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The Lie algebra admitted by the ternary differential system with nonlinearities of degree six

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Abstract. In this paper, the ternary differential system with nonlinearities of degree six is investigated. For this system, nine Lie operators providing a linear representation of one-parameter elementary groups are determined in the space of phase variables and coefficients. The commutators of these operators, which form a Lie algebra L_9, are constructed. The structure constants are determined, and the type of the Lie algebra is analyzed. By means of the Killing form, it is proved that the Lie algebra L_9 is reductive.
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Keywords: differential system, Lie algebra, continuous one-parameter elementary group, structure equation, associated application matrix, Killing's form.

Algebra Lie admisă de sistemul diferenţial ternar cu neliniarităţi de gradul șase

Rezumat. În acest articol a fost studiat sistemul diferenţial ternar cu neliniarităţi de gradul şase. Pentru acest sistem au fost determinaţi nouă operatori Lie de reprezentare liniară a grupurilor elementare uniparametrice în spaţiul coeficienţilor şi variabilelor de fază. Au fost construiţi comutatorii acestor operatori, ce formează o algebră Lie L_9. Au fost determinate constantele de structură şi studiat tipul algebrei Lie. Cu ajutorul formei lui Killing, a fost demonstrat că algebra Lie L_9 este reductivă.
Cuvinte-cheie: sistem diferenţial, algebră Lie, grup elementar continuu uniparametric, ecuaţie de structură, matricea aplicaţiei asociate, forma Killing.

1. Introduction

An important role in the theory of ordinary differential equations is attributed to the class of autonomous systems of first-order differential equations whose right-hand sides are polynomials with real coefficients. The study of these systems is actual because they arise in many problems of a theoretical or applied nature.

In this paper, we examine the differential system $s^3(1,6)$ of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta\mu\nu} x^\alpha x^\beta x^\gamma x^\delta x^\mu x^\nu \quad (j,\alpha,\beta,\gamma,\delta,\mu,\nu = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta\gamma\delta\mu\nu}$ is a symmetric tensor in lower indices in which the total convolution is done [1], [2], [3], [4]. The coefficients and variables of this system take values from the field of real numbers $\mathbb{R}$.

2. Lie operators of representation of elementary uniparametric transformation groups of the ternary differential system with nonlinearities of degree six

Remark 2.1. *In three-dimensional space, the number of elementary one-parameter continuous groups is determined by the following nine cases* [8]*:*

$$\bar{x}^1 = \lambda_1 x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3\ (\lambda_1 \in \mathbb{R} \setminus \{0\}), \tag{2}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = \lambda_2 x^2,\ \bar{x}^3 = x^3\ (\lambda_2 \in \mathbb{R} \setminus \{0\}), \tag{3}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \lambda_3 x^3\ (\lambda_3 \in \mathbb{R} \setminus \{0\}), \tag{4}$$

$$\bar{x}^1 = x^1 + \mu_1 x^2,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3\ (\mu_1 \in \mathbb{R}), \tag{5}$$

$$\bar{x}^1 = x^1 + \mu_2 x^3,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3\ (\mu_2 \in \mathbb{R}), \tag{6}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = \mu_3 x^1 + x^2,\ \bar{x}^3 = x^3\ (\mu_3 \in \mathbb{R}), \tag{7}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2 + \mu_4 x^3,\ \bar{x}^3 = x^3\ (\mu_4 \in \mathbb{R}), \tag{8}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \mu_5 x^1 + x^3\ (\mu_5 \in \mathbb{R}), \tag{9}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \mu_6 x^2 + x^3\ (\mu_6 \in \mathbb{R}), \tag{10}$$

the groups denoted respectively by H_i with parameters λ_i $(i = \overline{1,3})$ and H_{j+3} with parameters μ_j $(j = \overline{1,6})$, in the phase space $E^3(x)$ of system (1), where $x = (x^1, x^2, x^3)$.

It is known [2], [3], [7] that to each one-parameter transformation group acting in the space $E^{96}(x, a)$ of system (1), we can associate an operator of the form

$$X = \xi^j \frac{\partial}{\partial x^j} + \mathsf{d}\ \ (j = \overline{1,3}), \tag{11}$$

where

$$\mathsf{d} = \eta^j_\alpha \frac{\partial}{\partial a^j_\alpha} + \eta^j_{\alpha\beta\gamma\delta\mu\nu} \frac{\partial}{\partial a^j_{\alpha\beta\gamma\delta\mu\nu}}\ \ (j, \alpha, \beta, \gamma, \delta, \mu, \nu = \overline{1,3}), \tag{12}$$

and

$$\xi^j = \frac{\partial \bar{x}^j}{\partial \alpha}|_{\alpha=\alpha_0},\ \ \eta^j_\alpha = \frac{\partial \bar{a}^j_\alpha}{\partial \alpha}|_{\alpha=\alpha_0},\ \ \eta^j_{\alpha\beta\gamma\delta} = \frac{\partial \bar{a}^j_{\alpha\beta\gamma\delta\mu\nu}}{\partial \alpha}|_{\alpha=\alpha_0}, \tag{13}$$
$$(j, \alpha, \beta, \gamma, \delta, \mu, \nu = \overline{1,3})$$

when $\alpha_0 = 0$, and conversely.

The following theorem holds

Theorem 2.1. *The Lie operators corresponding to the representations of the groups H_i $(i=\overline{1,3})$ and H_{j+3} $(j=\overline{1,6})$, from (2) - (10), in the space of variables and coefficients $E^{96}(x,a)$ of system (1), are*

$$
\begin{aligned}
&X_1 = x^1\frac{\partial}{\partial x^1} + \mathsf{d}_1, \quad X_2 = x^2\frac{\partial}{\partial x^2} + \mathsf{d}_2, \quad X_3 = x^3\frac{\partial}{\partial x^3} + \mathsf{d}_3,\\
&X_4 = x^2\frac{\partial}{\partial x^1} + \mathsf{d}_4, \quad X_5 = x^3\frac{\partial}{\partial x^1} + \mathsf{d}_5, \quad X_6 = x^1\frac{\partial}{\partial x^2} + \mathsf{d}_6,\\
&X_7 = x^3\frac{\partial}{\partial x^2} + \mathsf{d}_7, \quad X_8 = x^1\frac{\partial}{\partial x^3} + \mathsf{d}_8, \quad X_9 = x^2\frac{\partial}{\partial x^3} + \mathsf{d}_9,
\end{aligned}
\tag{14}
$$

where

$$
\mathsf{d}_i = \mathsf{D}_i^{(1)} + \mathsf{D}_i^{(6)} \ \ (i=\overline{1,9}),
\tag{15}
$$

and

$$
\begin{aligned}
\mathsf{D}_1^{(1)} &= a_2^1\frac{\partial}{\partial a_2^1} + a_3^1\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_1^2} - a_1^3\frac{\partial}{\partial a_1^3},\\
\mathsf{D}_2^{(1)} &= -a_2^1\frac{\partial}{\partial a_2^1} + a_1^2\frac{\partial}{\partial a_1^2} + a_3^2\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_2^3},\\
\mathsf{D}_3^{(1)} &= -a_3^1\frac{\partial}{\partial a_3^1} - a_3^2\frac{\partial}{\partial a_3^2} + a_1^3\frac{\partial}{\partial a_1^3} + a_2^3\frac{\partial}{\partial a_2^3},\\
\mathsf{D}_4^{(1)} &= a_1^2\frac{\partial}{\partial a_1^1} + (a_2^2 - a_1^1)\frac{\partial}{\partial a_2^1} + a_3^2\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_2^2} - a_1^3\frac{\partial}{\partial a_2^3},\\
\mathsf{D}_5^{(1)} &= a_1^3\frac{\partial}{\partial a_1^1} + a_2^3\frac{\partial}{\partial a_2^1} + (a_3^3 - a_1^1)\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_3^2} - a_1^3\frac{\partial}{\partial a_3^3},\\
\mathsf{D}_6^{(1)} &= -a_2^1\frac{\partial}{\partial a_1^1} + (a_1^1 - a_2^2)\frac{\partial}{\partial a_1^2} + a_2^1\frac{\partial}{\partial a_2^2} + a_3^1\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_1^3},\\
\mathsf{D}_7^{(1)} &= -a_2^1\frac{\partial}{\partial a_3^1} + a_1^3\frac{\partial}{\partial a_1^2} + a_2^3\frac{\partial}{\partial a_2^2} + (a_3^3 - a_2^2)\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_3^3},\\
\mathsf{D}_8^{(1)} &= -a_3^1\frac{\partial}{\partial a_1^1} - a_3^2\frac{\partial}{\partial a_1^2} + (a_1^1 - a_3^3)\frac{\partial}{\partial a_1^3} + a_2^1\frac{\partial}{\partial a_2^3} + a_3^1\frac{\partial}{\partial a_3^3},\\
\mathsf{D}_9^{(1)} &= -a_3^1\frac{\partial}{\partial a_2^1} - a_3^2\frac{\partial}{\partial a_2^2} + a_1^2\frac{\partial}{\partial a_1^3} + (a_2^2 - a_3^3)\frac{\partial}{\partial a_2^3} + a_3^2\frac{\partial}{\partial a_3^3},
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\mathsf{D}_1^{(6)} = {}& -5a_{111111}^1\frac{\partial}{\partial a_{111111}^1} - 4a_{111112}^1\frac{\partial}{\partial a_{111112}^1} - 4a_{111113}^1\frac{\partial}{\partial a_{111113}^1} -\\
& -3a_{111122}^1\frac{\partial}{\partial a_{111122}^1} - 3a_{111123}^1\frac{\partial}{\partial a_{111123}^1} - 3a_{111133}^1\frac{\partial}{\partial a_{111133}^1} - 2a_{111222}^1\frac{\partial}{\partial a_{111222}^1} -\\
& -2a_{111223}^1\frac{\partial}{\partial a_{111223}^1} - 2a_{111233}^1\frac{\partial}{\partial a_{111233}^1} - 2a_{111333}^1\frac{\partial}{\partial a_{111333}^1} - a_{112222}^1\frac{\partial}{\partial a_{112222}^1} -
\end{aligned}
$$

$$-a^1_{112223}\frac{\partial}{\partial a^1_{112223}} - a^1_{112233}\frac{\partial}{\partial a^1_{112233}} - a^1_{112333}\frac{\partial}{\partial a^1_{112333}} - a^1_{113333}\frac{\partial}{\partial a^1_{113333}}+$$

$$+a^1_{222222}\frac{\partial}{\partial a^1_{222222}} + a^1_{222223}\frac{\partial}{\partial a^1_{222223}} + a^1_{222233}\frac{\partial}{\partial a^1_{222233}} + a^1_{222333}\frac{\partial}{\partial a^1_{222333}}+$$

$$+a^1_{223333}\frac{\partial}{\partial a^1_{223333}} + a^1_{233333}\frac{\partial}{\partial a^1_{233333}} + a^1_{333333}\frac{\partial}{\partial a^1_{333333}} - 6a^2_{111111}\frac{\partial}{\partial a^2_{111111}}-$$

$$-5a^2_{111112}\frac{\partial}{\partial a^2_{111112}} - 5a^2_{111113}\frac{\partial}{\partial a^2_{111113}} - 4a^2_{111122}\frac{\partial}{\partial a^2_{111122}} - 4a^2_{111123}\frac{\partial}{\partial a^2_{111123}}-$$

$$-4a^2_{111133}\frac{\partial}{\partial a^2_{111133}} - 3a^2_{111222}\frac{\partial}{\partial a^2_{111222}} - 3a^2_{111223}\frac{\partial}{\partial a^2_{111223}} - 3a^2_{111233}\frac{\partial}{\partial a^2_{111233}}-$$

$$-3a^2_{111333}\frac{\partial}{\partial a^2_{111333}} - 2a^2_{112222}\frac{\partial}{\partial a^2_{112222}} - 2a^2_{112223}\frac{\partial}{\partial a^2_{112223}} - 2a^2_{112233}\frac{\partial}{\partial a^2_{112233}}-$$

$$-2a^2_{112333}\frac{\partial}{\partial a^2_{112333}} - 2a^2_{113333}\frac{\partial}{\partial a^2_{113333}} - a^2_{122222}\frac{\partial}{\partial a^2_{122222}} - a^2_{122223}\frac{\partial}{\partial a^2_{122223}}-$$

$$-a^2_{122233}\frac{\partial}{\partial a^2_{122233}} - a^2_{122333}\frac{\partial}{\partial a^2_{122333}} - a^2_{123333}\frac{\partial}{\partial a^2_{123333}} - a^2_{133333}\frac{\partial}{\partial a^2_{133333}}-$$

$$-6a^3_{111111}\frac{\partial}{\partial a^3_{111111}} - 5a^3_{111112}\frac{\partial}{\partial a^3_{111112}} - 5a^3_{111113}\frac{\partial}{\partial a^3_{111113}} - 4a^3_{111122}\frac{\partial}{\partial a^3_{111122}}-$$

$$-4a^3_{111123}\frac{\partial}{\partial a^3_{111123}} - 4a^3_{111133}\frac{\partial}{\partial a^3_{111133}} - 3a^3_{111222}\frac{\partial}{\partial a^3_{111222}} - 3a^3_{111223}\frac{\partial}{\partial a^3_{111223}}-$$

$$-3a^3_{111233}\frac{\partial}{\partial a^3_{111233}} - 3a^3_{111333}\frac{\partial}{\partial a^3_{111333}} - 2a^3_{112222}\frac{\partial}{\partial a^3_{112222}} - 2a^3_{112223}\frac{\partial}{\partial a^3_{112223}}-$$

$$-2a^3_{112233}\frac{\partial}{\partial a^3_{112233}} - 2a^3_{112333}\frac{\partial}{\partial a^3_{112333}} - 2a^3_{113333}\frac{\partial}{\partial a^3_{113333}} - a^3_{122222}\frac{\partial}{\partial a^3_{122222}}-$$

$$-a^3_{122223}\frac{\partial}{\partial a^3_{122223}} - a^3_{122233}\frac{\partial}{\partial a^3_{122233}} - a^3_{122333}\frac{\partial}{\partial a^3_{122333}} - a^3_{123333}\frac{\partial}{\partial a^3_{123333}}-$$

$$-a^3_{133333}\frac{\partial}{\partial a^3_{133333}},$$

$$\mathsf{D}_2^{(6)} = -a^1_{111112}\frac{\partial}{\partial a^1_{111112}} - 2a^1_{111122}\frac{\partial}{\partial a^1_{111122}} - a^1_{111123}\frac{\partial}{\partial a^1_{111123}} - 3a^1_{111222}\frac{\partial}{\partial a^1_{111222}}-$$

$$-2a^1_{111223}\frac{\partial}{\partial a^1_{111223}} - a^1_{111233}\frac{\partial}{\partial a^1_{111233}} - 4a^1_{112222}\frac{\partial}{\partial a^1_{112222}} - 3a^1_{112223}\frac{\partial}{\partial a^1_{112223}}-$$

$$-2a^1_{112233}\frac{\partial}{\partial a^1_{112233}} - a^1_{112333}\frac{\partial}{\partial a^1_{112333}} - 5a^1_{122222}\frac{\partial}{\partial a^1_{122222}} - 4a^1_{122223}\frac{\partial}{\partial a^1_{122223}}-$$

$$-3a^1_{122233}\frac{\partial}{\partial a^1_{122233}} - 2a^1_{122333}\frac{\partial}{\partial a^1_{122333}} - a^1_{123333}\frac{\partial}{\partial a^1_{123333}} - 6a^1_{222222}\frac{\partial}{\partial a^1_{222222}}-$$

$$-5a^1_{222223}\frac{\partial}{\partial a^1_{222223}}-4a^1_{222233}\frac{\partial}{\partial a^1_{222233}}-3a^1_{222333}\frac{\partial}{\partial a^1_{222333}}-2a^1_{223333}\frac{\partial}{\partial a^1_{223333}}-$$
$$-a^1_{233333}\frac{\partial}{\partial a^1_{233333}}+a^2_{111111}\frac{\partial}{\partial a^2_{111111}}+a^2_{111113}\frac{\partial}{\partial a^2_{111113}}-a^2_{111122}\frac{\partial}{\partial a^2_{111122}}+$$
$$+a^2_{111133}\frac{\partial}{\partial a^2_{111133}}-2a^2_{111222}\frac{\partial}{\partial a^2_{111222}}-a^2_{111223}\frac{\partial}{\partial a^2_{111223}}+a^2_{111333}\frac{\partial}{\partial a^2_{111333}}-$$
$$-3a^2_{112222}\frac{\partial}{\partial a^2_{112222}}-2a^2_{112223}\frac{\partial}{\partial a^2_{112223}}-a^2_{112233}\frac{\partial}{\partial a^2_{112233}}+a^2_{113333}\frac{\partial}{\partial a^2_{113333}}-$$
$$-4a^2_{122222}\frac{\partial}{\partial a^2_{122222}}-3a^2_{122223}\frac{\partial}{\partial a^2_{122223}}-2a^2_{122233}\frac{\partial}{\partial a^2_{122233}}-a^2_{122333}\frac{\partial}{\partial a^2_{122333}}+$$
$$+a^2_{133333}\frac{\partial}{\partial a^2_{133333}}-5a^2_{222222}\frac{\partial}{\partial a^2_{222222}}-4a^2_{222223}\frac{\partial}{\partial a^2_{222223}}-3a^2_{222233}\frac{\partial}{\partial a^2_{222233}}-$$
$$-2a^2_{222333}\frac{\partial}{\partial a^2_{222333}}-a^2_{223333}\frac{\partial}{\partial a^2_{223333}}+a^2_{333333}\frac{\partial}{\partial a^2_{333333}}-a^3_{111112}\frac{\partial}{\partial a^3_{111112}}-$$
$$-2a^3_{111122}\frac{\partial}{\partial a^3_{111122}}-a^3_{111123}\frac{\partial}{\partial a^3_{111123}}-3a^3_{111222}\frac{\partial}{\partial a^3_{111222}}-2a^3_{111223}\frac{\partial}{\partial a^3_{111223}}-$$
$$-a^3_{111233}\frac{\partial}{\partial a^3_{111233}}-4a^3_{112222}\frac{\partial}{\partial a^3_{112222}}-3a^3_{112223}\frac{\partial}{\partial a^3_{112223}}-2a^3_{112233}\frac{\partial}{\partial a^3_{112233}}-$$
$$-a^3_{112333}\frac{\partial}{\partial a^3_{112333}}-5a^3_{122222}\frac{\partial}{\partial a^3_{122222}}-4a^3_{122223}\frac{\partial}{\partial a^3_{122223}}-3a^3_{122233}\frac{\partial}{\partial a^3_{122233}}-$$
$$-2a^3_{122333}\frac{\partial}{\partial a^3_{122333}}-a^3_{123333}\frac{\partial}{\partial a^3_{123333}}-6a^3_{222222}\frac{\partial}{\partial a^3_{222222}}-5a^3_{222223}\frac{\partial}{\partial a^3_{222223}}-$$
$$-4a^3_{222233}\frac{\partial}{\partial a^3_{222233}}-3a^3_{222333}\frac{\partial}{\partial a^3_{222333}}-2a^3_{223333}\frac{\partial}{\partial a^3_{223333}}-a^3_{233333}\frac{\partial}{\partial a^3_{233333}},$$
$$\mathsf{D}_3^{(6)}=-a^1_{111113}\frac{\partial}{\partial a^1_{111113}}-a^1_{111123}\frac{\partial}{\partial a^1_{111123}}-2a^1_{111133}\frac{\partial}{\partial a^1_{111133}}-a^1_{111223}\frac{\partial}{\partial a^1_{111223}}-$$
$$-2a^1_{111233}\frac{\partial}{\partial a^1_{111233}}-3a^1_{111333}\frac{\partial}{\partial a^1_{111333}}-a^1_{112223}\frac{\partial}{\partial a^1_{112223}}-2a^1_{112233}\frac{\partial}{\partial a^1_{112233}}-$$
$$-3a^1_{112333}\frac{\partial}{\partial a^1_{112333}}-4a^1_{113333}\frac{\partial}{\partial a^1_{113333}}-a^1_{122223}\frac{\partial}{\partial a^1_{122223}}-2a^1_{122233}\frac{\partial}{\partial a^1_{122233}}-$$
$$-3a^1_{122333}\frac{\partial}{\partial a^1_{122333}}-4a^1_{123333}\frac{\partial}{\partial a^1_{123333}}-5a^1_{133333}\frac{\partial}{\partial a^1_{133333}}-a^1_{222223}\frac{\partial}{\partial a^1_{222223}}-$$
$$-2a^1_{222233}\frac{\partial}{\partial a^1_{222233}}-3a^1_{222333}\frac{\partial}{\partial a^1_{222333}}-4a^1_{223333}\frac{\partial}{\partial a^1_{223333}}-5a^1_{233333}\frac{\partial}{\partial a^1_{233333}}-$$
$$-6a^1_{333333}\frac{\partial}{\partial a^1_{333333}}-a^2_{111113}\frac{\partial}{\partial a^2_{111113}}-a^2_{111123}\frac{\partial}{\partial a^2_{111123}}-2a^2_{111133}\frac{\partial}{\partial a^2_{111133}}-$$

$$-a^2_{111223}\frac{\partial}{\partial a^2_{111223}} - 2a^2_{111233}\frac{\partial}{\partial a^2_{111233}} - 3a^2_{111333}\frac{\partial}{\partial a^2_{111333}} - a^2_{112223}\frac{\partial}{\partial a^2_{112223}} -$$
$$-2a^2_{112233}\frac{\partial}{\partial a^2_{112233}} - 3a^2_{112333}\frac{\partial}{\partial a^2_{112333}} - 4a^2_{113333}\frac{\partial}{\partial a^2_{113333}} - a^2_{122223}\frac{\partial}{\partial a^2_{122223}} -$$
$$-2a^2_{122233}\frac{\partial}{\partial a^2_{122233}} - 3a^2_{122333}\frac{\partial}{\partial a^2_{122333}} - 4a^2_{123333}\frac{\partial}{\partial a^2_{123333}} - 5a^2_{133333}\frac{\partial}{\partial a^2_{133333}} -$$
$$-a^2_{222223}\frac{\partial}{\partial a^2_{222223}} - 2a^2_{222233}\frac{\partial}{\partial a^2_{222233}} - 3a^2_{222333}\frac{\partial}{\partial a^2_{222333}} - 4a^2_{223333}\frac{\partial}{\partial a^2_{223333}} -$$
$$-5a^2_{233333}\frac{\partial}{\partial a^2_{233333}} - 6a^2_{333333}\frac{\partial}{\partial a^2_{333333}} + a^3_{111111}\frac{\partial}{\partial a^3_{111111}} + a^3_{111112}\frac{\partial}{\partial a^3_{111112}} +$$
$$+a^3_{111122}\frac{\partial}{\partial a^3_{111122}} - a^3_{111133}\frac{\partial}{\partial a^3_{111133}} + a^3_{111222}\frac{\partial}{\partial a^3_{111222}} - a^3_{111233}\frac{\partial}{\partial a^3_{111233}} -$$
$$-2a^3_{111333}\frac{\partial}{\partial a^3_{111333}} + a^3_{112222}\frac{\partial}{\partial a^3_{112222}} - a^3_{112233}\frac{\partial}{\partial a^3_{112233}} - 2a^3_{112333}\frac{\partial}{\partial a^3_{112333}} -$$
$$-3a^3_{113333}\frac{\partial}{\partial a^3_{113333}} + a^3_{122222}\frac{\partial}{\partial a^3_{122222}} - a^3_{122233}\frac{\partial}{\partial a^3_{122233}} - 2a^3_{122333}\frac{\partial}{\partial a^3_{122333}} -$$
$$-3a^3_{123333}\frac{\partial}{\partial a^3_{123333}} - 4a^3_{133333}\frac{\partial}{\partial a^3_{133333}} + a^3_{222222}\frac{\partial}{\partial a^3_{222222}} - a^3_{222233}\frac{\partial}{\partial a^3_{222233}} -$$
$$-2a^3_{222333}\frac{\partial}{\partial a^3_{222333}} - 3a^3_{223333}\frac{\partial}{\partial a^3_{223333}} - 4a^3_{233333}\frac{\partial}{\partial a^3_{233333}} - 5a^3_{333333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}_4^{(6)} = a^2_{111111}\frac{\partial}{\partial a^1_{111111}} + (a^2_{111112} - a^1_{111111})\frac{\partial}{\partial a^1_{111112}} + a^2_{111113}\frac{\partial}{\partial a^1_{111113}} +$$
$$+(a^2_{111122} - 2a^1_{111112})\frac{\partial}{\partial a^1_{111122}} + (a^2_{111123} - a^1_{111113})\frac{\partial}{\partial a^1_{111123}} + a^2_{111133}\frac{\partial}{\partial a^1_{111133}} +$$
$$+(a^2_{111222} - 3a^1_{111122})\frac{\partial}{\partial a^1_{111222}} + (a^2_{111223} - 2a^1_{111123})\frac{\partial}{\partial a^1_{111223}} + (a^2_{111233} -$$
$$-a^1_{111133})\frac{\partial}{\partial a^1_{111233}} + a^2_{111333}\frac{\partial}{\partial a^1_{111333}} + (a^2_{112222} - 4a^1_{111222})\frac{\partial}{\partial a^1_{112222}} + (a^2_{112223} -$$
$$-3a^1_{111223})\frac{\partial}{\partial a^1_{112223}} + (a^2_{112233} - 2a^1_{111233})\frac{\partial}{\partial a^1_{112233}} + (a^2_{112333} - a^1_{111333})\frac{\partial}{\partial a^1_{112333}} +$$
$$+a^2_{113333}\frac{\partial}{\partial a^1_{113333}} + (a^2_{122222} - 5a^1_{112222})\frac{\partial}{\partial a^1_{122222}} + (a^2_{122223} - 4a^1_{112223})\frac{\partial}{\partial a^1_{122223}} +$$
$$+(a^2_{122233} - 3a^1_{112233})\frac{\partial}{\partial a^1_{122233}} + (a^2_{122333} - 2a^1_{112333})\frac{\partial}{\partial a^1_{122333}} + (a^2_{123333} -$$
$$-a^1_{113333})\frac{\partial}{\partial a^1_{123333}} + a^2_{133333}\frac{\partial}{\partial a^1_{133333}} + (a^2_{222222} - 6a^1_{122222})\frac{\partial}{\partial a^1_{222222}} + (a^2_{222223} -$$

$$-5a^1_{122223})\frac{\partial}{\partial a^1_{222223}}+(a^2_{222233}-4a^1_{122233})\frac{\partial}{\partial a^1_{222233}}+(a^2_{222333}-3a^1_{122333})\frac{\partial}{\partial a^1_{222333}}+$$

$$+(a^2_{223333}-2a^1_{123333})\frac{\partial}{\partial a^1_{223333}}+(a^2_{233333}-a^1_{133333})\frac{\partial}{\partial a^1_{233333}}+$$

$$+a^2_{333333}\frac{\partial}{\partial a^1_{333333}}-a^2_{111111}\frac{\partial}{\partial a^2_{111112}}-2a^2_{111112}\frac{\partial}{\partial a^2_{111122}}-a^2_{111113}\frac{\partial}{\partial a^2_{111123}}-$$

$$-3a^2_{111122}\frac{\partial}{\partial a^2_{111222}}-2a^2_{111123}\frac{\partial}{\partial a^2_{111223}}-a^2_{111133}\frac{\partial}{\partial a^2_{111233}}-4a^2_{111222}\frac{\partial}{\partial a^2_{112222}}-$$

$$-3a^2_{111223}\frac{\partial}{\partial a^2_{112223}}-2a^2_{111233}\frac{\partial}{\partial a^2_{112233}}-a^2_{111333}\frac{\partial}{\partial a^2_{112333}}-5a^2_{112222}\frac{\partial}{\partial a^2_{122222}}-$$

$$-4a^2_{112223}\frac{\partial}{\partial a^2_{122223}}-3a^2_{112233}\frac{\partial}{\partial a^2_{122233}}-2a^2_{112333}\frac{\partial}{\partial a^2_{122333}}-a^2_{113333}\frac{\partial}{\partial a^2_{123333}}-$$

$$-6a^2_{122222}\frac{\partial}{\partial a^2_{222222}}-5a^2_{122223}\frac{\partial}{\partial a^2_{222223}}-4a^2_{122233}\frac{\partial}{\partial a^2_{222233}}-3a^2_{122333}\frac{\partial}{\partial a^2_{222333}}-$$

$$-2a^2_{123333}\frac{\partial}{\partial a^2_{223333}}-a^2_{133333}\frac{\partial}{\partial a^2_{233333}}-a^3_{111111}\frac{\partial}{\partial a^3_{111112}}-2a^3_{111112}\frac{\partial}{\partial a^3_{111122}}-$$

$$-a^3_{111113}\frac{\partial}{\partial a^3_{111123}}-3a^3_{111122}\frac{\partial}{\partial a^3_{111222}}-2a^3_{111123}\frac{\partial}{\partial a^3_{111223}}-a^3_{111133}\frac{\partial}{\partial a^3_{111233}}-$$

$$-4a^3_{111222}\frac{\partial}{\partial a^3_{112222}}-3a^3_{111223}\frac{\partial}{\partial a^3_{112223}}-2a^3_{111233}\frac{\partial}{\partial a^3_{112233}}-a^3_{111333}\frac{\partial}{\partial a^3_{112333}}-$$

$$-5a^3_{112222}\frac{\partial}{\partial a^3_{122222}}-4a^3_{112223}\frac{\partial}{\partial a^3_{122223}}-3a^3_{112233}\frac{\partial}{\partial a^3_{122233}}-2a^3_{112333}\frac{\partial}{\partial a^3_{122333}}-$$

$$-a^3_{113333}\frac{\partial}{\partial a^3_{123333}}-6a^3_{122222}\frac{\partial}{\partial a^3_{222222}}-5a^3_{122223}\frac{\partial}{\partial a^3_{222223}}-4a^3_{122233}\frac{\partial}{\partial a^3_{222233}}-$$

$$-3a^3_{122333}\frac{\partial}{\partial a^3_{222333}}-2a^3_{123333}\frac{\partial}{\partial a^3_{223333}}-a^3_{133333}\frac{\partial}{\partial a^3_{233333}},$$

$$\mathsf{D}_5^{(6)}=a^3_{111111}\frac{\partial}{\partial a^1_{111111}}+a^3_{111112}\frac{\partial}{\partial a^1_{111112}}+(a^3_{111113}-a^1_{111111})\frac{\partial}{\partial a^1_{111113}}+$$

$$+a^3_{111122}\frac{\partial}{\partial a^1_{111122}}+(a^3_{111123}-a^1_{111112})\frac{\partial}{\partial a^1_{111123}}+(a^3_{111133}-2a^1_{111113})\frac{\partial}{\partial a^1_{111133}}+$$

$$+a^3_{111222}\frac{\partial}{\partial a^1_{111222}}+(a^3_{111223}-a^1_{111122})\frac{\partial}{\partial a^1_{111223}}+(a^3_{111233}-2a^1_{111123})\frac{\partial}{\partial a^1_{111233}}+$$

$$+(a^3_{111333}-3a^1_{111133})\frac{\partial}{\partial a^1_{111333}}+a^3_{112222}\frac{\partial}{\partial a^1_{112222}}+(a^3_{112223}-a^1_{111222})\frac{\partial}{\partial a^1_{112223}}+$$

$$+(a^3_{112233}-2a^1_{111223})\frac{\partial}{\partial a^1_{112233}}+(a^3_{112333}-3a^1_{111233})\frac{\partial}{\partial a^1_{112333}}+(a^3_{113333}-$$

$$-4a^1_{111333})\frac{\partial}{\partial a^1_{113333}}+a^3_{122222}\frac{\partial}{\partial a^1_{122222}}+(a^3_{122223}-a^1_{112222})\frac{\partial}{\partial a^1_{122223}}+(a^3_{122233}-$$

$$-2a^1_{112223})\frac{\partial}{\partial a^1_{122233}}+(a^3_{122333}-3a^1_{112233})\frac{\partial}{\partial a^1_{122333}}+(a^3_{123333}-4a^1_{112333})\frac{\partial}{\partial a^1_{123333}}+$$

$$+(a^3_{133333}-5a^1_{113333})\frac{\partial}{\partial a^1_{133333}}+a^3_{222222}\frac{\partial}{\partial a^1_{222222}}+(a^3_{222223}-a^1_{122222})\frac{\partial}{\partial a^1_{222223}}+$$

$$+(a^3_{222233}-2a^1_{122223})\frac{\partial}{\partial a^1_{222233}}+(a^3_{222333}-3a^1_{122233})\frac{\partial}{\partial a^1_{222333}}+(a^3_{223333}-$$

$$-4a^1_{122333})\frac{\partial}{\partial a^1_{223333}}+(a^3_{233333}-5a^1_{123333})\frac{\partial}{\partial a^1_{233333}}+(a^3_{333333}-6a^1_{133333})\frac{\partial}{\partial a^1_{333333}}-$$

$$-a^2_{111111}\frac{\partial}{\partial a^2_{111113}}-a^2_{111112}\frac{\partial}{\partial a^2_{111123}}-2a^2_{111113}\frac{\partial}{\partial a^2_{111133}}-a^2_{111122}\frac{\partial}{\partial a^2_{111223}}-$$

$$-2a^2_{111123}\frac{\partial}{\partial a^2_{111233}}-3a^2_{111133}\frac{\partial}{\partial a^2_{111333}}-a^2_{111222}\frac{\partial}{\partial a^2_{112223}}-2a^2_{111223}\frac{\partial}{\partial a^2_{112233}}-$$

$$-3a^2_{111233}\frac{\partial}{\partial a^2_{112333}}-4a^2_{111333}\frac{\partial}{\partial a^2_{113333}}-a^2_{112222}\frac{\partial}{\partial a^2_{122223}}-2a^2_{112223}\frac{\partial}{\partial a^2_{122233}}-$$

$$-3a^2_{112233}\frac{\partial}{\partial a^2_{122333}}-4a^2_{112333}\frac{\partial}{\partial a^2_{123333}}-5a^2_{113333}\frac{\partial}{\partial a^2_{133333}}-a^2_{122222}\frac{\partial}{\partial a^2_{222223}}-$$

$$-2a^2_{122223}\frac{\partial}{\partial a^2_{222233}}-3a^2_{122233}\frac{\partial}{\partial a^2_{222333}}-4a^2_{122333}\frac{\partial}{\partial a^2_{223333}}-5a^2_{123333}\frac{\partial}{\partial a^2_{233333}}-$$

$$-6a^2_{133333}\frac{\partial}{\partial a^2_{333333}}-a^3_{111111}\frac{\partial}{\partial a^3_{111113}}-a^3_{111112}\frac{\partial}{\partial a^3_{111123}}-2a^3_{111113}\frac{\partial}{\partial a^3_{111133}}-$$

$$-a^3_{111122}\frac{\partial}{\partial a^3_{111223}}-2a^3_{111123}\frac{\partial}{\partial a^3_{111233}}-3a^3_{111133}\frac{\partial}{\partial a^3_{111333}}-a^3_{111222}\frac{\partial}{\partial a^3_{112223}}-$$

$$-2a^3_{111223}\frac{\partial}{\partial a^3_{112233}}-3a^3_{111233}\frac{\partial}{\partial a^3_{112333}}-4a^3_{111333}\frac{\partial}{\partial a^3_{113333}}-a^3_{112222}\frac{\partial}{\partial a^3_{122223}}-$$

$$-2a^3_{112223}\frac{\partial}{\partial a^3_{122233}}-3a^3_{112233}\frac{\partial}{\partial a^3_{122333}}-4a^3_{112333}\frac{\partial}{\partial a^3_{123333}}-5a^3_{113333}\frac{\partial}{\partial a^3_{133333}}-$$

$$-a^3_{122222}\frac{\partial}{\partial a^3_{222223}}-2a^3_{122223}\frac{\partial}{\partial a^3_{222233}}-3a^3_{122233}\frac{\partial}{\partial a^3_{222333}}-4a^3_{122333}\frac{\partial}{\partial a^3_{223333}}-$$

$$-5a^3_{123333}\frac{\partial}{\partial a^3_{233333}}-6a^3_{133333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}^{(6)}_6=-6a^1_{111112}\frac{\partial}{\partial a^1_{111111}}-5a^1_{111122}\frac{\partial}{\partial a^1_{111112}}-5a^1_{111123}\frac{\partial}{\partial a^1_{111113}}-$$

$$-4a^1_{111222}\frac{\partial}{\partial a^1_{111122}}-4a^1_{111223}\frac{\partial}{\partial a^1_{111123}}-4a^1_{111233}\frac{\partial}{\partial a^1_{111133}}-3a^1_{112222}\frac{\partial}{\partial a^1_{111222}}-$$

$$-3a^1_{112223}\frac{\partial}{\partial a^1_{111223}}-3a^1_{112233}\frac{\partial}{\partial a^1_{111233}}-3a^1_{112333}\frac{\partial}{\partial a^1_{111333}}-2a^1_{122222}\frac{\partial}{\partial a^1_{112222}}-$$
$$-2a^1_{122223}\frac{\partial}{\partial a^1_{112223}}-2a^1_{122233}\frac{\partial}{\partial a^1_{112233}}-2a^1_{122333}\frac{\partial}{\partial a^1_{112333}}-2a^1_{123333}\frac{\partial}{\partial a^1_{113333}}-$$
$$-a^1_{222222}\frac{\partial}{\partial a^1_{122222}}-a^1_{222223}\frac{\partial}{\partial a^1_{122223}}-a^1_{222233}\frac{\partial}{\partial a^1_{122233}}-a^1_{222333}\frac{\partial}{\partial a^1_{122333}}-$$
$$-a^1_{223333}\frac{\partial}{\partial a^1_{123333}}-a^1_{233333}\frac{\partial}{\partial a^1_{133333}}+(a^1_{111111}-6a^2_{111112})\frac{\partial}{\partial a^2_{111111}}+(a^1_{111112}-$$
$$-5a^2_{111122})\frac{\partial}{\partial a^2_{111112}}+(a^1_{111113}-5a^2_{111123})\frac{\partial}{\partial a^2_{111113}}+(a^1_{111122}-4a^2_{111222})\frac{\partial}{\partial a^2_{111122}}+$$
$$+(a^1_{111123}-4a^2_{111223})\frac{\partial}{\partial a^2_{111123}}+(a^1_{111133}-4a^2_{111233})\frac{\partial}{\partial a^2_{111133}}+(a^1_{111222}-$$
$$-3a^2_{112222})\frac{\partial}{\partial a^2_{111222}}+(a^1_{111223}-3a^2_{112223})\frac{\partial}{\partial a^2_{111223}}+(a^1_{111233}-3a^2_{112233})\frac{\partial}{\partial a^2_{111233}}+$$
$$+(a^1_{111333}-3a^2_{112333})\frac{\partial}{\partial a^2_{111333}}+(a^1_{112222}-2a^2_{122222})\frac{\partial}{\partial a^2_{112222}}+(a^1_{112223}-$$
$$-2a^2_{122223})\frac{\partial}{\partial a^2_{112223}}+(a^1_{112233}-2a^2_{122233})\frac{\partial}{\partial a^2_{112233}}+(a^1_{112333}-2a^2_{122333})\frac{\partial}{\partial a^2_{112333}}+$$
$$+(a^1_{113333}-2a^2_{123333})\frac{\partial}{\partial a^2_{113333}}+(a^1_{122222}-a^2_{222222})\frac{\partial}{\partial a^2_{122222}}+(a^1_{122223}-$$
$$-a^2_{222223})\frac{\partial}{\partial a^2_{122223}}+(a^1_{122233}-a^2_{222233})\frac{\partial}{\partial a^2_{122233}}+(a^1_{122333}-a^2_{222333})\frac{\partial}{\partial a^2_{122333}}+$$
$$+(a^1_{123333}-a^2_{223333})\frac{\partial}{\partial a^2_{123333}}+(a^1_{133333}-a^2_{233333})\frac{\partial}{\partial a^2_{133333}}+a^1_{222222}\frac{\partial}{\partial a^2_{222222}}+$$
$$+a^1_{222223}\frac{\partial}{\partial a^2_{222223}}+a^1_{222233}\frac{\partial}{\partial a^2_{222233}}+a^1_{222333}\frac{\partial}{\partial a^2_{222333}}+a^1_{223333}\frac{\partial}{\partial a^2_{223333}}+$$
$$+a^1_{233333}\frac{\partial}{\partial a^2_{233333}}+a^1_{333333}\frac{\partial}{\partial a^2_{333333}}-6a^3_{111112}\frac{\partial}{\partial a^3_{111111}}-5a^3_{111122}\frac{\partial}{\partial a^3_{111112}}-$$
$$-5a^3_{111123}\frac{\partial}{\partial a^3_{111113}}-4a^3_{111222}\frac{\partial}{\partial a^3_{111122}}-4a^3_{111223}\frac{\partial}{\partial a^3_{111123}}-4a^3_{111233}\frac{\partial}{\partial a^3_{111133}}-$$
$$-3a^3_{112222}\frac{\partial}{\partial a^3_{111222}}-3a^3_{112223}\frac{\partial}{\partial a^3_{111223}}-3a^3_{112233}\frac{\partial}{\partial a^3_{111233}}-3a^3_{112333}\frac{\partial}{\partial a^3_{111333}}-$$
$$-2a^3_{122222}\frac{\partial}{\partial a^3_{112222}}-2a^3_{122223}\frac{\partial}{\partial a^3_{112223}}-2a^3_{122233}\frac{\partial}{\partial a^3_{112233}}-2a^3_{122333}\frac{\partial}{\partial a^3_{112333}}-$$
$$-2a^3_{123333}\frac{\partial}{\partial a^3_{113333}}-a^3_{222222}\frac{\partial}{\partial a^3_{122222}}-a^3_{222223}\frac{\partial}{\partial a^3_{122223}}-a^3_{222233}\frac{\partial}{\partial a^3_{122233}}-$$

$$-a^3_{222333}\frac{\partial}{\partial a^3_{122333}} - a^3_{223333}\frac{\partial}{\partial a^3_{123333}} - a^3_{233333}\frac{\partial}{\partial a^3_{133333}},$$

$$\mathsf{D}^{(6)}_7 = -a^1_{111112}\frac{\partial}{\partial a^1_{111113}} - a^1_{111122}\frac{\partial}{\partial a^1_{111123}} - 2a^1_{111123}\frac{\partial}{\partial a^1_{111133}} - a^1_{111222}\frac{\partial}{\partial a^1_{111223}} -$$

$$-2a^1_{111223}\frac{\partial}{\partial a^1_{111233}} - 3a^1_{111233}\frac{\partial}{\partial a^1_{111333}} - a^1_{112222}\frac{\partial}{\partial a^1_{112223}} - 2a^1_{112223}\frac{\partial}{\partial a^1_{112233}} -$$

$$-3a^1_{112233}\frac{\partial}{\partial a^1_{112333}} - 4a^1_{112333}\frac{\partial}{\partial a^1_{113333}} - a^1_{122222}\frac{\partial}{\partial a^1_{122223}} - 2a^1_{122223}\frac{\partial}{\partial a^1_{122233}} -$$

$$-3a^1_{122233}\frac{\partial}{\partial a^1_{122333}} - 4a^1_{122333}\frac{\partial}{\partial a^1_{123333}} - 5a^1_{123333}\frac{\partial}{\partial a^1_{133333}} - a^1_{222222}\frac{\partial}{\partial a^1_{222223}} -$$

$$-2a^1_{222223}\frac{\partial}{\partial a^1_{222233}} - 3a^1_{222233}\frac{\partial}{\partial a^1_{222333}} - 4a^1_{222333}\frac{\partial}{\partial a^1_{223333}} - 5a^1_{223333}\frac{\partial}{\partial a^1_{233333}} -$$

$$-6a^1_{233333}\frac{\partial}{\partial a^1_{333333}} + a^3_{111111}\frac{\partial}{\partial a^2_{111111}} + a^3_{111112}\frac{\partial}{\partial a^2_{111112}} + (-a^2_{111112} + a^3_{111113})\frac{\partial}{\partial a^2_{111113}} +$$

$$+a^3_{111122}\frac{\partial}{\partial a^2_{111122}} + (-a^2_{111122} + a^3_{111123})\frac{\partial}{\partial a^2_{111123}} + (-2a^2_{111123} + a^3_{111133})\frac{\partial}{\partial a^2_{111133}} +$$

$$+a^3_{111222}\frac{\partial}{\partial a^2_{111222}} + (-a^2_{111222} + a^3_{111223})\frac{\partial}{\partial a^2_{111223}} + (-2a^2_{111223} + a^3_{111233})\frac{\partial}{\partial a^2_{111233}} +$$

$$+(-3a^2_{111233} + a^3_{111333})\frac{\partial}{\partial a^2_{111333}} + a^3_{112222}\frac{\partial}{\partial a^2_{112222}} + (-a^2_{112222} + a^3_{112223})\frac{\partial}{\partial a^2_{112223}} +$$

$$+(-2a^2_{112223} + a^3_{112233})\frac{\partial}{\partial a^2_{112233}} + (-3a^2_{112233} + a^3_{112333})\frac{\partial}{\partial a^2_{112333}} + (-4a^2_{112333} +$$

$$+a^3_{113333})\frac{\partial}{\partial a^2_{113333}} + a^3_{122222}\frac{\partial}{\partial a^2_{122222}} + (-a^2_{122222} + a^3_{122223})\frac{\partial}{\partial a^2_{122223}} + (-2a^2_{122223} +$$

$$+a^3_{122233})\frac{\partial}{\partial a^2_{122233}} + (-3a^2_{122233} + a^3_{122333})\frac{\partial}{\partial a^2_{122333}} + (-4a^2_{122333} + a^3_{123333})\frac{\partial}{\partial a^2_{123333}} +$$

$$+(-5a^2_{123333} + a^3_{133333})\frac{\partial}{\partial a^2_{133333}} + a^3_{222222}\frac{\partial}{\partial a^2_{222222}} + (-a^2_{222222} + a^3_{222223})\frac{\partial}{\partial a^2_{222223}} +$$

$$+(-2a^2_{222223} + a^3_{222233})\frac{\partial}{\partial a^2_{222233}} + (-3a^2_{222233} + a^3_{222333})\frac{\partial}{\partial a^2_{222333}} + (-4a^2_{222333} +$$

$$+a^3_{223333})\frac{\partial}{\partial a^2_{223333}} + (-5a^2_{223333} + a^3_{233333})\frac{\partial}{\partial a^2_{233333}} + (-6a^2_{233333} + a^3_{333333})\frac{\partial}{\partial a^2_{333333}} -$$

$$-a^3_{111112}\frac{\partial}{\partial a^3_{111113}} - a^3_{111122}\frac{\partial}{\partial a^3_{111123}} - 2a^3_{111123}\frac{\partial}{\partial a^3_{111133}} - a^3_{111222}\frac{\partial}{\partial a^3_{111223}} -$$

$$-2a^3_{111223}\frac{\partial}{\partial a^3_{111233}} - 3a^3_{111233}\frac{\partial}{\partial a^3_{111333}} - a^3_{112222}\frac{\partial}{\partial a^3_{112223}} - 2a^3_{112223}\frac{\partial}{\partial a^3_{112233}} -$$

$$-3a^3_{112233}\frac{\partial}{\partial a^3_{112333}}-4a^3_{112333}\frac{\partial}{\partial a^3_{113333}}-a^3_{122222}\frac{\partial}{\partial a^3_{122223}}-2a^3_{122223}\frac{\partial}{\partial a^3_{122233}}-$$
$$-3a^3_{122233}\frac{\partial}{\partial a^3_{122333}}-4a^3_{122333}\frac{\partial}{\partial a^3_{123333}}-5a^3_{123333}\frac{\partial}{\partial a^3_{133333}}-a^3_{222222}\frac{\partial}{\partial a^3_{222223}}-$$
$$-2a^3_{222223}\frac{\partial}{\partial a^3_{222233}}-3a^3_{222233}\frac{\partial}{\partial a^3_{222333}}-4a^3_{222333}\frac{\partial}{\partial a^3_{223333}}-5a^3_{223333}\frac{\partial}{\partial a^3_{233333}}-$$
$$-6a^3_{233333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}_8^{(6)}=-6a^1_{111113}\frac{\partial}{\partial a^1_{111111}}-5a^1_{111123}\frac{\partial}{\partial a^1_{111112}}-5a^1_{111133}\frac{\partial}{\partial a^1_{111113}}-4a^1_{111223}\frac{\partial}{\partial a^1_{111122}}-$$
$$-4a^1_{111233}\frac{\partial}{\partial a^1_{111123}}-4a^1_{111333}\frac{\partial}{\partial a^1_{111133}}-3a^1_{112223}\frac{\partial}{\partial a^1_{111222}}-3a^1_{112233}\frac{\partial}{\partial a^1_{111223}}-$$
$$-3a^1_{112333}\frac{\partial}{\partial a^1_{111233}}-3a^1_{113333}\frac{\partial}{\partial a^1_{111333}}-2a^1_{122223}\frac{\partial}{\partial a^1_{112222}}-2a^1_{122233}\frac{\partial}{\partial a^1_{112223}}-$$
$$-2a^1_{122333}\frac{\partial}{\partial a^1_{112233}}-2a^1_{123333}\frac{\partial}{\partial a^1_{112333}}-2a^1_{133333}\frac{\partial}{\partial a^1_{113333}}-a^1_{222223}\frac{\partial}{\partial a^1_{122222}}-$$
$$-a^1_{222233}\frac{\partial}{\partial a^1_{122223}}-a^1_{222333}\frac{\partial}{\partial a^1_{122233}}-a^1_{223333}\frac{\partial}{\partial a^1_{122333}}-a^1_{233333}\frac{\partial}{\partial a^1_{123333}}-$$
$$-a^1_{333333}\frac{\partial}{\partial a^1_{133333}}-6a^2_{111113}\frac{\partial}{\partial a^2_{111111}}-5a^2_{111123}\frac{\partial}{\partial a^2_{111112}}-5a^2_{111133}\frac{\partial}{\partial a^2_{111113}}-$$
$$-4a^2_{111223}\frac{\partial}{\partial a^2_{111122}}-4a^2_{111233}\frac{\partial}{\partial a^2_{111123}}-4a^2_{111333}\frac{\partial}{\partial a^2_{111133}}-3a^2_{112223}\frac{\partial}{\partial a^2_{111222}}-$$
$$-3a^2_{112233}\frac{\partial}{\partial a^2_{111223}}-3a^2_{112333}\frac{\partial}{\partial a^2_{111233}}-3a^2_{113333}\frac{\partial}{\partial a^2_{111333}}-2a^2_{122223}\frac{\partial}{\partial a^2_{112222}}-$$
$$-2a^2_{122233}\frac{\partial}{\partial a^2_{112223}}-2a^2_{122333}\frac{\partial}{\partial a^2_{112233}}-2a^2_{123333}\frac{\partial}{\partial a^2_{112333}}-2a^2_{133333}\frac{\partial}{\partial a^2_{113333}}-$$
$$-a^2_{222223}\frac{\partial}{\partial a^2_{122222}}-a^2_{222233}\frac{\partial}{\partial a^2_{122223}}-a^2_{222333}\frac{\partial}{\partial a^2_{122233}}-a^2_{223333}\frac{\partial}{\partial a^2_{122333}}-$$
$$-a^2_{233333}\frac{\partial}{\partial a^2_{123333}}-a^2_{333333}\frac{\partial}{\partial a^2_{133333}}+(a^1_{111111}-6a^3_{111113})\frac{\partial}{\partial a^3_{111111}}+(a^1_{111112}-$$
$$-5a^3_{111123})\frac{\partial}{\partial a^3_{111112}}+(a^1_{111113}-5a^3_{111133})\frac{\partial}{\partial a^3_{111113}}+(a^1_{111122}-4a^3_{111223})\frac{\partial}{\partial a^3_{111122}}+$$
$$+(a^1_{111123}-4a^3_{111233})\frac{\partial}{\partial a^3_{111123}}+(a^1_{111133}-4a^3_{111333})\frac{\partial}{\partial a^3_{111133}}+(a^1_{111222}-$$
$$-3a^3_{112223})\frac{\partial}{\partial a^3_{111222}}+(a^1_{111223}-3a^3_{112233})\frac{\partial}{\partial a^3_{111223}}+(a^1_{111233}-3a^3_{112333})\frac{\partial}{\partial a^3_{111233}}+$$

$$+(a^1_{111333}-3a^3_{113333})\frac{\partial}{\partial a^3_{111333}}+(a^1_{112222}-2a^3_{122223})\frac{\partial}{\partial a^3_{112222}}+(a^1_{112223}-$$

$$-2a^3_{122233})\frac{\partial}{\partial a^3_{112223}}+(a^1_{112233}-2a^3_{122333})\frac{\partial}{\partial a^3_{112233}}+(a^1_{112333}-2a^3_{123333})\frac{\partial}{\partial a^3_{112333}}+$$

$$+(a^1_{113333}-2a^3_{133333})\frac{\partial}{\partial a^3_{113333}}+(a^1_{122222}-a^3_{222223})\frac{\partial}{\partial a^3_{122222}}+(a^1_{122223}-$$

$$-a^3_{222233})\frac{\partial}{\partial a^3_{122223}}+(a^1_{122233}-a^3_{222333})\frac{\partial}{\partial a^3_{122233}}+(a^1_{122333}-a^3_{223333})\frac{\partial}{\partial a^3_{122333}}+$$

$$+(a^1_{123333}-a^3_{233333})\frac{\partial}{\partial a^3_{123333}}+(a^1_{133333}-a^3_{333333})\frac{\partial}{\partial a^3_{133333}}+a^1_{222222}\frac{\partial}{\partial a^3_{222222}}+$$

$$+a^1_{222223}\frac{\partial}{\partial a^3_{222223}}+a^1_{222233}\frac{\partial}{\partial a^3_{222233}}+a^1_{222333}\frac{\partial}{\partial a^3_{222333}}+a^1_{223333}\frac{\partial}{\partial a^3_{223333}}+$$

$$+a^1_{233333}\frac{\partial}{\partial a^3_{233333}}+a^1_{333333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}^{(6)}_9=-a^1_{111113}\frac{\partial}{\partial a^1_{111112}}-2a^1_{111123}\frac{\partial}{\partial a^1_{111122}}-a^1_{111133}\frac{\partial}{\partial a^1_{111123}}-3a^1_{111223}\frac{\partial}{\partial a^1_{111222}}-$$

$$-2a^1_{111233}\frac{\partial}{\partial a^1_{111223}}-a^1_{111333}\frac{\partial}{\partial a^1_{111233}}-4a^1_{112223}\frac{\partial}{\partial a^1_{112222}}-3a^1_{112233}\frac{\partial}{\partial a^1_{112223}}-$$

$$-2a^1_{112333}\frac{\partial}{\partial a^1_{112233}}-a^1_{113333}\frac{\partial}{\partial a^1_{112333}}-5a^1_{122223}\frac{\partial}{\partial a^1_{122222}}-4a^1_{122233}\frac{\partial}{\partial a^1_{122223}}-$$

$$-3a^1_{122333}\frac{\partial}{\partial a^1_{122233}}-2a^1_{123333}\frac{\partial}{\partial a^1_{122333}}-a^1_{133333}\frac{\partial}{\partial a^1_{123333}}-6a^1_{222223}\frac{\partial}{\partial a^1_{222222}}-$$

$$-5a^1_{222233}\frac{\partial}{\partial a^1_{222223}}-4a^1_{222333}\frac{\partial}{\partial a^1_{222233}}-3a^1_{223333}\frac{\partial}{\partial a^1_{222333}}-2a^1_{233333}\frac{\partial}{\partial a^1_{223333}}-$$

$$-a^1_{333333}\frac{\partial}{\partial a^1_{233333}}-a^2_{111113}\frac{\partial}{\partial a^2_{111112}}-2a^2_{111123}\frac{\partial}{\partial a^2_{111122}}-a^2_{111133}\frac{\partial}{\partial a^2_{111123}}-$$

$$-3a^2_{111223}\frac{\partial}{\partial a^2_{111222}}-2a^2_{111233}\frac{\partial}{\partial a^2_{111223}}-a^2_{111333}\frac{\partial}{\partial a^2_{111233}}-4a^2_{112223}\frac{\partial}{\partial a^2_{112222}}-$$

$$-3a^2_{112233}\frac{\partial}{\partial a^2_{112223}}-2a^2_{112333}\frac{\partial}{\partial a^2_{112233}}-a^2_{113333}\frac{\partial}{\partial a^2_{112333}}-5a^2_{122223}\frac{\partial}{\partial a^2_{122222}}-$$

$$-4a^2_{122233}\frac{\partial}{\partial a^2_{122223}}-3a^2_{122333}\frac{\partial}{\partial a^2_{122233}}-2a^2_{123333}\frac{\partial}{\partial a^2_{122333}}-a^2_{133333}\frac{\partial}{\partial a^2_{123333}}-$$

$$-6a^2_{222223}\frac{\partial}{\partial a^2_{222222}}-5a^2_{222233}\frac{\partial}{\partial a^2_{222223}}-4a^2_{222333}\frac{\partial}{\partial a^2_{222233}}-3a^2_{223333}\frac{\partial}{\partial a^2_{222333}}-$$

$$-2a^2_{233333}\frac{\partial}{\partial a^2_{223333}}-a^2_{333333}\frac{\partial}{\partial a^2_{233333}}+a^2_{111111}\frac{\partial}{\partial a^3_{111111}}+(a^2_{111112}-$$

$$-a^3_{111113})\frac{\partial}{\partial a^3_{111112}}+a^2_{111113}\frac{\partial}{\partial a^3_{111113}}+(a^2_{111122}-2a^3_{111123})\frac{\partial}{\partial a^3_{111122}}+(a^2_{111123}-$$
$$-a^3_{111133})\frac{\partial}{\partial a^3_{111123}}+a^2_{111133}\frac{\partial}{\partial a^3_{111133}}+(a^2_{111222}-3a^3_{111223})\frac{\partial}{\partial a^3_{111222}}+(a^2_{111223}-$$
$$-2a^3_{111233})\frac{\partial}{\partial a^3_{111223}}+(a^2_{111233}-a^3_{111333})\frac{\partial}{\partial a^3_{111233}}+a^2_{111333}\frac{\partial}{\partial a^3_{111333}}+(a^2_{112222}-$$
$$-4a^3_{112223})\frac{\partial}{\partial a^3_{112222}}+(a^2_{112223}-3a^3_{112233})\frac{\partial}{\partial a^3_{112223}}+(a^2_{112233}-2a^3_{112333})\frac{\partial}{\partial a^3_{112233}}+$$
$$+(a^2_{112333}-a^3_{113333})\frac{\partial}{\partial a^3_{112333}}+a^2_{113333}\frac{\partial}{\partial a^3_{113333}}+(a^2_{122222}-5a^3_{122223})\frac{\partial}{\partial a^3_{122222}}+$$
$$+(a^2_{122223}-4a^3_{122233})\frac{\partial}{\partial a^3_{122223}}+(a^2_{122233}-3a^3_{122333})\frac{\partial}{\partial a^3_{122233}}+(a^2_{122333}-$$
$$-2a^3_{123333})\frac{\partial}{\partial a^3_{122333}}+(a^2_{123333}-a^3_{133333})\frac{\partial}{\partial a^3_{123333}}+a^2_{133333}\frac{\partial}{\partial a^3_{133333}}+(a^2_{222222}-$$
$$-6a^3_{222223})\frac{\partial}{\partial a^3_{222222}}+(a^2_{222223}-5a^3_{222233})\frac{\partial}{\partial a^3_{222223}}+(a^2_{222233}-4a^3_{222333})\frac{\partial}{\partial a^3_{222233}}+$$
$$+(a^2_{222333}-3a^3_{223333})\frac{\partial}{\partial a^3_{222333}}+(a^2_{223333}-2a^3_{233333})\frac{\partial}{\partial a^3_{223333}}+$$
$$+(a^2_{233333}-a^3_{333333})\frac{\partial}{\partial a^3_{233333}}+a^2_{333333}\frac{\partial}{\partial a^3_{333333}}. \tag{17}$$

Proof. We will examine the transformation group H_1, given by formulas (2), in the space of variables and coefficients of system (1). Then, this system will be written in the form

$$\frac{d\bar{x}^j}{dt}=\bar{a}^j_\alpha x^\alpha+\bar{a}^j_{\alpha\beta\gamma\delta\mu\nu}\bar{x}^\alpha\bar{x}^\beta\bar{x}^\gamma\bar{x}^\delta\bar{x}^\mu\bar{x}^\nu \quad (j,\alpha,\beta,\gamma,\delta,\mu,\nu=\overline{1,3}), \tag{18}$$

where

$$\bar{a}^1_1=a^1_1,\ \bar{a}^1_2=\lambda_1 a^1_2,\ \bar{a}^1_3=\lambda_1 a^1_3,\ \bar{a}^2_1=\frac{a^2_1}{\lambda_1},\ \bar{a}^2_2=a^2_2,\ \bar{a}^2_3=a^2_3,\ \bar{a}^3_1=\frac{a^3_1}{\lambda_1},\ \bar{a}^3_2=a^3_2,$$
$$\bar{a}^3_3=a^3_3,\ \bar{a}^1_{111111}=\frac{a^1_{111111}}{\lambda_1^5},\ \bar{a}^1_{111112}=\frac{a^1_{111112}}{\lambda_1^4},\ \bar{a}^1_{111122}=\frac{a^1_{111122}}{\lambda_1^3},\ \bar{a}^1_{111222}=\frac{a^1_{111222}}{\lambda_1^2},$$
$$\bar{a}^1_{112222}=\frac{a^1_{112222}}{\lambda_1},\ \bar{a}^1_{122222}=a^1_{122222},\ \bar{a}^1_{222222}=\lambda_1 a^1_{222222},\ \bar{a}^1_{111113}=\frac{a^1_{111113}}{\lambda_1^4},$$
$$\bar{a}^1_{111123}=\frac{a^1_{111123}}{\lambda_1^3},\ \bar{a}^1_{111223}=\frac{a^1_{111223}}{\lambda_1^2},\ \bar{a}^1_{112223}=\frac{a^1_{112223}}{\lambda_1},\ \bar{a}^1_{122223}=a^1_{122223},$$
$$\bar{a}^1_{222223}=\lambda_1 a^1_{222223},\ \bar{a}^1_{111133}=a^1_{111133}\frac{1}{\lambda_1^3},\ \bar{a}^1_{111233}=a^1_{111233}\frac{1}{\lambda_1^2},\ \bar{a}^1_{112233}=\frac{a^1_{112233}}{\lambda_1},$$
$$\bar{a}^1_{122233}=a^1_{122233},\ \bar{a}^1_{222233}=\lambda_1 a^1_{222233},\ \bar{a}^1_{111333}=\frac{a^1_{111333}}{\lambda_1^2},\ \bar{a}^1_{112333}=\frac{a^1_{112333}}{\lambda_1},$$

$$\bar{a}^1_{122333} = a^1_{122333},\ \bar{a}^1_{222333} = \lambda_1 a^1_{222333},\ \bar{a}^1_{113333} = \frac{a^1_{113333}}{\lambda_1},\ \bar{a}^1_{123333} = a^1_{123333},$$

$$\bar{a}^1_{223333} = \lambda_1 a^1_{223333},\ \bar{a}^1_{133333} = a^1_{133333},\ \bar{a}^1_{233333} = \lambda_1 a^1_{233333},\ \bar{a}^1_{333333} = \lambda_1 a^1_{333333},$$

$$\bar{a}^2_{111111} = \frac{a^2_{111111}}{\lambda_1^6},\ \bar{a}^2_{111112} = \frac{a^2_{111112}}{\lambda_1^5},\ \bar{a}^2_{111122} = \frac{a^2_{111122}}{\lambda_1^4},\ \bar{a}^2_{111222} = \frac{a^2_{111222}}{\lambda_1^3},$$

$$\bar{a}^2_{112222} = \frac{a^2_{112222}}{\lambda_1^2},\ \bar{a}^2_{122222} = \frac{a^2_{122222}}{\lambda_1},\ \bar{a}^2_{222222} = a^2_{222222},\ \bar{a}^2_{111113} = \frac{a^2_{111113}}{\lambda_1^5},$$

$$\bar{a}^2_{111123} = \frac{a^2_{111123}}{\lambda_1^4},\ \bar{a}^2_{111223} = \frac{a^2_{111223}}{\lambda_1^3},\ \bar{a}^2_{112223} = \frac{a^2_{112223}}{\lambda_1^2},\ \bar{a}^2_{122223} = \frac{a^2_{122223}}{\lambda_1},$$

$$\bar{a}^2_{222223} = a^2_{222223},\ \bar{a}^2_{111133} = \frac{a^2_{111133}}{\lambda_1^4},\ \bar{a}^2_{111233} = \frac{a^2_{111233}}{\lambda_1^3},\ \bar{a}^2_{112233} = \frac{a^2_{112233}}{\lambda_1^2},$$

$$\bar{a}^2_{122233} = \frac{a^2_{122233}}{\lambda_1},\ \bar{a}^2_{222233} = a^2_{222233},\ \bar{a}^2_{111333} = \frac{a^2_{111333}}{\lambda_1^3},\ \bar{a}^2_{112333} = \frac{a^2_{112333}}{\lambda_1^2},$$

$$\bar{a}^2_{122333} = \frac{a^2_{122333}}{\lambda_1},\ \bar{a}^2_{222333} = a^2_{222333},\ \bar{a}^2_{113333} = \frac{a^2_{113333}}{\lambda_1^2},\ \bar{a}^2_{123333} = \frac{a^2_{123333}}{\lambda_1},$$

$$\bar{a}^2_{223333} = a^2_{223333},\ \bar{a}^2_{133333} = \frac{a^2_{133333}}{\lambda_1},\ \bar{a}^2_{233333} = a^2_{233333},\ \bar{a}^2_{333333} = a^2_{333333},$$

$$\bar{a}^3_{111111} = \frac{a^3_{111111}{}^{6}}{\lambda_1},\ \bar{a}^3_{111112} = \frac{a^3_{111112}}{\lambda_1^5},\ \bar{a}^3_{111122} = \frac{a^3_{111122}}{\lambda_1^4},\ \bar{a}^3_{111222} = \frac{a^3_{111222}}{\lambda_1^3},$$

$$\bar{a}^3_{112222} = \frac{a^3_{112222}}{\lambda_1^2},\ \bar{a}^3_{122222} = \frac{a^3_{122222}}{\lambda_1},\ \bar{a}^3_{222222} = a^3_{222222},\ \bar{a}^3_{111113} = \frac{a^3_{111113}}{\lambda_1^5},$$

$$\bar{a}^3_{111123} = \frac{a^3_{111123}}{\lambda_1^4},\ \bar{a}^3_{111223} = \frac{a^3_{111223}}{\lambda_1^3},\ \bar{a}^3_{112223} = \frac{a^3_{112223}}{\lambda_1^2},\ \bar{a}^3_{122223} = \frac{a^3_{122223}}{\lambda_1},$$

$$\bar{a}^3_{222223} = a^3_{222223},\ \bar{a}^3_{111133} = \frac{a^3_{111133}}{\lambda_1^4},\ \bar{a}^3_{111233} = \frac{a^3_{111233}}{\lambda_1^3},\ \bar{a}^3_{112233} = \frac{a^3_{112233}}{\lambda_1^2},$$

$$\bar{a}^3_{122233} = \frac{a^3_{122233}}{\lambda_1},\ \bar{a}^3_{222233} = a^3_{222233},\ \bar{a}^3_{111333} = \frac{a^3_{111333}}{\lambda_1^3},\ \bar{a}^3_{112333} = \frac{a^3_{112333}}{\lambda_1^2},$$

$$\bar{a}^3_{122333} = \frac{a^3_{122333}}{\lambda_1},\ \bar{a}^3_{222333} = a^3_{222333},\ \bar{a}^3_{113333} = \frac{a^3_{113333}}{\lambda_1^2},\ \bar{a}^3_{123333} = \frac{a^3_{123333}}{\lambda_1},$$

$$\bar{a}^3_{223333} = a^3_{223333},\ \bar{a}^3_{133333} = \frac{a^3_{133333}}{\lambda_1},\ \bar{a}^3_{233333} = a^3_{233333},\ \bar{a}^3_{333333} = a^3_{333333}.$$

Considering these equalities and introducing into (13) the notations $\alpha = \exp(\bar{\lambda}_1)$ [**?**], where $\exp(\bar{\lambda}_1) = |\lambda_1|$, we obtain

$$\xi^1 = x^1,\ \ \xi^2 = \xi^3 = 0,$$

$$\eta^1_1 = 0,\ \ \eta^1_2 = a^1_2,\ \ \eta^1_3 = a^1_3,\ \ \eta^2_1 = -a^2_1,\ \ \eta^2_2 = \eta^2_3 = 0,\ \ \eta^3_1 = -a^3_1,\ \ \eta^3_2 = \eta^3_3 = 0,$$

$$\eta^1_{111111} = -5a^1_{111111},\ \eta^1_{111112} = -4a^1_{111112},\ \eta^1_{111122} = -3a^1_{111122},\ \eta^1_{111222} = -2a^1_{111222},$$

$$\eta^1_{112222} = -a^1_{112222},\ \eta^1_{122222} = 0,\ \eta^1_{222222} = a^1_{222222},\ \eta^1_{111113} = -4a^1_{111113},$$

$$\eta^1_{111123} = -3a^1_{111123},\ \eta^1_{111223} = -2a^1_{111223},\ \eta^1_{112223} = -a^1_{112223},\ \eta^1_{122223} = 0,$$

$$\eta^1_{222223} = a^1_{222223},\ \eta^1_{111133} = -3a^1_{111133},\ \eta^1_{111233} = -2a^1_{111233},\ \eta^1_{112233} = -a^1_{112233},$$

$$\eta^1_{122233} = 0,\ \eta^1_{222233} = a^1_{222233},\ \eta^1_{111333} = -2a^1_{111333},\ \eta^1_{112333} = -a^1_{112333},$$

$$\eta^1_{122333} = 0,\ \eta^1_{222333} = a^1_{222333},\ \eta^1_{113333} = -a^1_{113333},\ \eta^1_{123333} = 0,$$

$$\eta^1_{223333} = a^1_{223333},\ \eta^1_{133333} = 0,\ \eta^1_{233333} = a^1_{233333},\ \eta^1_{333333} = a^1_{333333},$$

$$\eta^2_{111111} = -6a^2_{111111},\ \eta^2_{111112} = -5a^2_{111112},\ \eta^2_{111122} = -4a^2_{111122},\ \eta^2_{111222} = -3a^2_{111222},$$

$$\eta^2_{112222} = -2a^2_{112222},\ \eta^2_{122222} = -a^2_{122222},\ \eta^2_{222222} = 0,\ \eta^2_{111113} = -5a^2_{111113},$$

$$\eta^2_{111123} = -4a^2_{111123},\ \eta^2_{111223} = -3a^2_{111223},\ \eta^2_{112223} = -2a^2_{112223},\ \eta^2_{122223} = -a^2_{122223},$$

$$\eta^2_{222223} = 0,\ \eta^2_{111133} = -4a^2_{111133},\ \eta^2_{111233} = -3a^2_{111233},\ \eta^2_{112233} = -2a^2_{112233},$$

$$\eta^2_{122233} = -a^2_{122233},\ \eta^2_{222233} = 0,\ \eta^2_{111333} = -3a^2_{111333},\ \eta^2_{112333} = -2a^2_{112333},$$

$$\eta^2_{122333} = -a^2_{122333},\ \eta^2_{222333} = 0,\ \eta^2_{113333} = -2a^2_{113333},\ \eta^2_{123333} = -a^2_{123333},$$

$$\eta^2_{223333} = 0,\ \eta^2_{133333} = -a^2_{133333},\ \eta^2_{233333} = 0,\ \eta^2_{333333} = 0,$$

$$\eta^3_{111111} = -6a^3_{111111},\ \eta^3_{111112} = -5a^3_{111112},\ \eta^3_{111122} = -4a^3_{111122},\ \eta^3_{111222} = -3a^3_{111222},$$

$$\eta^3_{112222} = -2a^3_{112222},\ \eta^3_{122222} = -a^3_{122222},\ \eta^3_{222222} = 0,\ \eta^3_{111113} = -5a^3_{111113},$$

$$\eta^3_{111123} = -4a^3_{111123},\ \eta^3_{111223} = -3a^3_{111223},\ \eta^3_{112223} = -2a^3_{112223},\ \eta^3_{122223} = -a^3_{122223},$$

$$\eta^3_{222223} = 0,\ \eta^3_{111133} = -4a^3_{111133},\ \eta^3_{111233} = -3a^3_{111233},\ \eta^3_{112233} = -2a^3_{112233},$$

$$\eta^3_{122233} = -a^3_{122233},\ \eta^3_{222233} = 0,\ \eta^3_{111333} = -3a^3_{111333},\ \eta^3_{112333} = -2a^3_{112333},$$

$$\eta^3_{122333} = -a^3_{122333},\ \eta^3_{222333} = 0,\ \eta^3_{113333} = -2a^3_{113333},\ \eta^3_{123333} = -a^3_{123333},$$

$$\eta^3_{223333} = 0,\ \eta^3_{133333} = -a^3_{133333},\ \eta^3_{233333} = 0,\ \eta^3_{333333} = 0.$$

By substituting the values of these coordinates into (11) - (12), we obtain the operator X_1 and d_1, from (14) - (16).

Similarly, one can prove the determination of the operators X_k and d_k $(k = \overline{2,9})$, from (14) - (16), for the transformation groups H_i $(i = \overline{2,3})$ and H_{j+3} $(j = \overline{1,6})$, when $\mu_j = 0$ $(j = \overline{1,6})$, given by formulas (3) - (10), in the space of variables and coefficients $E^{96}(x,a)$ of system (1). Theorem 1 is proved. □

3. The Lie algebra L_9 of the admissible representation operators for the differential system with nonlinearities of degree six

For the operators X_i $(\overline{1,9})$ from (14) - (17), admitted by the system $s^3(1,6)$, the following holds

Lemma 3.1. *The operators X_i $(\overline{1,9})$, from (14) - (17), form a Lie algebra L_9 with the structure equations*

$$
\begin{gathered}
[X_1, X_4] = -X_4, \quad [X_1, X_5] = -X_5, \quad [X_1, X_6] = X_6, \quad [X_1, X_8] = X_8, \quad [X_2, X_4] = X_4,\\
[X_2, X_6] = -X_6, \quad [X_2, X_7] = -X_7, \quad [X_2, X_9] = X_9, \quad [X_3, X_5] = X_5, \quad [X_3, X_7] = X_7,\\
[X_3, X_8] = -X_8, \quad [X_3, X_9] = -X_9, \quad [X_4, X_6] = X_2 - X_1, \quad [X_4, X_7] = -X_5,\\
[X_4, X_9] = X_8, \quad [X_5, X_6] = X_7, \quad [X_5, X_8] = X_3 - X_1, \quad [X_5, X_9] = -X_4,\\
[X_6, X_9] = X_8, \quad [X_7, X_8] = -X_6, \quad [X_7, X_9] = X_3 - X_2,
\end{gathered}
\tag{19}
$$

and for other commutators $[X_i, X_j] = 0$.

We introduce the notations

$$
\begin{gathered}
Y_1 = X_9, \quad Y_2 = X_1 - X_3, \quad Y_3 = X_1 - X_2, \quad Y_4 = X_4, \quad Y_5 = X_5, \quad Y_6 = X_6,\\
Y_7 = X_7, \quad Y_8 = X_8, \quad Y_9 = X_1 + X_2 + X_3.
\end{gathered}
\tag{20}
$$

The following lemma holds.

Lemma 3.2. *The Lie algebra $L_9\langle X_1, \dots, X_9\rangle$, with structure equations (19), is isomorphic to the Lie algebra $L_9\langle Y_1, \dots, Y_9\rangle$ with the following commutator table:*

Table 1. The commutators of the Lie algebra $L_9\langle Y_1, \dots, Y_9\rangle$

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
Y_1	0	$-Y_1$	Y_1	0	Y_4	$-Y_8$	$Y_2 - Y_3$	0	0
Y_2	Y_1	0	0	$-Y_4$	$-2Y_5$	Y_6	$-Y_7$	$2Y_8$	0
Y_3	$-Y_1$	0	0	$-2Y_4$	$-Y_5$	$2Y_6$	Y_7	$-Y_8$	0
Y_4	0	Y_4	$2Y_4$	0	0	$-Y_3$	$-Y_5$	Y_1	0
Y_5	$-Y_4$	$2Y_5$	Y_5	0	0	Y_7	0	$-Y_2$	0
Y_6	Y_8	$-Y_6$	$-2Y_6$	Y_3	$-Y_7$	0	0	0	0
Y_7	$-Y_2 + Y_3$	Y_7	$-Y_7$	Y_5	0	0	0	$-Y_6$	0
Y_8	0	$-2Y_8$	$-Y_8$	$-Y_1$	Y_2	0	Y_6	0	0
Y_9	0	0	0	0	0	0	0	0	0

where the non-zero structure constants, except for those determined by antisymmetry with respect to the lower indices, are

$$
\begin{gathered}
C_{12}^{1} = -1, \quad C_{13}^{1} = 1, \quad C_{15}^{4} = 1, \quad C_{16}^{8} = -1, \quad C_{17}^{2} = 1, \quad C_{17}^{3} = -1, \quad C_{24}^{4} = -1, \\
C_{25}^{5} = -2, \quad C_{26}^{6} = 1, \quad C_{27}^{7} = -1, \quad C_{28}^{8} = 2, \quad C_{34}^{4} = -2, \quad C_{35}^{5} = -1, \quad C_{36}^{6} = 2, \\
C_{37}^{7} = 1, \quad C_{38}^{8} = 1, \quad C_{46}^{3} = -1, \quad C_{47}^{5} = -1, \quad C_{48}^{1} = 1, \quad C_{56}^{7} = 1, \\
C_{58}^{2} = -1, \quad C_{78}^{6} = -1.
\end{gathered} \tag{21}
$$

From Table 1 we observe that

$$L_9 = \langle Y_1, \ldots, Y_9 \rangle = L_8 \langle Y_1, \ldots, Y_8 \rangle \oplus \{Y_9\}, \tag{22}$$

where $Z = \{Y_9\}$ is the non-zero center of the Lie algebra $L_9 = \langle Y_1, \ldots, Y_9 \rangle$.

To determine the type of the Lie algebra $L_8 \langle Y_1, \ldots, Y_8 \rangle$, we construct the associated application matrix [2], [3]:

$(ad(X)) =$

$$
= \begin{pmatrix}
x^3 - x^2 & x^1 & -x^1 & x^8 & 0 & 0 & 0 & -x^4 \\
x^7 & 0 & 0 & 0 & -x^8 & 0 & -x^1 & x^5 \\
-x^7 & 0 & 0 & -x^6 & 0 & x^4 & x^1 & 0 \\
x^5 & -x^4 & -2x^4 & x^2 + 2x^3 & -x^1 & 0 & 0 & 0 \\
0 & -2x^5 & -x^5 & -x^7 & 2x^2 + x^3 & 0 & x^4 & 0 \\
0 & x^6 & 2x^6 & 0 & 0 & -x^2 - 2x^3 & -x^8 & x^7 \\
0 & -x^7 & x^7 & 0 & x^6 & -x^5 & x^2 - x^3 & 0 \\
-x^6 & 2x^8 & x^8 & 0 & 0 & x^1 & 0 & -2x^2 - x^3
\end{pmatrix} \tag{23}
$$

In a similar way, we construct the associated application matrix for the element $Y \in L_8$:

$(ad(Y)) =$

$$
= \begin{pmatrix}
y^3 - y^2 & y^1 & -y^1 & y^8 & 0 & 0 & 0 & -y^4 \\
y^7 & 0 & 0 & 0 & -y^8 & 0 & -y^1 & y^5 \\
-y^7 & 0 & 0 & -y^6 & 0 & y^4 & y^1 & 0 \\
y^5 & -y^4 & -2y^4 & y^2 + 2y^3 & -y^1 & 0 & 0 & 0 \\
0 & -2y^5 & -y^5 & -y^7 & 2y^2 + y^3 & 0 & y^4 & 0 \\
0 & y^6 & 2y^6 & 0 & 0 & -y^2 - 2y^3 & -y^8 & y^7 \\
0 & -y^7 & y^7 & 0 & y^6 & -y^5 & y^2 - y^3 & 0 \\
-y^6 & 2y^8 & y^8 & 0 & 0 & y^1 & 0 & -2y^2 - y^3
\end{pmatrix} \tag{24}
$$

Using the structure constants $C_{\alpha\beta}$ from (21) and the matrices (23) - (24), we obtain the trace of the product matrix $tr((ad(X))(ad(Y)))$. By means of the trace of the matrix product $tr((ad(X))(ad(Y)))$, we determine the Killing form [2], [3]:

$$K < X, Y >= tr((ad(X))(ad(Y))). \tag{25}$$

Therefore

$$K < X, Y >= 6x^1y^7 + 12x^2y^2 + 6x^2y^3 + 6x^3y^2 + 12x^3y^3 + 6x^4y^6 + 6x^5y^8 + 6x^6y^4 + \\ +6x^7y^1 + 6x^8y^5. \tag{26}$$

Taking into account the Killing form (26), we obtain that

$$\begin{gathered} K_{17} = K_{23} = K_{32} = K_{46} = K_{58} = K_{64} = K_{71} = K_{85} = 6, \\ K_{22} = K_{33} = 12, \end{gathered} \tag{27}$$

and for other elements $K_{\alpha\beta} = 0$, $(\alpha, \beta = \overline{1, 8})$ for which we have

$$det(K_{\alpha\beta}) = det \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 12 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \end{pmatrix} \neq 0. \tag{28}$$

4. Conclusions

Since $det(K_{\alpha\beta}) \neq 0$, it follows that the Killing's form is non-degenerate, and in accordance with Cartan's criterion [3], we obtain that the Lie algebra L_8 is semisimple. Therefore, based on relation (22), we obtain that the Lie algebra L_9 is a reductive Lie algebra.

The Lie algebra L_9 obtained for the ternary differential system with nonlinearities of degree six (1) is isomorphic to the Lie algebra L_9 admitted by the ternary differential system with quadratic nonlinearities, from [8].

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