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# Lie algebra admitted by the differential system $s^3(3)$ and functional basis of invariants polynomials

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**Abstract.** The Lie algebra of the representation of the centro-affine group in the space of coefficients, phase variables and covariant vector coordinates for the ternary differential system with third-degree homogeneities $s^3(3)$ was obtained. Using this algebra, the number of elements in the functional basis of invariant centro-affine polynomials (invariants, comitants, contravariants, mixed comitants) was determined and irreductible elements from these bases were obtained, and for the comitants and contravariants a functional basis was constructed.

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**Keywords:** ternary differential systems, Lie algebras, functional bases, invariants, comitants, contravariants, mixed comitants.

# Algebra Lie admisă de sistemul diferențial $s^3(3)$ şi baze funcţionale de polinoame invariante

**Rezumat.** A fost obţinută algebra Lie a grupului centroafin de reprezentare în spaţiul coeficienţilor, variabilelor de fază şi coordonatele vectorului covariant pentru sistemul diferenţial ternar cu omogenităţi de gradul trei $s^3(3)$. Cu ajutorul acestei algebre a fost determinat numărul de elemente din baza funcţională a polinoamelor centroafin invariante (invarianţi, comitanţi, contravarianţi, comitanţi micşti) şi obţinute elemente ireductibile din aceste baze, iar pentru comitanţi şi contravarianţi o bază funcţională a lor.

**Cuvinte-cheie:** sisteme diferențiale ternare, algebre Lie, baze funcționale, invarianți, comitanți, contravarianți, comitanți micști.

## 1. Introduction

We examine the ternary differential system $s^3(3)$ of the form

$$\frac{dx^j}{dt} = a^j_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma \equiv P^j \ (j, \alpha, \beta, \gamma = \overline{1,3}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma}$ is symmetrical in lower indices in which the complete convolution holds. Let $GL(3, \mathbb{R})$ be the group of centro-affine transformations of the phase variables of the system (1).

Ternary differential systems are applied in diverse fields, including physics (rigid body dynamics, laser physics, geophysics, and circuit theory), biology (population dynamics and infectious disease models), economics, and engineering (missile defense systems and optimal control of physical processes like continuous casting).

Ternary cubic differential system has applications in many domains, for example: physic Kirchhoff equation for elastic rods with intrinsic curvature [1].

It is known [9] that the centro-affine group of transformations $GL(3,\mathbb{R})$ can be expressed as the product of the following elementary one-parameter transformations:

$$\begin{gathered}
\overline{x}^1 = \lambda_1 x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = x^3\ (\lambda_1 \in \mathbb{R}\backslash\{0\});\\
\overline{x}^1 = x^1,\ \overline{x}^2 = \lambda_2 x^2,\ \overline{x}^3 = x^3\ (\lambda_2 \in \mathbb{R}\backslash\{0\});\\
\overline{x}^1 = x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = \lambda_3 x^3\ (\lambda_3 \in \mathbb{R}\backslash\{0\});\\
\overline{x}^1 = x^1 + \mu_1 x_2,\ \overline{x}^2 = x^2,\ \overline{x}^3 = x^3;\\
\overline{x}^1 = x^1 + \mu_2 x_3,\ \overline{x}^2 = x^2,\ \overline{x}^3 = x^3;\\
\overline{x}^1 = x^1,\ \overline{x}^2 = \mu_3 x^1 + x^2,\ \overline{x}^3 = x^3;\\
\overline{x}^1 = x^1,\ \overline{x}^2 = x^2 + \mu_4 x^3,\ \overline{x}^3 = x^3;\\
\overline{x}^1 = x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = \mu_5 x^1 + x^3;\\
\overline{x}^1 = x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = \mu_6 x^2 + x^3,
\end{gathered} \tag{2}$$

which are denoted by $H_i$ with parameter $\lambda_i$ $(i = \overline{1,3})$ and $H_{j+3}$ with parameter $\mu_j$ $(j = \overline{1,6})$ in the space of the phase variables $E^3(x)$ of the system (1), where $x = (x^1, x^2, x^3)$.

We consider one-parameter linear transformations

$$\overline{x}^j = f^j(x,\alpha)\ (j = \overline{1,3}), \tag{3}$$

where $\alpha$ is a real parameter continuously changing over a certain interval of $\mathbb{R}$.

Applying the transformation (3) to the system (1) we obtain the system

$$\dot{\overline{x}}^j = \overline{a}^j_{\alpha\beta\gamma}\overline{x}^\alpha\overline{x}^\beta\overline{x}^\gamma\ (j,\alpha,\beta,\gamma = \overline{1,3}), \tag{4}$$

where the coefficients $\overline{a}^j_{\alpha\beta\gamma}$ are expressed linearly by the coefficients $a^j_{\alpha\beta\gamma}$.

Let us denote by $a$ and $\overline{a}$ the sets of coefficients of the right-hand sides in (1) and (4), respectively. It is clear that each coefficient of the set $\overline{a}$ is a linear function of the elements of the set $a$ with coefficients depending on the parameter $\alpha$.

We write this as

$$\overline{a}^j_{j_1 j_2 j_3} = g^j_{j_1 j_2 j_3}(a,\alpha)\ (j, j_1, j_2, j_3 = \overline{1,3}). \tag{5}$$

Let us assume that the identity transformation in the group with formulas (3) ) is achieved in the value of the parameter $\alpha = 0$. Having expanded the functions $\overline{x}^j$ $(j = \overline{1,3})$ and $\overline{a}^j_{j_1 j_2 j_3}$ $(k \in A)$ from (4), (5) into a Taylor series in the parameter $\alpha$ in some neighborhood of $\alpha = 0$, we obtain

$$\overline{x}^j = x^j + \xi^j(x)\alpha + o(\alpha)\ (j = \overline{1,3}), \tag{6}$$

$$\overline{a}^j_{j_1 j_2 j_3} = a^j_{j_1 j_2 j_3} + \eta^j_{j_1 j_2 j_3}(a)\alpha + o(\alpha)\ (k \in A), \tag{7}$$

where

$$\xi^j(x) = \frac{\partial \overline{x}^j(x,\alpha)}{\partial \alpha}|_{\alpha=0},\ \eta^j_{j_1 j_2 j_3}(a) = \frac{\partial \overline{a}^j_{j_1 j_2 j_3}}{\partial \alpha}|_{\alpha=0}\ (j, j_1, j_2, j_3 = \overline{1,3};\ k \in A). \tag{8}$$

In the case when we have a representation of $r$ linear one-parameter groups of the form (3) in the space of variables and coefficients of the system (1), then their operators are written in the form

$$\upsilon_i = \xi^j_i(x)\frac{\partial}{\partial x^j} + \mathfrak{d}_i\ (i = \overline{1,r}; j = \overline{1,3}), \tag{9}$$

where

$$\mathfrak{d}_i = [\eta^j_{j_1 j_2 j_3 (a)}]_i \frac{\partial}{\partial a^j_{j_1, j_2, j_3}}\ (i = \overline{1,r};\ j, j_1, j_2, j_3 = \overline{1,3};\ k \in A). \tag{10}$$

To the representations of the groups (2) in the space of the coefficients and the phase variables $E^{33}(x,a)$ of the system (1) correspond the operators of the form

$$\begin{gathered}
X = \xi^1\frac{\partial}{\partial x^1} + \xi^2\frac{\partial}{\partial x^2} + \xi^3\frac{\partial}{\partial x^3} + \eta^1_{111}\frac{\partial}{\partial a^1_{111}} + \eta^1_{112}\frac{\partial}{\partial a^1_{112}} + \eta^1_{113}\frac{\partial}{\partial a^1_{113}} + \\
+\eta^1_{122}\frac{\partial}{\partial a^1_{122}} + \eta^1_{123}\frac{\partial}{\partial a^1_{123}} + \eta^1_{133}\frac{\partial}{\partial a^1_{133}} + \eta^1_{222}\frac{\partial}{\partial a^1_{222}} + \eta^1_{223}\frac{\partial}{\partial a^1_{223}} + \eta^1_{233}\frac{\partial}{\partial a^1_{233}} + \\
+\eta^1_{333}\frac{\partial}{\partial a^1_{333}} + \eta^2_{111}\frac{\partial}{\partial a^2_{111}} + \eta^2_{112}\frac{\partial}{\partial a^2_{112}} + \eta^2_{113}\frac{\partial}{\partial a^2_{113}} + \eta^2_{122}\frac{\partial}{\partial a^2_{122}} + \eta^2_{123}\frac{\partial}{\partial a^2_{123}} + \\
+\eta^2_{133}\frac{\partial}{\partial a^2_{133}} + \eta^2_{222}\frac{\partial}{\partial a^2_{222}} + \eta^2_{223}\frac{\partial}{\partial a^2_{223}} + \eta^2_{233}\frac{\partial}{\partial a^2_{233}} + \eta^2_{333}\frac{\partial}{\partial a^2_{333}} + \eta^3_{111}\frac{\partial}{\partial a^3_{111}} + \\
+\eta^3_{112}\frac{\partial}{\partial a^3_{112}} + \eta^3_{113}\frac{\partial}{\partial a^3_{113}} + \eta^3_{122}\frac{\partial}{\partial a^3_{122}} + \eta^3_{123}\frac{\partial}{\partial a^3_{123}} + \eta^3_{133}\frac{\partial}{\partial a^3_{133}} + \eta^3_{222}\frac{\partial}{\partial a^3_{222}} + \\
+\eta^3_{223}\frac{\partial}{\partial a^3_{223}} + \eta^3_{233}\frac{\partial}{\partial a^3_{233}} + \eta^3_{333}\frac{\partial}{\partial a^3_{333}},
\end{gathered} \tag{11}$$

where

$$\xi^j = \frac{\partial \overline{x}^j}{\partial \alpha}|_{\alpha=0},\ \eta^j_{\alpha\beta\gamma} = \frac{\partial \overline{a}^j_{\alpha\beta\gamma}}{\partial \alpha}|_{\alpha=0}. \tag{12}$$

## 2. Lie operators of representation of the centro-afine group $GL(3,\mathbb{R})$ in the space of phase variables and coefficients of the system $s^3(3)$

We analyze the representation of the group $H_1$ from (2) in the space of phase variables and coefficients $E^{33}(x, a)$ of the system (1).

The system (1) after transformation obtains the form

$$\begin{gathered}
\frac{d\overline{x}^1}{dt} = \overline{a^1_{111}}(\overline{x}^1)^3 + \overline{a^1_{112}}(\overline{x}^1)^2\overline{x}^2 + \overline{a^1_{122}}x^1(\overline{x}^2)^2 + \\
+\overline{a^1_{222}}(\overline{x}^2)^3 + \overline{a^1_{113}}(\overline{x}^1)^2\overline{x}^3 + \overline{a^1_{123}}\overline{x}^1\overline{x}^2\overline{x}^3 + \overline{a^1_{223}}(\overline{x}^2)^2\overline{x}^3 + \\
+\overline{a^1_{133}}\overline{x}^1(\overline{x}^3)^2 + \overline{a^1_{233}}\overline{x}^2(\overline{x}^3)^2 + \overline{a^1_{333}}(\overline{x}^3)^3, \\
\frac{d\overline{x}^2}{dt} = \overline{a^2_{111}}(\overline{x}^1)^3 + \overline{a^2_{112}}(\overline{x}^1)^2\overline{x}^2 + \overline{a^2_{122}}\overline{x}^1(\overline{x}^2)^2 + \\
+\overline{a^2_{222}}(\overline{x}^2)^3 + \overline{a^2_{3}}\overline{x}^3 + \overline{a^2_{113}}(\overline{x}^1)^2\overline{x}^3 + \overline{a^2_{123}}\overline{x}^1\overline{x}^2\overline{x}^3 + \overline{a^2_{223}}(\overline{x}^2)^2\overline{x}^3 + \\
+\overline{a^2_{133}}\overline{x}^1(\overline{x}^3)^2 + \overline{a^2_{233}}\overline{x}^2(\overline{x}^3)^2 + \overline{a^2_{333}}(\overline{x}^3)^3, \\
\frac{d\overline{x}^3}{dt} = \overline{a^3_{111}}(\overline{x}^1)^3 + \overline{a^3_{112}}(\overline{x}^1)^2\overline{x}^2 + \overline{a^3_{122}}\overline{x}^1(\overline{x}^2)^2 + \\
+\overline{a^3_{222}}(\overline{x}^2)^3 + \overline{a^3_{113}}(\overline{x}^1)^2\overline{x}^3 + \overline{a^3_{123}}\overline{x}^1\overline{x}^2\overline{x}^3 + \overline{a^3_{223}}(\overline{x}^2)^2\overline{x}^3 + \overline{a^3_{133}}\overline{x}^1(\overline{x}^3)^2 + \\
+\overline{a^3_{233}}\overline{x}^2(\overline{x}^3)^2 + \overline{a^3_{333}}(\overline{x}^3)^3,
\end{gathered} \tag{13}$$

where

$$\overline{a^1_{111}} = \frac{a^1_{111}}{\lambda_1^2},\ \overline{a^1_{112}} = 3\frac{a^1_{112}}{\lambda_1},\ \overline{a^1_{122}} = 3a^1_{122},$$

$$\overline{a^1_{222}} = \lambda_1 a^1_{222},\ \overline{a^1_{113}} = 3\frac{a^1_{113}}{\lambda_1},\ \overline{a^1_{123}} = 6a^1_{123},\ \overline{a^1_{223}} = 3\lambda_1 a^1_{223},\ \overline{a^1_{133}} = 3a^1_{133},$$

$$\overline{a^1_{233}} = 3\lambda_1 a^1_{233},\ \overline{a^1_{333}} = \lambda_1 a^1_{333},$$

$$\overline{a^2_{111}} = \frac{a^2_{111}}{\lambda_1^3},\ \overline{a^2_{112}} = 3\frac{a^2_{112}}{\lambda_1^2},\ \overline{a^2_{223}} = 3\frac{a^2_{122}}{\lambda_1},$$

$$\overline{a^2_{222}} = a^2_{222},\ \overline{a^2_3} = a^2_3,\ \overline{a^2_{113}} = 3\frac{a^2_{113}}{\lambda_1^2},\ \overline{a^2_{123}} = 6\frac{a^2_{123}}{\lambda_1},\ \overline{a^2_{223}} = 3a^2_{223},$$

$$\overline{a^2_{133}} = 3\frac{a^2_{133}}{\lambda_1},\ \overline{a^2_{233}} = 3a^2_{233},\ \overline{a^2_{333}} = a^2_{333},$$

$$\overline{a^3_{111}} = \frac{a^3_{111}}{\lambda_1^3},\ \overline{a^3_{112}} = 3\frac{a^3_{112}}{\lambda_1^2},\ \overline{a^3_{122}} = 3\frac{a^3_{122}}{\lambda_1},$$

$$\overline{a^3_{222}} = a^3_{222},\ \overline{a^3_{113}} = 3\frac{a^3_{113}}{\lambda_1^2},\ \overline{a^3_{123}} = 6\frac{a^3_{123}}{\lambda_1},\ \overline{a^3_{223}} = 3a^3_{223},\ \overline{a^3_{133}} = 3\frac{a^3_{133}}{\lambda_1},$$

$$\overline{a^3_{233}} = 3a^3_{233},\ \overline{a^3_{333}} = a^3_{333},\ |\lambda_1| = exp(\overline{\lambda_1}). \tag{14}$$

For $\overline{\lambda_1} = 0$ we obtain the following non-zero coordinates of the Lie operator corresponding to the group $H_1$:

$$\begin{gathered}
\xi^1 = x^1, \eta^1_{111} = -2a^1_{111}, \eta^1_{112} = -a^1_{112}, \eta^1_{113} = -a^1_{113},\\
\eta^1_{222} = a^1_{222}, \eta^1_{223} = a^1_{223}, \eta^1_{233} = a^1_{233}, \eta^1_{333} = a^1_{333}, \eta^2_{111} = -3a^2_{111},\\
\eta^2_{112} = -2a^2_{112}, \eta^2_{113} = -2a^2_{113}, \eta^2_{122} = -a^2_{122}, \eta^2_{123} = -a^2_{123}, \eta^2_{133} = -a^2_{133},\\
\eta^3_{111} = -3a^3_{111}, \eta^3_{112} = -2a^3_{112}, \eta^3_{113} = -2a^3_{113}, \eta^3_{122} = -a^3_{122},\\
\eta^3_{123} = -a^3_{123}, \eta^3_{133} = -a^3_{133}.
\end{gathered} \tag{15}$$

Thus, for the group $H_1$ we obtain the following operator:

$$\upsilon_1 = x^1\frac{\partial}{\partial x^1} + \mathfrak{d}_1, \tag{16}$$

where

$$\begin{gathered}
\mathfrak{d}_1 = -2a^1_{111}\frac{\partial}{\partial a^1_{111}} - a^1_{112}\frac{\partial}{\partial a^1_{112}} - a^1_{113}\frac{\partial}{\partial a^1_{113}} + a^1_{222}\frac{\partial}{\partial a^1_{222}} +\\
+a^1_{223}\frac{\partial}{\partial a^1_{223}} + a^1_{233}\frac{\partial}{\partial a^1_{233}} + a^1_{333}\frac{\partial}{\partial a^1_{333}} - 3a^2_{111}\frac{\partial}{\partial a^2_{111}} -\\
-2a^2_{112}\frac{\partial}{\partial a^2_{112}} - 2a^2_{113}\frac{\partial}{\partial a^2_{113}} - a^2_{122}\frac{\partial}{\partial a^2_{122}} - a^2_{123}\frac{\partial}{\partial a^2_{123}} -\\
-a^2_{133}\frac{\partial}{\partial a^2_{133}} - 3a^3_{111}\frac{\partial}{\partial a^3_{111}} - 2a^3_{112}\frac{\partial}{\partial a^3_{112}} - 2a^3_{113}\frac{\partial}{\partial a^3_{113}} -\\
-a^3_{122}\frac{\partial}{\partial a^3_{122}} - a^3_{123}\frac{\partial}{\partial a^3_{123}} - a^3_{133}\frac{\partial}{\partial a^3_{133}}.
\end{gathered} \tag{17}$$

Similarly for the groups $H_i$ $(i = 2,3)$ and $H_{j+3}$ $(j = \overline{1,6})$ we obtain the following operators:

$$H_2 \leftrightarrow \upsilon_2 = x^2\frac{\partial}{\partial x^2} + \mathfrak{d}_2, \tag{18}$$

where

$$\begin{gathered}\mathfrak{d}_2 = -a^1_{112}\frac{\partial}{\partial a^1_{112}} - 2a^1_{122}\frac{\partial}{\partial a^1_{122}} - a^1_{123}\frac{\partial}{\partial a^1_{123}} - 3a^1_{222}\frac{\partial}{\partial a^1_{222}} - \\ -2a^1_{223}\frac{\partial}{\partial a^1_{223}} - a^1_{233}\frac{\partial}{\partial a^1_{233}} + a^2_{111}\frac{\partial}{\partial a^2_{111}} + a^2_{113}\frac{\partial}{\partial a^2_{113}} - \\ -a^2_{122}\frac{\partial}{\partial a^2_{122}} + a^2_{133}\frac{\partial}{\partial a^2_{133}} - 2a^2_{222}\frac{\partial}{\partial^2_{222}} - a^2_{223}\frac{\partial}{\partial a^2_{223}} + \\ +a^2_{333}\frac{\partial}{\partial a^2_{333}} - a^3_{112}\frac{\partial}{\partial a^3_{112}} - 2a^3_{122}\frac{\partial}{\partial a^3_{122}} - a^3_{123}\frac{\partial}{\partial a^3_{123}} - \\ -3a^3_{222}\frac{\partial}{\partial a^3_{222}} - 2a^3_{223}\frac{\partial}{\partial a^3_{223}}.\end{gathered} \tag{19}$$

$$H_3 \leftrightarrow \upsilon_3 = x^3\frac{\partial}{\partial x^3} + \mathfrak{d}_3, \tag{20}$$

where

$$\begin{gathered}\mathfrak{d}_3 = -a^1_{113}\frac{\partial}{\partial a^1_{113}} - a^1_{123}\frac{\partial}{\partial a^1_{123}} - 2a^1_{133}\frac{\partial}{\partial a^1_{133}} - a^1_{223}\frac{\partial}{\partial a^1_{223}} - \\ -2a^1_{233}\frac{\partial}{\partial a^1_{233}} - 3a^1_{333}\frac{\partial}{\partial a^1_{333}} - a^2_{113}\frac{\partial}{\partial a^2_{113}} - a^2_{123}\frac{\partial}{\partial a^2_{123}} - \\ -2a^2_{133}\frac{\partial}{\partial a^2_{133}} - a^2_{223}\frac{\partial}{\partial a^2_{223}} - 2a^2_{233}\frac{\partial}{\partial a^2_{233}} - 3a^2_{333}\frac{\partial}{\partial a^2_{333}} + \\ +a^3_{111}\frac{\partial}{\partial a^3_{111}} + a^3_{112}\frac{\partial}{\partial a^3_{112}} + a^3_{122}\frac{\partial}{\partial a^3_{122}} - \\ -a^3_{133}\frac{\partial}{\partial a^3_{133}} + a^3_{222}\frac{\partial}{\partial a^3_{222}} - a^3_{233}\frac{\partial}{\partial a^3_{233}} - 2a^3_{333}\frac{\partial}{\partial a^3_{333}}.\end{gathered} \tag{21}$$

$$H_4 \leftrightarrow \upsilon_4 = x^2\frac{\partial}{\partial x^1} + \mathfrak{d}_4, \tag{22}$$

where

$$
\begin{aligned}
\mathfrak{d}_4 = {} & a^2_{111}\frac{\partial}{\partial a^1_{111}} + (a^2_{112} - a^1_{111})\frac{\partial}{\partial a^1_{112}} + a^2_{113}\frac{\partial}{\partial a^1_{113}} + \\
& +(a^2_{122} - 2a^1_{112})\frac{\partial}{\partial a^1_{122}} + (a^2_{123} - a^1_{113})\frac{\partial}{\partial a^1_{123}} + a^2_{133}\frac{\partial}{\partial a^1_{133}} + \\
& +(a^2_{223} - 2a^1_{123})\frac{\partial}{\partial a^1_{223}} + (a^2_{233} - a^1_{133})\frac{\partial}{\partial a^1_{233}} + a^2_{333}\frac{\partial}{\partial a^1_{333}} - \\
& -a^2_{111}\frac{\partial}{\partial a^2_{112}} - 2a^2_{112}\frac{\partial}{\partial a^2_{112}} - a^2_{113}\frac{\partial}{\partial a^2_{123}} - 3a^2_{122}\frac{\partial}{\partial a^2_{222}} - \\
& -2a^2_{123}\frac{\partial}{\partial a^2_{223}} - a^2_{133}\frac{\partial}{\partial a^2_{233}} - a^3_{111}\frac{\partial}{\partial a^3_{112}} - 2a^3_{112}\frac{\partial}{\partial a^3_{122}} - \\
& -a^3_{113}\frac{\partial}{\partial a^3_{123}} - 3a^3_{122}\frac{\partial}{\partial a^3_{222}} - 2a^3_{123}\frac{\partial}{\partial a^3_{223}} - a^3_{133}\frac{\partial}{\partial a^3_{233}}.
\end{aligned}
\tag{23}
$$

$$
H_5 \leftrightarrow \upsilon_5 = x^2\frac{\partial}{\partial x^1} + \mathfrak{d}_5, \tag{24}
$$

where

$$
\begin{aligned}
\mathfrak{d}_5 = {} & a^2_{111}\frac{\partial}{\partial a^1_{111}} + (a^2_{112} - a^1_{111})\frac{\partial}{\partial a^1_{112}} + a^2_{113}\frac{\partial}{\partial a^1_{113}} + \\
& +(a^2_{122} - 2a^1_{112})\frac{\partial}{\partial a^1_{122}} + (a^2_{123} - a^1_{113})\frac{\partial}{\partial a^1_{123}} + a^2_{133}\frac{\partial}{\partial a^1_{133}} + \\
& +(a^2_{223} - 2a^1_{123})\frac{\partial}{\partial a^1_{223}} + (a^2_{233} - a^1_{133})\frac{\partial}{\partial a^1_{233}} + a^2_{333}\frac{\partial}{\partial a^1_{333}} - \\
& -a^2_{111}\frac{\partial}{\partial a^2_{112}} - 2a^2_{112}\frac{\partial}{\partial a^2_{112}} - a^2_{113}\frac{\partial}{\partial a^2_{123}} - 3a^2_{122}\frac{\partial}{\partial a^2_{222}} - \\
& -2a^2_{123}\frac{\partial}{\partial a^2_{223}} - a^2_{133}\frac{\partial}{\partial a^2_{233}} - a^3_{111}\frac{\partial}{\partial a^3_{112}} - 2a^3_{112}\frac{\partial}{\partial a^3_{122}} - \\
& -a^3_{113}\frac{\partial}{\partial a^3_{123}} - 3a^3_{122}\frac{\partial}{\partial a^3_{222}} - 2a^3_{123}\frac{\partial}{\partial a^3_{223}} - a^3_{133}\frac{\partial}{\partial a^3_{233}}.
\end{aligned}
\tag{25}
$$

$$
H_6 \leftrightarrow \upsilon_6 = x^1\frac{\partial}{\partial x^2} + \mathfrak{d}_6, \tag{26}
$$

where

$$
\begin{aligned}
\mathfrak{d}_6 = {} & -3a^1_{112}\frac{\partial}{\partial a^1_{111}} - 2a^1_{122}\frac{\partial}{\partial a^1_{112}} - 2a^1_{123}\frac{\partial}{\partial a^1_{113}} - a^1_{222}\frac{\partial}{\partial a^1_{122}} - \\
& -a^1_{223}\frac{\partial}{\partial a^1_{123}} - a^1_{233}\frac{\partial}{\partial a^1_{133}} + (a^1_{111} - 3a^2_{112})\frac{\partial}{\partial a^2_{111}} + (a^1_{112} - 2a^2_{122})\frac{\partial}{\partial a^2_{112}} + \\
& +(a^1_{113} - 2a^2_{123})\frac{\partial}{\partial a^2_{113}} + (a^1_{122} - a^2_{222})\frac{\partial}{\partial a^2_{122}} + (a^1_{123} - a^2_{223})\frac{\partial}{\partial a^2_{123}} + \\
& +(a^1_{133} - a^2_{233})\frac{\partial}{\partial a^2_{133}} + a^1_{222}\frac{\partial}{\partial a^2_{222}} + a^1_{223}\frac{\partial}{\partial a^2_{223}} + a^1_{233}\frac{\partial}{\partial a^2_{233}} + \\
& +a^1_{333}\frac{\partial}{\partial a^2_{333}} - 3a^3_{112}\frac{\partial}{\partial a^3_{111}} - 2a^3_{122}\frac{\partial}{\partial a^3_{112}} - 2a^3_{123}\frac{\partial}{\partial a^3_{113}} - \\
& -a^3_{222}\frac{\partial}{\partial a^3_{122}} - a^3_{223}\frac{\partial}{\partial a^3_{123}} - a^3_{233}\frac{\partial}{\partial a^3_{133}}.
\end{aligned} \tag{27}
$$

$$
H_7 \leftrightarrow \upsilon_7 = x^3\frac{\partial}{\partial x^2} + \mathfrak{d}_7, \tag{28}
$$

where

$$
\begin{aligned}
\mathfrak{d}_7 = {} & -a^1_{112}\frac{\partial}{\partial a^1_{113}} - a^1_{122}\frac{\partial}{\partial a^1_{123}} - 2a^1_{123}\frac{\partial}{\partial a^1_{133}} - a^1_{222}\frac{\partial}{\partial a^1_{223}} - \\
& -2a^1_{223}\frac{\partial}{\partial a^1_{233}} - 3a^1_{233}\frac{\partial}{\partial a^1_{333}} + a^3_{111}\frac{\partial}{\partial a^2_{111}} + a^3_{112}\frac{\partial}{\partial a^2_{112}} + \\
& +(a^3_{113} - a^2_{112})\frac{\partial}{\partial a^2_{113}} + a^3_{122}\frac{\partial}{\partial a^2_{122}} + (a^3_{123} - a^2_{122})\frac{\partial}{\partial a^2_{123}} + (a^3_{133} - \\
& -2a^2_{123})\frac{\partial}{\partial a^2_{133}} + a^3_{222}\frac{\partial}{\partial a^2_{222}} + (a^3_{223} - a^2_{222})\frac{\partial}{\partial a^2_{223}} + \\
& +(a^3_{233} - 2a^2_{223})\frac{\partial}{\partial a^2_{233}} + (a^3_{333} - 3a^2_{233})\frac{\partial}{\partial a^2_{333}} - a^3_{112}\frac{\partial}{\partial a^3_{113}} - \\
& -a^3_{122}\frac{\partial}{\partial a^3_{123}} - 2a^3_{123}\frac{\partial}{\partial a^3_{133}} - a^3_{222}\frac{\partial}{\partial a^3_{223}} - 2a^3_{223}\frac{\partial}{\partial a^3_{233}} - 3a^3_{233}\frac{\partial}{\partial a^3_{333}}.
\end{aligned} \tag{29}
$$

$$
H_8 \leftrightarrow \upsilon_8 = x^1\frac{\partial}{\partial x^3} + \mathfrak{d}_8, \tag{30}
$$

where

$$
\begin{aligned}
\mathfrak{d}_8 = &-3a^1_{113}\frac{\partial}{\partial a^1_{111}} - 2a^1_{123}\frac{\partial}{\partial a^1_{112}} - 2a^1_{133}\frac{\partial}{\partial a^1_{113}} - a^1_{223}\frac{\partial}{\partial a^1_{122}} - \\
&-a^1_{233}\frac{\partial}{\partial a^1_{123}} - a^1_{333}\frac{\partial}{\partial a^1_{133}} - 3a^2_{113}\frac{\partial}{\partial a^2_{111}} - 2a^2_{123}\frac{\partial}{\partial a^2_{112}} - \\
&-2a^2_{133}\frac{\partial}{\partial a^2_{113}} - a^2_{223}\frac{\partial}{\partial a^2_{122}} - a^2_{233}\frac{\partial}{\partial a^2_{123}} - a^2_{333}\frac{\partial}{\partial a^2_{133}} + \\
&+(a^1_{111} - 3a^3_{113})\frac{\partial}{\partial a^3_{111}} + (a^1_{112} - 2a^3_{123})\frac{\partial}{\partial a^3_{112}} + (a^1_{113} - 2a^3_{133})\frac{\partial}{\partial a^3_{113}} + \\
&+(a^1_{122} - a^3_{223})\frac{\partial}{\partial a^3_{122}} + (a^1_{123} - a^3_{233})\frac{\partial}{\partial a^3_{123}} + (a^1_{133} - a^3_{333})\frac{\partial}{\partial a^3_{133}} + \\
&+a^1_{222}\frac{\partial}{\partial a^3_{222}} + a^1_{223}\frac{\partial}{\partial a^3_{223}} + a^1_{233}\frac{\partial}{\partial a^3_{233}} + a^1_{333}\frac{\partial}{\partial a^3_{333}}.
\end{aligned}
\tag{31}
$$

$$
H_9 \leftrightarrow \upsilon_9 = x^2\frac{\partial}{\partial x^3} + \mathfrak{d}_9,
\tag{32}
$$

where

$$
\begin{aligned}
\mathfrak{d}_9 = &-a^1_{113}\frac{\partial}{\partial a^1_{112}} - 2a^1_{123}\frac{\partial}{\partial a^1_{122}} - a^1_{133}\frac{\partial}{\partial a^1_{123}} - 3a^1_{223}\frac{\partial}{\partial a^1_{222}} - \\
&-2a^1_{233}\frac{\partial}{\partial a^1_{223}} - a^1_{333}\frac{\partial}{\partial a^1_{233}} - a^2_{113}\frac{\partial}{\partial a^2_{112}} - 2a^2_{123}\frac{\partial}{\partial a^2_{122}} - a^2_{133}\frac{\partial}{\partial a^2_{123}} - \\
&-3a^2_{223}\frac{\partial}{\partial a^2_{222}} - 2a^2_{233}\frac{\partial}{\partial a^2_{223}} - a^2_{333}\frac{\partial}{\partial a^2_{233}} + a^2_{111}\frac{\partial}{\partial a^3_{111}} + \\
&+(a^2_{112} - a^3_{113})\frac{\partial}{\partial a^3_{112}} + a^2_{113}\frac{\partial}{\partial a^3_{113}} + (a^2_{122} - 2a^3_{123})\frac{\partial}{\partial a^3_{122}} + \\
&+(a^2_{123} - a^3_{133})\frac{\partial}{\partial a^3_{123}} + a^2_{133}\frac{\partial}{\partial a^3_{133}} + (a^2_{222} - 3a^3_{223})\frac{\partial}{\partial a^3_{222}} + \\
&+(a^2_{223} - 2a^3_{233})\frac{\partial}{\partial a^3_{223}} + (a^2_{233} - a^3_{333})\frac{\partial}{\partial a^3_{233}} + a^2_{333}\frac{\partial}{\partial a^3_{333}}.
\end{aligned}
\tag{33}
$$

According to the work [9] we obtain

**Theorem 2.1.** *To the representation of the centro-affine group $GL(3,\mathbb{R})$ in the space of variables and coefficients of the system* (1) *correspond the Lie operators*

$$
\begin{gathered}
\upsilon_1 = x^1\frac{\partial}{\partial x^1} + \mathfrak{d}_1, \upsilon_2 = x^2\frac{\partial}{\partial x^2} + \mathfrak{d}_2, \upsilon_3 = x^3\frac{\partial}{\partial x^3} + \mathfrak{d}_3,\\
\upsilon_4 = x^2\frac{\partial}{\partial x^1} + \mathfrak{d}_4, \upsilon_5 = x^3\frac{\partial}{\partial x^1} + \mathfrak{d}_5, \upsilon_6 = x^1\frac{\partial}{\partial x^2} + \mathfrak{d}_6,\\
\upsilon_7 = x^3\frac{\partial}{\partial x^2} + \mathfrak{d}_7, \upsilon_8 = x^1\frac{\partial}{\partial x^3} + \mathfrak{d}_8, \upsilon_9 = x^2\frac{\partial}{\partial x^3} + \mathfrak{d}_9,
\end{gathered} \tag{34}
$$

*where $\mathfrak{d}_i$ are the operators of the representation of the group $GL(3,\mathbb{R})$ in the space of coefficients of the system* (1) *and are from* (17)–(33).

For the system (1) the determining equations are [9]

$$
\xi^i_{x^k}P^k = \xi^j P^i_{x^j} + \mathfrak{d}(P^i)\ (i,j,k=\overline{1,3}), \tag{35}
$$

where $\xi^i_{x^k} = \frac{\partial \xi^i}{\partial x^k}$ şi $P^i_{x^j} = \frac{\partial P^i}{\partial x^j}$.

**Remark 2.1.** *The operators* (34) *satisfy the system of determining equations* (35) *for the system* (1) *and, consequently, are admitted by this system.* [9].

## 3. $\mathbf{GL(3,\mathbb{R})}$-Lie algebra admitted by the differential system $\mathbf{s^3(3)}$

**Lemma 3.1.** *The operators* (34) *form a nine-dimensional Lie algebra $L_9 < v_1, ..., v_9 >$ with structural equations*

$$
\begin{gathered}
[\upsilon_1,\upsilon_4] = -\upsilon_4, [\upsilon_1,\upsilon_5] = -\upsilon_5, [\upsilon_1,\upsilon_6] = \upsilon_6, [\upsilon_1,\upsilon_8] = \upsilon_8, [\upsilon_2,\upsilon_4] = \upsilon_4,\\
[\upsilon_2,\upsilon_6] = -\upsilon_6, [\upsilon_2,\upsilon_7] = -\upsilon_7, [\upsilon_2,\upsilon_9] = \upsilon_9, [\upsilon_3,\upsilon_5] = \upsilon_5, [\upsilon_3,\upsilon_7] = \upsilon_7,\\
[\upsilon_3,\upsilon_8] = -\upsilon_8, [\upsilon_3,\upsilon_9] = -\upsilon_9, [\upsilon_4,\upsilon_6] = \upsilon_2-\upsilon_1, [\upsilon_4,\upsilon_7] = -\upsilon_5,\\
[\upsilon_4,\upsilon_8] = \upsilon_9, [\upsilon_5,\upsilon_6] = \upsilon_7, [\upsilon_5,\upsilon_8] = \upsilon_3-\upsilon_1, [\upsilon_5,\upsilon_9] = -\upsilon_4,\\
[\upsilon_6,\upsilon_9] = \upsilon_8, [\upsilon_7,\upsilon_8] = -\upsilon_6, [\upsilon_7,\upsilon_9] = \upsilon_3-\upsilon_2,
\end{gathered} \tag{36}
$$

(*the rest* $[\upsilon_i,\upsilon_j] = 0$).

If we denote by

$$
\begin{gathered}
\omega_1 = \upsilon_9,\ \omega_2 = \upsilon_1-\upsilon_3,\ \omega_3 = \upsilon_1-\upsilon_2,\ \omega_4 = \upsilon_4,\ \omega_5 = \upsilon_5,\ \omega_6 = \upsilon_6,\\
\omega_7 = \upsilon_7,\ \omega_8 = \upsilon_8,\ \omega_9 = \upsilon_1+\upsilon_2+\upsilon_3,
\end{gathered} \tag{37}
$$

then we obtain

**Lemma 3.2.** *$GL(3,\mathbb{R})$-Lie algebra $L_9 < \upsilon_1, ..., \upsilon_9 >$ with structural equations* (36) *is isomorphic to the Lie algebra $L_9 < \omega_1, ..., \omega_9 >$ with structural equations*

$$\begin{gathered}
[\omega_1,\omega_2] = -\omega_1,\ [\omega_1,\omega_3] = \omega_1,\ [\omega_1,\omega_5] = \omega_4,\ [\omega_1,\omega_6] = -\omega_8,\\
[\omega_1,\omega_7] = \omega_2 - \omega_3,\ [\omega_2,\omega_4] = -\omega_4,\ [\omega_2,\omega_5] = -2\omega_5,\ [\omega_2,\omega_6] = \omega_6,\\
[\omega_2,\omega_7] = -\omega_7,\ [\omega_2,\omega_8] = 2\omega_8,\ [\omega_3,\omega_4] = -2\omega_4,\ [\omega_3,\omega_5] = -\omega_5,\\
[\omega_3,\omega_6] = 2\omega_6,\ [\omega_3,\omega_7] = \omega_7,\ [\omega_3,\omega_8] = \omega_8,\ [\omega_4,\omega_6] = -\omega_6,\\
[\omega_4,\omega_7] = -\omega_5,\ [\omega_4,\omega_8] = \omega_1,\ [\omega_5,\omega_6] = \omega_7,\ [\omega_5,\omega_8] = -\omega_2,\\
[\omega_7,\omega_8] = -\omega_6, ([\omega_i,\omega_j] = 0 \ \textit{otherwise}).
\end{gathered} \tag{38}$$

For the Lie algebra $L_9 < \omega_1, ..., \omega_9 >$, non-zero structural constants up to antisymmetry in lower indices are:

$$\begin{gathered}
C^1_{12} = -1,\ C^1_{13} = 1,\ C^4_{15} = 1,\ C^8_{16} = -1,\ C^2_{17} = 1, C^3_{17} = -1,\\
C^4_{24} = -1,\ C^5_{25} = -2,\ C^6_{26} = 1,\ C^7_{27} = -1,\ C^8_{28} = 2,\ C^4_{34} = -2,\\
C^5_{35} = -1,\ C^6_{36} = 2,\ C^7_{37} = 1,\ C^8_{38} = 1,\ C^3_{46} = -1,\ C^5_{47} = -1,\\
C^1_{48} = 1,\ C^7_{56} = 1,\ C^2_{58} = -1,\ C^6_{78} = -1.
\end{gathered} \tag{39}$$

According to (38) we have

$$L_9 < \omega_1, ..., \omega_9 >= L_8 < \omega_1, ..., \omega_8 > \oplus Z < \omega_9 >, \tag{40}$$

where $Z < \omega_9 >$ is a non-zero center of the Lie algebra $L_9 < \omega_1, ..., \omega_9 >$.

Next we determine the type of the algebra $L_8 < \omega_1, ..., \omega_8 >$.

The Killing form [2] of this algebra is

$$K < v, w >= K_{\alpha\beta} x^\alpha y^\beta,\ K_{\alpha\beta} = C^\tau_{\sigma\alpha} C^\sigma_{\tau\beta}\ (\alpha,\beta,\tau,\sigma = \overline{1,8}). \tag{41}$$

From (41) using (39) we obtain

$$\begin{gathered}
K < v, w >= 6x^7y^1 + 12x^2y^2 + 6x^3y^2 + 6x^2y^3 + 12x^3y^3+\\
+6x^6y^4 + 6x^8y^5 + 6x^4y^6 + 6x^1y^7 + 6x^5y^8.
\end{gathered}$$

For it we have $det(K_{\alpha\beta}) \neq 0$, consequently, the Killing form is non-degenerate [2].

According to Cartan's criterion [2], Lie algebra $L_8 < \omega_1, ..., \omega_8 >$ is semisimple.

From (40) and definition of reductive Lie algebra we obtain

**Theorem 3.1.** *The algebra $L_9 < \upsilon_1, ..., \upsilon_9 >$ is a reductive Lie algebra and corresponds to the centro-affine group $GL(3,\mathbb{R})$.*

This algebra is isomorphic to the $GL(3,\mathbb{R})$-Lie algebra admitted by the differential system $s^3(1,2)$ [9].

## 4. On the number of center-affine invariant elements for the differential system $s^3(3)$

Based on the ideas from the papers [3]–[8] some ternary differential systems were examined in the works [8]–[10] under different aspects. The problem of constructing the functional bases of invariants, comitants, contravariants and mixed comitants for different ternary differential systems is a rather important one.

Consider the center-affine group $GL(3,\mathbb{R})$ given by transformations $q$:

$$\bar{x}^j = q^j_\alpha x^\alpha \quad (\Delta = det(q^j_\alpha) \neq 0) \quad (j,\alpha = \overline{1,3}). \tag{42}$$

**Definition 4.1.** *According to* [3] *the vector of phase variables* $x = (x^1, x^2, x^3)$ *of the system* (1)*, which changes by formulas* (42) *is usually called* ***contravariant****.*

Solving the equation (42) with respect to $x^j$ we obtain

$$x^j = p^j_r \bar{x}^r \quad (r,j = \overline{1,3}), \tag{43}$$

where

$$p^r_j q^j_s = \delta^r_s \tag{44}$$

is the Kroniker's symbol, for which we have

$$\delta^r_s = \begin{cases} 1, & \text{for } r = s, \\ 0, & \text{for } r \neq s. \end{cases} \tag{45}$$

**Definition 4.2.** *According to* [3] *the vector of phase variables* $u = (u_1, u_2, u_3)$ *of the system* (1) *which changes according to the formulas*

$$\overline{u}_r = p^j_r u_j \quad (r,j = \overline{1,3}), \tag{46}$$

*where* $p^j_r$ *are from* (43) *is called* ***covariant****, and it is also called* ***contragradient*** *with the vector* $x$*.*

Geometrically, the vector $x = (x^1, x^2, x^3)$ is a point in a three-dimensional space, and the vector $u = (u_1, u_2, u_3)$ - is a line.

**Definition 4.3.** *According to* [3] *we say that the polynomial* $\varkappa(x,u,a)$ *of the coefficients of the system* (1) *and of the coordinates of the vectors* $x$ *and* $u$ *is called a* ***mixed comitant*** *of the system* (1) *with respect to the center-affine group* $GL(3,\mathbb{R})$ *if the following identity holds*

$$s(\bar{x},\bar{u},\bar{a}) = \Delta^{-g} s(x,u,a) \tag{47}$$

*for all $q$ from $GL(3,\mathbb{R})$ and every coordinates of the vectors $x$ and $u$, as well as all the coefficients $a$ of the system* (1)*, where $g$ is an integer number called* ***the weight of comitant****.*

If the mixt comitant $s(x,u,a)$ does not depend on the coordinates of the vector $u$, then following the traditions of the academic C. Sibirschi school, we call it simply **comitant**, and we will note it by $k(x,u,a)$. If $s(x,u,a)$ does not depend on the coordinates of the vector $x$ we call it **contravariant** [3] and we will note it by $r(x,u,a)$. If $s(x,u,a)$ does not depend on the coordinates of the vectors $x$ and $u$, then we will call it **invariant** of the system (1) with respect to the $GL(3,\mathbb{R})$ group and we will note it by $j(x,u,a)$.

If the mixed comitant $s$ has the weight $g = 0$, then it is called *absolute*, otherwise *relative*.

**Definition 4.4.** *We will say that the set of polynomial mixed comitants $\{s_k(x,u,a),\ k \in B\}$, of the system* (1)*, with respect to the $GL(3,\mathbb{R})$ group forms a functional basis of the comitants of the system* (1) *with respect to $GL(3,\mathbb{R})$ group, if any comitant $s_r(x,u,a)$ $r \notin B$ of the system* (1)*, with respect to the $GL(3,\mathbb{R})$ group, is expressed as a unique function from the comitants $s_k(x,u,a)$. Here $B$ is a set of finite or infinite natural numbers.*

**Definition 4.5.** *The functional basis of the mixed comitants of the system* (1) *with respect to the $GL(3,\mathbb{R})$ group is called minimal, if excluding any comitant from the basis, it ceases to form a functional basis.*

Similary, it is defined the functional basis of the comitants, contravariants and invariants of the system (1) with respect to the $GL(3,\mathbb{R})$ group.

**Definition 4.6.** *The polynomial $s(x,u,a)$ is called irreducible, if it is not expressed polynomially by invariants, comitants, contravariants and mixed comitants, of the same or lower degree.*

**Theorem 4.1.** [9] For the polynomial $s(x,u,a)$ in the coefficients of the system (1) and the coordinates of the vectors $x$ and $u$ to be a mixed comitant of the system (1) with the weight $g$ and respect to the $GL(3,R)$ group, it is necessary and sufficient that it satisfies the equations

$$\mathcal{V}_i(s) = -gs \ \ (i = \overline{1,3}),\ \ \mathcal{V}_j(s) = 0 \ \ (j = \overline{4,9}), \tag{48}$$

where $\mathcal{V}_i \quad (i = \overline{1,9})$ have the form

$$\begin{aligned} \mathcal{V}_1 &= \upsilon_1 - u_1 \frac{\partial}{\partial u_1}, \ \mathcal{V}_2 = \upsilon_2 - u_2 \frac{\partial}{\partial u_2}, \ \mathcal{V}_3 = \upsilon_3 - u_3 \frac{\partial}{\partial u_3}, \\ \mathcal{V}_4 &= \upsilon_4 - u_1 \frac{\partial}{\partial u_2}, \ \mathcal{V}_5 = \upsilon_5 - u_1 \frac{\partial}{\partial u_3}, \ \mathcal{V}_6 = \upsilon_6 - u_2 \frac{\partial}{\partial u_1}, \\ \mathcal{V}_7 &= \upsilon_7 - u_2 \frac{\partial}{\partial u_3}, \ \mathcal{V}_8 = \upsilon_8 - u_3 \frac{\partial}{\partial u_1}, \ \mathcal{V}_9 = \upsilon_9 - u_3 \frac{\partial}{\partial u_2}, \end{aligned} \tag{49}$$

where $\upsilon_i \quad (i = \overline{1,9})$ are from (34).

**Note 4.1.** *From the theory of systems of differential equations* (*see for example* [3]), *it follows that the maximum number of functionally independent solution* (*invariants* $j$, *comitants* $k$, *contravariants* $r$ *and mixed comitants* $s$) *is equal to*

$$\mu = N - rank M + 1 \ (g \neq 0), \ (\mu = N - rank M \ (g = 0)), \tag{50}$$

*where* $N$ *is the number of variables in the system of equations,* $M$ *is the matrix constructed on the coordinates of the differential operators* (34).

**Theorem 4.2.** *The number of the invariant center-affine polynomials of the differential system* $s^3(3)$ *that form the functional bases of invariants* $\mu(j)$, *comitants* $\mu(k)$, *contravariants* $\mu(r)$ *and mixed comitants* $\mu(s)$, *respectively, is given in the following table*

| $\mu(j)$ | $\mu(k)$ | $\mu(r)$ | $\mu(s)$ |
|---|---|---|---|
| 22 | 25 | 25 | 28 |

The number of this invariant elements is necessary for constructing their functional bases.

## 5. An elements from the functional bases of invariant polynomials of the differential system $s^3(3)$

According to the basic theorem of the theory of invariant tensors [4], of invariant centro-affine polynomials, these invariants are constructed using the product of the tensor coefficients, using the alternation operation (using the unit vector), followed by the total convolution operation (summation by an upper and a lower index).

In the case of the system $s^3(3)$ the general form of the mixed comitant is

$$\underbrace{a^{-}_{---} a^{-}_{---} \cdots a^{-}_{---}}_{d} \underbrace{x^{-} x^{-} \cdots x^{-}}_{\delta} \underbrace{u_{-} u_{-} \cdots u_{-}}_{\omega} \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3} \ldots \varepsilon^{\omega_1\omega_2\omega_3}}_{g_1} \underbrace{\varepsilon_{\mu_1\mu_2\mu_3} \cdots \varepsilon_{\upsilon_1\upsilon_2\upsilon_3}}_{g_2}, \tag{51}$$

where $d$, $\delta$, $\omega$, $g_1$, $g_2$ are natural numbers, which indicates the number of elements in the brace.

Particular cases of the mixed comitant are the invariant ($\delta = \omega = 0$), the comitant ($\omega = 0$) and the contravariant ($\delta = 0$).

We will study the construction of invariant centro-affine polynomials of the system $s^3(3)$ of the form (51) through the prism of the type below

$$(\delta,\ d;\ \omega), \tag{52}$$

where $\delta$ is the degree with respect to the coordinates of the phase variable vector $x = (x_1, x_3, x_3)$, $d$ - the degree with respect to the tensor coefficients $a^j_{\alpha\beta\gamma}$ of the system $s^3(3)$, and $\omega$ is the degree with respect to the coordinates of the covariant vector $u = (u_1, u_2, u_3)$.

Next, we will investigate the centro-affine invariants of the system $s^3(3)$ for which ($\delta = \omega = g_2 = 0$), and according to the form (51) will be written as

$$\underbrace{a^{-}_{---}a^{-}_{---}\cdots a^{-}_{---}}_{d}\ \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3}\ldots\varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}, \tag{53}$$

where each upper index meets once again lower. Following this property we obtain that $3d = d + 3g_1$. From here we have

$$2d = 3g_1, \tag{54}$$

where $d$ and $g_1$ are natural numbers.

Then from the equality (54) it follows that

**Remark 5.1.** *For the product* (53) *to be a centro-affine invariant of the system* $s^3(3)$ *it is necessary and sufficient that d be a multiple of* 3 *and* $g_1$ *be a multiple of* 2.

We will mention that $\varepsilon^{\alpha_1\alpha_2\alpha_3}, \ldots, \varepsilon^{\omega_1\omega_2\omega_3}$ ($\varepsilon_{\mu_1\mu_2\mu_3}, \ldots, \varepsilon_{\upsilon_1\upsilon_2\upsilon_3}$) are called the unit trivectors with coordinates

$$\begin{gathered}\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1,\\ \varepsilon^{\alpha_1\alpha_2\alpha_3} = 0 \text{ otherwise},\\ (\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1,\\ \varepsilon_{\mu_1\mu_2\mu_3} = 0 \text{ otherwise}).\end{gathered} \tag{55}$$

Next we will focus on the construction of the invariants of the system $s^3(3)$. In this case the type (52) of the invariants will be written as

$$(0, d; 0). \tag{56}$$

Therefore, the lowest degree of the invariant (51) of the system $s^3(3)$ has the type

$$(0, 3; 0).$$

In this case we can write unambiguously

$$a^{-}_{---}a^{-}_{---}a^{-}_{---}\varepsilon^{\alpha_1\alpha_2\alpha_3}\varepsilon^{\omega_1\omega_2\omega_3}. \tag{57}$$

The question arises, how will the indices of the expression (57) be trained in the total convolution?

In this situation, Aronhold's symbolic is welcome [3].

Invariants of the type (56) for the system $s^3(3)$ can be written using Aronhold's symbolic [3], from which it is easy to pass to the tensor form and vice versa.

**Remark 5.2.** *We will describe the work with Aronhold's symbolic. We introduce for the tensor coefficients of the system* $s^3(3)$ *the symbolic Aronhold equality*

$$a^j_{\alpha\beta\gamma} = a^j \cdot a_\alpha \cdot a_\beta \cdot a_\gamma \ (j,\alpha,\beta,\gamma = \overline{1,3}), \tag{58}$$

*which we will denote by the* ***ideal vector*** $a$ *(*$a^j, a_\alpha, a_\beta, a_\gamma$*-its coordinates). According to the above mentioned, the invariants of the system* $s^3(3)$ *are a product of tensor coefficients. In this case, we will examine the removal of some misunderstandings in the product of tensors with Aronhold's symbolic.*

*Suppose, for example, that for the tensor* $a^j_{\alpha\beta\gamma}$ *it is necessary to write the product*

$$a^j_{\alpha\beta\gamma}a^k_{\delta\mu\nu}. \tag{59}$$

*Taking into account* (58) *with the* ***ideal vector*** $a$ *(the symbols* $a^j$, $a_\alpha$, $a_\beta$, $a_\gamma$ *do not make sense by themselves, only their product does), we obtain that for* (59) *the symbolic notation has the form*

$$a^j a_\alpha a_\beta a_\gamma a^k a_\delta a_\mu a_\nu, \tag{60}$$

*which can mean, for example,* $a^j_{\alpha\delta\mu}a^k_{\beta\gamma\nu}$, *as well as* $a^j_{\alpha\beta\delta}a^k_{\gamma\mu\nu}$ *and so on. To remove this polysemy, another ideal vector* $b$ *is introduced for which analogously we have*

$$a^j_{\alpha\beta\gamma} = b^j b_\alpha b_\beta b_\gamma. \tag{61}$$

*In this way, the* ***ideal vectors*** $a$ *and* $b$ *are different, but the products* (58) *and* (61) *are the same. Such ideal vectors* $a$ *and* $b$ *are called* ***parallel ideal vectors****, or they are also called* ***parallel symbols****. Then for the product* (59) *we obtain the exclusion of any misunderstanding and will write*

$$a^j_{\alpha\beta\gamma}a^k_{\delta\mu\nu} = a^j a_\alpha a_\beta a_\gamma b^k b_\delta b_\mu b_\nu,$$

*or*

$$a^j_{\alpha\beta\gamma}a^k_{\delta\mu\nu} = b^j b_\alpha b_\beta b_\gamma a^k a_\delta a_\mu a_\nu.$$

*These two symbolic writing are equal in rights.*

*In the case of the cubic product* (57) *of the coordinates of the tensor coefficient* $a^j_{\alpha\beta\gamma}$ *it is necessary to introduce three parallel symbols to each other and so on, when we have products of higher degree.*

*We proceed analogously in the case of the symbolic writing of polynomial differential systems with different tensor coefficients.*

Now we will examine the product (57) with three parallel symbols $a$, $b$, $c$ by writing the first expression

$$j_{1,3} = (\widetilde{a}a)(\widetilde{b}b)(\widetilde{c}c)[abc]^2, \tag{62}$$

where

$$(\widetilde{a}a) = a^1a_1 + a^2a_2 + a^3a_3,\ (\widetilde{b}b) = b^1b_1 + b^2b_2 + b^3b_3,$$
$$(\widetilde{c}c) = c^1c_1 + c^2c_2 + c^3c_3,$$
$$[abc] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \tag{63}$$

with the tensorial product

$$j_{1,3} = a^\alpha_{\alpha ps}a^\beta_{\beta qt}a^\gamma_{\gamma ru}\varepsilon^{pqr}\varepsilon^{stu}. \tag{64}$$

Examining all the possibilities of placing the parallel symbols $a$, $b$, $c$ in the round brackets (in those squares only the sign can be changed, but since it is squared, this does not matter, but it is necessary that the same symbol does not occur in the square bracket, because a priori it is zero) we obtain the following expressions:

$$\begin{aligned} j_{2,3} &= (\widetilde{a}a)(\widetilde{b}c)(\widetilde{c}b)[abc]^2, \\ j_3 &= (\widetilde{a}b)(\widetilde{b}a)(\widetilde{c}c)[abc]^2, \\ j_{3,3} &= (\widetilde{a}b)(\widetilde{b}c)(\widetilde{c}a)[abc]^2, \\ j_4 &= (\widetilde{a}c)(\widetilde{b}a)(\widetilde{c}b)[abc]^2, \\ j_5 &= (\widetilde{a}c)(\widetilde{b}b)(\widetilde{c}a)[abc]^2. \end{aligned} \tag{65}$$

Interchanging the parallel symbols we obtain

$$\underset{a \leftrightarrow b}{j_{2,3} = j_5},\ \underset{a \leftrightarrow c}{j_{2,3} = j_3},\ \underset{a \leftrightarrow b}{j_{3,3} = j_4}. \tag{66}$$

From these 5 expressions form (68) only two are obtained, to which the tensor expressions correspond

$$j_{2,3} = a^\alpha_{\alpha ps}a^\beta_{\gamma qt}a^\gamma_{\beta ru}\varepsilon^{pqr}\varepsilon^{stu},\ j_{3,3} = a^\alpha_{\beta ps}a^\beta_{\gamma qt}a^\gamma_{\alpha ru}\varepsilon^{pqr}\varepsilon^{stu}. \tag{67}$$

Using the Jacobi matrix constructed on these elements we determine that they are functionally independent.

The next type that exists is $(0,6;0)$, using the procedure described above, 15 functionally independent invariants were determined

$$
\begin{aligned}
j_{4,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\gamma\beta_2\gamma_2} a^{\delta}_{\delta\alpha_2\beta_3} a^{\mu}_{\mu\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\ \alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{5,3} &= a^{\alpha}_{\nu\alpha_1\gamma_1} a^{\beta}_{\gamma\alpha_2,\delta_1} a^{\gamma}_{\delta\alpha_3\gamma_2} a^{\delta}_{]\beta\beta_1,\delta_2} a^{\mu}_{\alpha\beta_2\gamma_3} a^{\nu}_{\nu\beta_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{6,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\mu\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{7,3} &= a^{\alpha}_{\gamma\alpha_1\gamma_1} a^{\beta}_{\delta\beta_1\delta_1} a^{\gamma}_{\beta\beta_2\gamma_2} a^{\delta}_{\alpha\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\mu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{8,3} &= a^{\alpha}_{\mu\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\beta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{9,3} &= a^{\alpha}_{\nu\alpha_1\gamma_1} a^{\beta}_{\delta\beta_1\delta_1} a^{\gamma}_{\mu\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\beta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{10,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\alpha\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\gamma\alpha_3\delta_2} a^{\nu}_{\delta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{11,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\mu\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\delta\alpha_3\delta_2} a^{\nu}_{\alpha\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{12,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\gamma\beta_1\delta_1} a^{\gamma}_{\nu\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\delta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{13,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\alpha\alpha_2\beta_3} a^{\mu}_{\mu\alpha_3\delta_2} a^{\nu}_{\gamma\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{14,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\gamma\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\delta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{15,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\gamma\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{16,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\gamma\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{17,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\delta\beta_1\delta_1} a^{\gamma}_{\mu\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{18,3} &= a^{\alpha}_{\gamma\alpha_1\gamma_1} a^{\beta}_{\beta\beta_1\delta_1} a^{\gamma}_{\alpha\beta_2\gamma_2} a^{\delta}_{\delta\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\mu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3}.
\end{aligned}
\tag{68}
$$

**Remark 5.3.** *In this way,* 18 *functionally independent irreducible invariants were constructed that are contained in a functional basis of the invariants of the ternary differential system* $s^3(3)$.

Analogously, we construct the centro-affine comitants of the system $s^3(3)$, which according to the form (51) will be written

$$
\underbrace{x^{-}x^{-}\cdots x^{-}}_{\delta}\ \underbrace{a^{-}_{---}a^{-}_{---}\cdots a^{-}_{---}}_{d}\ \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3}\ldots\varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}\ \underbrace{\varepsilon_{\mu_1\mu_2\mu_3}\cdots\varepsilon_{\upsilon_1\upsilon_2\upsilon_3}}_{g_2}, \tag{69}
$$

where each upper index meets once again lower. We obtain that $3d + 3g_2 = \delta + d + 3g_1$. From here we have

$$
2d = \delta + 3g_1 - 3g_2, \tag{70}
$$

where $d$, $\delta$, $g_1$ and $g_2$ are nonegative integer numbers.

The first possible type is $(2,1;0)$, of this type we have only one comitant

$$k_{1,3} = a^{\alpha}_{\alpha\beta\gamma}x^{\beta}x^{\gamma}. \tag{71}$$

There are two comitants of the type $(4,2;0)$

$$k_{2,3} = a^{\alpha}_{\beta\gamma\mu}a^{\beta}_{\alpha\delta\nu}x^{\gamma}x^{\delta}x^{\mu}x^{\nu},\ k_{3,3} = a^{\alpha}_{\alpha\beta\gamma}a^{\beta}_{\delta\mu\theta}x^{\gamma}x^{\delta}x^{\mu}x^{\theta}. \tag{72}$$

There are four comitants of the type $(3,3;0)$

$$\begin{aligned} k_{4,3} &= a^{\alpha}_{\beta\delta\mu}a^{\beta}_{\gamma\nu\pi}a^{\gamma}_{\alpha\varrho\delta}x^{\delta}x^{\nu}x^{\varrho}\varepsilon^{\mu\pi\delta},\ k_{5,3} = a^{\alpha}_{\alpha\beta\delta}a^{\beta}_{\gamma\mu\pi}a^{\gamma}_{\nu\varrho\sigma}x^{\mu}x^{\nu}x^{\varrho}\varepsilon^{\delta\pi\sigma},\\ k_{6,3} &= a^{\alpha}_{\beta\delta\pi}a^{\beta}_{\alpha\gamma\mu}a^{\gamma}_{\nu\varrho\sigma}x^{\delta}x^{\nu}x^{\varrho}\varepsilon^{\pi\mu\sigma},\ k_{7,3} = a^{\alpha}_{\alpha\gamma\delta}a^{\beta}_{\beta\mu\nu}a^{\gamma}_{\pi\varrho\sigma}x^{\mu}x^{\pi}x^{\varrho}\varepsilon^{\delta\nu\sigma}. \end{aligned} \tag{73}$$

There are two comitants of the type $(6,3;0)$

$$\begin{aligned} k_{8,3} &= a^{\alpha}_{\beta\nu\tau}a^{\beta}_{\alpha\gamma\phi}a^{\gamma}_{\delta\mu\psi}x^{\delta}x^{\mu}x^{\nu}x^{\tau}x^{\phi}x^{\psi},\\ k_{9,3} &= a^{\alpha}_{\gamma\tau p}a^{\beta}_{\mu\phi q}a^{\gamma}_{\nu\psi r}x^{\delta}x^{\mu}x^{\nu}x^{\tau}x^{\phi}x^{\psi}\varepsilon^{pqr}\varepsilon_{\alpha\beta\gamma}. \end{aligned} \tag{74}$$

**Theorem 5.1.** *The invariants $j_{1,3} - j_{16,3}$ from* (68) *together with the comitants $k_{1,3} - k_{9,3}$ from* (71) – (74) *form a functional basis of the irreducible comitants of the ternary differential system $s^3(3)$.*

Analogously, we construct the centro-affine contravariants of the system $s^3(3)$ for which ($\delta = g_2 = 0$), and according to the form (51) there will be written

$$\underbrace{a^{-}_{---}a^{-}_{---}\cdots a^{-}_{---}}_{d}\ \underbrace{u_{-}u_{-}\cdots u_{-}}_{\omega}\ \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3}\ldots\varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}, \tag{75}$$

where each index from above meets again from below. We get $3d + \omega = d + 3g_1$. From here we have

$$2d = 3g_1 - \omega, \tag{76}$$

where $d$, $\omega$ and $g_1$ are natural numbers.

Two contravariants of the type $(0,2;2)$ were constructed

$$\begin{aligned} r_{1,3} &= a^{\alpha}_{\alpha\alpha_1\beta_1}a^{\beta}_{\beta\alpha_2\beta_2}u_{\alpha_3}u_{\beta_3}\varepsilon^{\alpha_1\alpha_2\alpha_3}\varepsilon^{\beta_1\beta_2\beta_3},\\ r_{2,3} &= a^{\alpha}_{\beta\alpha_1\beta_1}a^{\beta}_{\alpha\alpha_2\beta_2}u_{\alpha_3}u_{\beta_3}\varepsilon^{\alpha_1\alpha_2\alpha_3}\varepsilon^{\beta_1\beta_2\beta_3}. \end{aligned} \tag{77}$$

Four contravariants of the type $(0,3;3)$ were constructed

$$\begin{aligned} r_{3,3} &= a^{\alpha}_{\alpha\beta\alpha_1}a^{\beta}_{\gamma\alpha_2\beta_1}a^{\gamma}_{\beta_2\alpha_3\beta_3}u_{\gamma_1}u_{\gamma_2}u_{\gamma_3}\varepsilon^{\alpha_1\alpha_3\gamma_1}\varepsilon^{\alpha_2\beta_2\gamma_2}\varepsilon^{\beta_1\beta_3\gamma_3},\\ r_{4,3} &= a^{\alpha}_{\alpha\gamma\alpha_1}a^{\beta}_{\beta\alpha_2\beta_1}a^{\gamma}_{\beta_2\alpha_3\beta_3}u_{\gamma_1}u_{\gamma_2}u_{\gamma_3}\varepsilon^{\alpha_1\alpha_3\gamma_1}\varepsilon^{\alpha_2\beta_2\gamma_2}\varepsilon^{\beta_1\beta_3\gamma_3},\\ r_{5,3} &= a^{\alpha}_{\beta\gamma\alpha_1}a^{\beta}_{\alpha\alpha_2\beta_1}a^{\gamma}_{\beta_2\alpha_3\beta_3}u_{\gamma_1}u_{\gamma_2}u_{\gamma_3}\varepsilon^{\alpha_1\alpha_3\gamma_1}\varepsilon^{\alpha_2\beta_2\gamma_2}\varepsilon^{\beta_1\beta_3\gamma_3},\\ r_{6,3} &= a^{\alpha}_{\beta\alpha_1\beta_1}a^{\beta}_{\gamma\alpha_2\beta_2}a^{\gamma}_{\alpha\alpha_3\beta_3}u_{\gamma_1}u_{\gamma_2}u_{\gamma_3}\varepsilon^{\alpha_1\alpha_3\gamma_1}\varepsilon^{\beta_1\beta_2\gamma_2}\varepsilon^{\alpha_2\beta_3\gamma_3}. \end{aligned} \tag{78}$$

One contravariant of the type $(0,4;1)$ was constructed

$$r_{7,3}=a^{\alpha}_{\beta\alpha_1\beta_1}a^{\beta}_{\gamma\alpha_2\beta_2}a^{\gamma}_{\delta\alpha_3\beta_3}a^{\delta}_{\alpha\gamma_1\gamma_2}u_{\gamma_3}\varepsilon^{\alpha_1\alpha_2\alpha_3}\varepsilon^{\beta_1\beta_2\gamma_1}\varepsilon^{\beta_3\gamma_2\gamma_3}. \tag{79}$$

**Theorem 5.2.** *The invariants $j_{1,3}-j_{18,3}$ from* (68) *together with the contravariants $r_{1,3}-r_{7,3}$ from* (77)–(79) *form a functional basis of the irreducible contravariants of the ternary differential system $s^3(3)$.*

Analogously, we construct the centro-affine mixed comitants of the system $s^3(3)$ with the type (51), therefore we have

$$2d=\delta+3g_1-3g_2-\omega, \tag{80}$$

where $d$, $\delta$, $\omega$, $g_1$ and $g_2$ are nonnegative integer numbers.

One mixed comitant of the type $(3,1;1)$ was constructed

$$s_{1,3}=a^{\alpha}_{\beta\gamma\delta}x^{\beta}x^{\gamma}x^{\delta}u_{\alpha}. \tag{81}$$

Two mixed comitants of the type $(1,3;1)$ were constructed

$$\begin{aligned} s_{2,3}&=a^{\alpha}_{\alpha\alpha_1\beta_1}a^{\beta}_{\beta\alpha_2\beta_2}a^{\gamma}_{\gamma_1\gamma_2\gamma_3}x^{\gamma_1}u_{\gamma}\varepsilon^{\alpha_1\alpha_2\gamma_2}\varepsilon^{\beta_1\beta_2\gamma_3},\\ s_{3,3}&=a^{\alpha}_{\beta\alpha_1\beta_1}a^{\beta}_{\gamma\alpha_2\beta_2}a^{\gamma}_{\gamma_1\gamma_2\gamma_3}x^{\gamma_1}u_{\alpha}\varepsilon^{\alpha_1\alpha_2\gamma_2}\varepsilon^{\beta_1\beta_2\gamma_3}. \end{aligned} \tag{82}$$

One mixed comitant of the type $(2,3;2)$ was constructed

$$s_{4,3}=a^{\alpha}_{\gamma\alpha_1\beta_1}a^{\beta}_{\alpha_2\beta_2\gamma_1}a^{\gamma}_{\alpha_3\beta_3\gamma_2}x^{\alpha_2}x^{\alpha_3}u_{\alpha}u_{\beta}\varepsilon^{\alpha_1\beta_2\beta_3}\varepsilon_{\beta_1\gamma_1\gamma_2}. \tag{83}$$

One mixed comitant of the type $(3,3;3)$ was constructed

$$s_{5,3}=a^{\alpha}_{\alpha_1\beta_1\gamma_1}a^{\beta}_{\alpha_2\beta_2\gamma_2}a^{\gamma}_{\alpha_3\beta_3\gamma_3}x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}u_{\alpha}u_{\beta}u_{\gamma}\varepsilon^{\beta_1\beta_2\beta_3}\varepsilon^{\gamma_1\gamma_2\gamma_3}. \tag{84}$$

**Remark 5.4.** *In this way,* 5 *functionally independent irreducible mixed comitants were constructed that are contained in a functional basis of the mixed comitants of the ternary differential system $s^3(3)$.*

**Remark 5.5.** *To prove the functional independence of the invariant polynomials, the rank of the Jacobi matrix of these elements was calculated at the point*

$$\begin{gathered} x^1=1,\ x^2=1,\ x^3=1,\ u_1=1,\ u_2=1,\ u_3=1,\ a^1_{111}=1,\ a^1_{112}=0,\ a^1_{113}=1,\\ a^1_{122}=1,\ a^1_{123}=-1,\ a^1_{133}=-1,\ a^1_{222}=0,\ a^1_{223}=0,\ a^1_{233}=1,\ a^1_{333}=1,\\ a^2_{111}=1,\ a^2_{112}=-1,\ a^2_{113}=0,\ a^2_{122}=1,\ a^2_{123}=0,\ a^2_{133}=1,\ a^2_{222}=0,\\ a^2_{223}=0,\ a^2_{233}=-1,\ a^2_{333}=1,\ a^3_{111}=1,\ a^3_{112}=-1,\ a^3_{113}=0,\ a^3_{122}=-1,\\ a^3_{123}=0,\ a^3_{133}=-1,\ a^3_{222}=0,\ a^3_{223}=0,\ a^3_{233}=1,\ a^3_{333}=-1. \end{gathered}$$

## References


[1] Goriely, A.. *Integrability and nonintegrability of dynamical systems*. USA, University of Arizona, 2001, 415 p.

[2] Ovsyannikov, L.V. *Group analysis of differential equations*. Academic Press, 1982, 432 p.

[3] Gurevich, G.B. *Fundation of the theory of algebraic invariants*. Groningen: Noordhoff, 1964, 429 p.

[4] Sibirsky, K.S. *Introduction to the algebraic theory of invariants of differential equations*. Manchester-New York, Manchester University Press, 1988, VII, 169 p.

[5] Vulpe, N.I. *Polynomial bases of comitants of differential systems and their applications in qualitative theory*. Chişinău, Shtiintsa, 1986, 171 p. (in Russian).

[6] Popa, M.N. *Algebraic methods for differential systems*. Piteshti University, Ser. Appl. Ind. Math., 2004, vol. 15, 340 p. (in Romanian).

[7] Popa, M.N. *Applications of algebras to differential systems*. Chişinău, IMI, AŞM, 2001, 224 p. (in Russian).

[8] Diaconescu, O. *Lie algebras and invariant integrals for polynomial differential systems*. PhD thesis, Chişinău, 2008, 126 p. (in Russian).

[9] Gherştega, N. *Lie algebras for the three-dimensional differential system and applications*. PhD thesis, Chişinău, 2006, 133 p. (in Russian).

[10] Neagu, N. *Lie algebras and invariants for differential systems with projections on some mathematical models*. PhD thesis, Chişinău, 2017, 125 p. (in Romanian).


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