } p \ge 2,\\ l^{\frac{2-p}{2p}}\, tg\dfrac{\pi}{2p}, & \text{for } 1 < p \le 2,\end{cases} \tag{4.7}$$

By K_0 in the space $L_p^l(\Gamma_0)$ we mean the operator $\operatorname{diag}(K_0, \dots, K_0)$. Let $f = i(b + e)^{-1}(b - e)$. The matrix f is self-adjoint and for any $t \in \Gamma_0$ its spectrum lies on the segment $\left[-tg\frac{\alpha}{2}, tg\frac{\alpha}{2}\right]$. Let $\varphi = \{\varphi_j\}_{j=1}^{l}$ and $f\varphi = \{\psi_j\}_{j=1}^{l}$, then

$$\sum_{j=1}^{l} |\psi_j(t)| \le tg^2\frac{\alpha}{2}\sum_{j=1}^{l} |\varphi_j(t)|^2.$$

Therefore, $\|f\| \le tg\frac{\alpha}{2}$. From relations (4.7) and the assumptions of the lemma it follows that if $\|f(S_{\Gamma_0} - K_0)\| < 1$. Thus, the operator $I - ifS_{\Gamma_0}$ is Noetherian (invertible). Just as the multiplication operator on the matrix $b + e$ is invertible, it follows that the operator B is Noetherian. The lemma is proved. □

Let

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \qquad (t_k \in \Gamma - 1 < \beta_k < p - 1).$$

We denote by $G_{\delta,\rho}$ the set of all matrices a in $L_\infty^{l\times l}(\Gamma)$ that satisfy the following conditions:

(i) for any $\tau \in \Gamma \setminus \{t_1, \dots, t_n\}$ there is a neighborhood $u(\tau)$ of this point and two g_τ^+ and g_τ^- such that $(g_\tau^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma)$;

(ii) for $\triangledown t \in u(\tau)$ the matrix $g_\tau^-(t)a(t)g_\tau^+(t)$ is unitary and its spectrum lies in an angle $\lambda_\tau(\delta)$ with the vertex at the point $z = 0$ and of measure δ:

$$\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)\big) \subset \lambda_\tau(\delta);$$

(iii) there is a neighborhood $u(t_k)$ of the point t_k $(k = 1, \dots, n)$ and the pair of matrices g_k^+ and g_k^-, $(g_k^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma)$, $(g_k^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma)$ such that, for any $\triangledown t \in u(t_k)$, the matrix

$$g_\tau^-(t)a(t)g_\tau^+(t)\, g_\tau^-(t)a(t)g_\tau^+(t)$$

is unitary and

$$\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)\big) \subset \lambda_k(\delta) \quad (\textit{for } t \in u^+(t_k)),$$

$$\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)e^{-\frac{2\pi i\beta_k}{p}}\big) \subset \lambda_k(\delta) \quad (\textit{for } t \in u^-(t_k)),$$

where

$$u^+(t_k) = \{t : t \in u(t_k),\ t > t_k\}, \qquad u^-(t_k) = \{t : t \in u(t_k),\ t < t_k\}.$$

It follows from Theorem 4.4.

Theorem 4.7. *Let $a \in G_{\delta,\rho}$, where*

$$tg\frac{\alpha}{2} = \min\left(l^{\frac{2-p}{2p}} tg\frac{\pi}{2p}, l^{\frac{2-p}{2p}} ctg\frac{\pi}{2p}\right).$$

Then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, \rho)$.

5. Observations on the case of the unbounded contour. Generalizations

Let Γ be an unbounded and admissible contour, $0 \notin \Gamma$ and

$$\rho(z) = |z|^\beta \prod_{k=1}^{n} |z - z_k|^{\beta_k} \quad (z_k \in \Gamma,\ -1 < \beta_k < p-1,\ -1 < \beta + \sum_{k=1}^{n} \beta_k < p-1).$$

In the space $L_p^l(\Gamma, \rho)$, we consider the operator

$$A = aP_\Gamma + Q_\Gamma,$$

where $a, b \in L_\infty^{l\times l}(\bar\Gamma)$ $(\bar\Gamma = \Gamma \cup \infty)$. With the help of the invertible operator B, defined by the relation

$$(B\varphi)(z) = \frac{1}{z}\,\varphi\left(\frac{1}{z}\right), \tag{5.1}$$

the study of the operator A in the space $L_p^l(\Gamma, \rho)$ reduces to the study of the operator

$$\tilde A = \tilde a P_{\tilde\Gamma} + \tilde b Q_{\tilde\Gamma},$$

where $\tilde\Gamma = \{t : t = \frac{1}{z},\ z \in \bar\Gamma\}$, $\tilde a(t) = a\left(\frac{1}{t}\right)$, $\tilde b(t) = b\left(\frac{1}{t}\right)$, in the space $L_p^l(\tilde\Gamma, \tilde\rho)$ with the weight

$$\tilde\rho(z) = |z|^{\beta_0} \prod_{k=1}^{n} |z - z_k|^{\beta_k}, \qquad \left(\beta_0 = p - 2 - \beta - \sum_{k=1}^{n} \beta_k,\ t_k = \frac{1}{z_k}\right).$$

Thus, the theorems presented for the operator $\tilde A$ in the space $L_p^l(\tilde\Gamma, \tilde\rho)$ are easily transposed to the operator A in the space $L_p^l(\Gamma, \rho)$. Similar results to the Theorems 4.2 – 4.5 hold for the operator $A = aP_\Gamma + Q_\Gamma$ in the case of the unbounded contour, which we will not explain. We will focus on the case where the integration contour Γ consists of a finite number of bounded and unbounded curves.

So, let Γ be such a contour,

$$\Gamma = \bigcup_{j=0}^{m} \Gamma_j,$$

where Γ_0 is unbounded and admissible and Γ_j $(j = 1, \dots, m)$ are bounded. For a start, we will assume that all the curves Γ_j are closed and disjoint. If

$$\rho(z) = |z|^{\beta} \prod_{k=1}^{n} |z - z_k|^{\beta_k} \quad (0 \notin \Gamma),$$

then by ρ_j $(j = 0, 1, \dots, m)$ we will denote the restriction of the function $\rho(z)$ on Γ_j $(j = 0, 1, \dots, m)$. We also denote by R_j the multiplication operator to the characteristic function of the contour Γ_j:

$$(R_j\varphi)(z) = \begin{cases} \varphi(z), & \text{for } z \in \Gamma_j, \\ 0, & \text{for } z \in \Gamma \setminus \Gamma_j. \end{cases}$$

We consider the operator

$$A = aP_\Gamma + Q_\Gamma,$$

where $a, b \in L_\infty^{l\times l}(\overline{\Gamma})$. As

$$\sum_{k=0}^{m} R_k = I,$$

then the operator A can be written in the form

$$A = \sum_{j=0}^{m} R_j A \sum_{k=0}^{m} R_k = \sum_{j,k=0}^{m} R_j A R_k. \tag{5.2}$$

The operators R_jAR_j represent the restriction of A on the contour Γ_j $(j = 0, i, \dots, m)$ that were studied in the previous sections. As $\Gamma_j \cap \Gamma_k = \varnothing$ for $j \neq k$, the operator R_jAR_k $(j \neq k)$ is an integral operator with continuous kernel on $\Gamma_j \times \Gamma_k$ and, therefore, is compact in the space $L_p^l(\Gamma, \rho)$. From these arguments it follows the following assertion.

Theorem 5.1. *The operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, \rho)$ if and only if the operators*

$$R_jAR_j = A = aP_{\Gamma_j} + Q_{\Gamma_j}$$

are Noetherian, respectively, in the spaces $L_p^l(\Gamma_j, \rho_j)$. In addition,

$$\operatorname{Ind} A\big|_{L_p^l(\Gamma,\rho)} = \sum_{j=0}^{m} \operatorname{Ind} A_j\big|_{L_p^l(\Gamma_j,\rho_j)}. \tag{5.3}$$

On the other hand, with the help of the results of the paper [6] we can extend the obtained results to the case where the contour Γ also contains open Lyapunov curves on portions, and by appealing to Theorems 4.2, 4.6 and the local principle, we can consider the case where the curves curbele Γ_j $(j = 0, 1, \dots, m)$ have a finite number of intersection points, in particular the point $z = \infty$. In this paper we will not dwell on these details.

6. Conclusions

In the paper, the notions of equivalence and local inversibility were effectively applied to the determination of Noetherian conditions for singular integral operators with scalar or matrix coefficients, bounded and measurable. In certain cases, specified in the paper, the obtained leads are necessary and sufficient in which the studied operators are Noetherian. The Noetherian conditions, as well as the index of the operators, depend on the space in which the research is carried out, but also on the sizes of the angles formed by the contour at its corner points. At the same time, an important role in establishing the main theorems in this paper was played by the sizes of the essential norms of the Riesz projection operators, established by the author in the papers [8], [9, [11], as well as by the authors of the papers [7], [13], [14], [17]. The methods and ideas used, and the results presented in this paper can be used in the subsequent study of singular operators in order to expand the class of integration contours, research spaces, and operator coefficients.

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