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On methods of constructing AD-quasigroups

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Abstract. We research T- *quasigroups* with AD identity and Schweitzer identity. The necessary and sufficient condition has been determined that in the T- *quasigroup* G of the form $x \cdot y = \varphi(x) + \psi(y)$, AD identity and the Schweitzer identity are true. We have used the concept of a special direct product of an topological Abelian group G and proved that a binary operation can be defined on the set $G \times G$, such that the new algebraic structure is a non-associative topological quasigroups obeying medial, semimedial and AD laws.

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Keywords: T-quasigroup, AG-quasigroup, AD-quasigroup, Schweitzer quasigroup, medial, semimedial quasigroup, topological quasigroup.

Asupra metodelor de construcție a quasigrupurilor AD

Rezumat. Lucrarea este dedicata studiului T- *quasigrupurilor* cu identitate AD şi identitate Schweitzer. A fost determinată condiţia necesară şi suficientă ca în T- *quasigrupul* G de forma $x \cdot y = \varphi(x) + \psi(y)$, identitatea AD si identitatea Schweitzer să fie adevarate. A fost utilizat conceptul de produs direct special al unui grup abelian topologic G şi s-a demonstrat ca o operaţie binară poate fi definită pe mulţimea $G \times G$, astfel încât noua structură algebrică să fie un quasigrup topologic neasociativ care este medial, semimedial şi AD.

Cuvinte-cheie: T-quasigrup, AG-quasigrup, AD-quasigrup, quasigrup Schweitzer, quasigrup medial, quasigrup semimedial, quasigrup topologic.

1. Introduction

In this paper we study T-*quasigroups* with AD identity (or Abel-Grassmann's right invertive groupoid identity) $a \cdot bc = c \cdot ba$ and Schweitzer identity $xy \cdot xz = zy$.

A T-*quasigroup* is a specific category of quasigroups $(G, \cdot)$ that can be expressed in terms of automorphisms of an abelian group $(G, +)$, using the relation $x \cdot y = \varphi(x) + \psi(y) + a$, where φ and ψ are automorphisms, and a is a fixed element.

We will say that the isotope $(G, \cdot)$ is generated by the automorphisms φ, ψ and a fixed element $a \in G$ of the Abelian group $(G, +)$, and, we write $(G, \cdot) = g(G, +, \varphi, \psi, a)$.

It is shown that if G is a T-*quasigroup*, then G is AD-*quasigroup* if and only if for any of its T-*forms* $(G,+,\varphi,\psi)$, $\varphi(x)=\psi^2(x)$.

At the same time it is proven that in T-*quasigroup* $(G,\cdot)$ of the form $(G,+,\varphi,\psi)$ Schweitzer identity is true if and only if $\varphi^2(y)+\psi\varphi(y)=0$, $\varphi(\psi(x))=\psi(x)$, $\psi^2(z)=\varphi(z)$.

Based on the demonstrated statements, examples of AD and Sweitzer quasigroups are constructed.

A new construction method of medial, semimedial and AD topological quasigroups has been proposed.

The authors have used the concept of a special direct product of an topological Abelian group G and proved that a binary operation can be defined on the set $G\times G$, such that the new algebraic structure is a non-associative topological quasigroups obeying medial, semimedial and AD laws.

The results established are related to the results of L. Chiriac and N. Josu in [1, 2] and to the research papers [3, 4, 7, 8, 9, 10].

2. Basic Notions

In this section we recall some fundamental definitions and notations [5, 6, 11].

A non-empty set G is said to be a *groupoid* with respect to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a,b) of elements of G there is a unique element $ab\in G$.

A quasigroup is a binary algebraic structure in which one-sided multiplication is a bijection in that all equations of the form $ax=b$ and $ya=b$ have unique solutions.

A groupoid G is called a primitive groupoid with divisions, if there exist two binary operation $l:G\times G\to G$, $r:G\times G\to G$ such that $l(a,b)\cdot a=b, a\cdot r(a,b)=b$ for all $a,b\in G$. Thus, a primitive groupoid with divisions is a universal algebra with three binary operations.

A primitive groupoid G with divisions is called a quasigroup if the equations $ax=b$ and $ya=b$ have unique solutions. In a quasigroup G the divisions l,r are unique. If the multiplication operation in a quasigroup $(G,\cdot)$ with a topology is continuous, then G is called a semitopoligical quasigroup. If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a topological quasigroup.

An element $e\in G$ is called an *identity* if $ex=xe=x$ every $x\in G$. A quasigroup $(G,\cdot)$ with an identity element $e\in G$ is called a loop.

A groupoid $(G,\cdot)$ is called *medial* if it satisfies the law $xy\cdot zt=xz\cdot yt$ for all $x,y,z,t\in G$. A groupoid $(G,\cdot)$ is called *paramedial* if it satisfies the law $xy\cdot zt=ty\cdot zx$ for all $x,y,z,t\in G$.

A groupoid $(G,\cdot)$ is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

A groupoid $(G,\cdot)$ is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(ab)\cdot c = (cb)\cdot a$ for all $a, b, c \in G$.

An Abel-Grassmann's right invertive groupoid (also known as a right almost semigroup or RA-semigroup) is a groupoid that satisfies the right invertive law $a(bc) = c(ba)$. This structure is the dual of the AG-groupoid (or left invertive law). We will use a right invertive groupoid as AD groupoid.

A groupoid $(G,\cdot)$ is called *AD-groupoid* if it satisfies the law $a \cdot bc = c \cdot ba$ for all $a, b, c \in G$.

A *Schweitzer* quasigroup $(G,\cdot)$ is a quasigroup that satisfies the Schweitzer identity $(xy)\cdot(xz) = zy$ for all elements $x, y, z \in G$.

A *Neumann* quasigroup $(G,\cdot)$ is a quasigroup that satisfies the Neumann identity $x(yz \cdot yx) = z$ for all elements $x, y, z \in G$.

Left semi-medial identity in a groupoid $(G,\cdot)$ has the following form: $xx \cdot zt = xy \cdot xz$ for all $x, y, z, t \in G$. R.H. Bruck [15] used this identity to define commutative Moufang loops in the class of loops.

3. T-quasigroups with AD and Sweitzer identities

In this section we study some aspects of charaterization of abelian groups isotopic to T-$quasigroups$.

Definition 3.1. *Quasigroup $(G,\cdot)$ is a T-$quasigroup$ if and only if there exists an abelian group $(G,+)$, its automorphisms φ and ψ, and a fixed element $a \in G$ such that $x \cdot y = \varphi(x) + \psi(y) + a$ for all $x, y \in G$.*

Under the conditions of Definition 3.1 we shall say that the isotope $(G,\cdot)$ is generated by the automorphisms φ, ψ and a fixed element $a \in G$ of the abelian group $(G,+)$, and, we write $(G,\cdot) = g(G,+,\varphi,\psi,a)$.

We study the problem formulated below.

Problem 3.1. *Let $(Q,\cdot)$ be a T-$quasigroup$. Under which conditions the Q is a quasigroup (of its T-$forms$ $(Q,+,\varphi,\psi,a)$) satisfying the identities P_i of some algebraic structure, where $i = 1, 2, ..., k$?*

In [12, 13], V. Shcherbacov and his students studied T-quasigroups with Schroder identity, Stein 2-nd and 3-rd identity.

We examine the T-$quasigroups$ with AD and Schweitzer identities.

Theorem 3.1. *Let $(G,\cdot)$ be a T-$quasigroup$. Then $(G,\cdot)$ is AD-$quasigroup$ if and only if for any of its T-$forms$ $(G,+,\varphi,\psi)$, $\varphi(x)=\psi(x)^2$.*

Proof. We rewrite the identity of the AD-$quasigroup$

$$x\cdot yz = z\cdot yx, \tag{1}$$

in the following form:

$$\varphi(x)+\psi(yz)=\varphi(z)+\psi(yx). \tag{2}$$

From (2) we have

$$\varphi(x)+\psi(\varphi(y)+\psi(z))=\varphi(z)+\psi(\varphi(y)+\psi(x)). \tag{3}$$

$$\varphi(x)+\psi\varphi(y)+\psi^2(z)=\varphi(z)+\psi\varphi(y)+\psi^2(x). \tag{4}$$

$$\varphi(x)+\psi^2(z)=\varphi(z)+\psi^2(x). \tag{5}$$

If we substitute $x=0$ in equality (5), then we obtain

$$\psi^2(z)=\varphi(z). \tag{6}$$

If in equality (5) $z=0$, then we obtain

$$\varphi(x)=\psi^2(x). \tag{7}$$

Converse. Substituting the expression $x\cdot y=\varphi(x)+\psi(y)$ in identity (1), then we get (4). Substituting equalities (6) and (7) in (4), we obtain that identity (4) is true. In this way, in this case, we have that identity (1) is true. The proof is complete. □

Example 3.1. *Examine the group $\mathbb{Z}_{17}$ of residues modulo 17. Define the quasigroup $(G,\cdot)$. We define the binary operation $x\cdot y=16x+4y(mod17)$ for all $x,y\in G$. Then $(G,\cdot)$ is an $AD-quasigroup$. Check. Let $x\cdot yz=z\cdot yx$. Then $16x+4(16y+4z)=16z+4(16y+4x)(mod17)$, $16x+64y+16z=16z+64y+16x(mod17)$, $0=0(mod17)$.*

Example 3.2. *Examine the group $\mathbb{Z}_{27}$ of residues modulo 27. Define the quasigroup $(G,\cdot)$, where binary operation $x\cdot y=25x+5y(mod27)$ for all $x,y\in G$. Then $(G,\cdot)$ is an $AD-quasigroup$. Check. Let $x\cdot yz=z\cdot yx$. Then $25x+5(25y+5z)=25z+5(25y+5x)(mod27)$, $25x+125y+25z=25z+125y+25x(mod27)$, $0=0(mod27)$.*

Using Theorem 3.1 from [7], we can construct the example of AG-quasigroups. A T-$quasigroup$ $(G,\cdot)$ is an AG-quasigroup if and only if the following condition holds for its automorphisms:$\varphi^2(x)=\psi(x)$ for all $x\in G$

Example 3.3. *Examine the group $\mathbb{Z}_{37}$ of residues modulo 37. Define the quasigroup $(G,\cdot)$. We define the binary operation $x \cdot y = 6x + 36y(mod37)$ for all $x, y \in G$. Then $(G,\cdot)$ is an AG-quasigroup. Check. Let $(x \cdot y) \cdot z = (z \cdot y) \cdot x$. Then $6(6x+36y)+36z = 6(6z+36y)+36x(mod37)$, $36x+108y+36z = 36z+108y+36x(mod37)$, $0 = 0(mod37)$.*

Example 3.4. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G,+) = (\mathbb{Z}_5,+)$, $\varphi(x) = 4x$, $\psi(x) = 2x$. Then $x \cdot y = 4x + 2y$ and $\varphi(x) = \psi^2(x)$. Hence, $(G,\cdot) = g(G,+,\varphi,\psi)$ is an AD-quasigroup. Below we have constructed the Cayley table for AD-quasigroup $(G,\cdot)$*

$(\cdot)$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

In this case, $(G,\cdot) = g(G,+,4x,2x)$ is an AD-quasigroup, but it is not AG-quasigroup. For example $(2 \cdot 4) \cdot 3 \neq (3 \cdot 4) \cdot 2$.

Example 3.5. *Let $(G,+) = (\mathbb{Z}_5,+)$, $\varphi(x) = 3x$, $\psi(x) = 2x$. Then $x \cdot y = 3x + 2y$ and $\varphi(x) \neq \psi^2(x)$. Hence, $(G,\cdot) = g(G,+,\varphi,\psi)$ is not an AD-quasigroup. Below we have constructed the Cayley table for quasigroup $(G,\cdot)$, where $a \cdot (bc) = c \cdot (ba)$ does not hold in $(G,\cdot)$. For example, $3 \cdot (4 \cdot 2) \neq 2 \cdot (4 \cdot 3)$. At the same time, $(G,\cdot) = g(G,+,3x,2x)$ is not AG-quasigroup, for example, $(3 \cdot 4) \cdot 2 \neq (2 \cdot 4) \cdot 3$.*

$(\cdot)$	0	1	2	3	4
0	0	2	4	1	3
1	3	0	2	4	1
2	1	3	0	2	4
3	4	1	3	0	2
4	2	4	1	3	0

Theorem 3.2. *Let $(G,\cdot)$ be a T-quasigroup. Then $(G,\cdot)$ is Schweitzer quasigroup if and only if for any of its T-forms $(G,+,\varphi,\psi)$, $\varphi^2(y) + \psi\varphi(y) = 0$, $\varphi(\psi(x)) = \psi(x)$, $\psi^2(z) = \varphi(z)$.*

Proof. The identity of the Schweitzer quasigroup

$$(yx) \cdot (yz) = zx, \tag{8}$$

we rewrite in the following form:

$$\varphi(yx) + \psi(yz) = \varphi(z) + \psi(x). \tag{9}$$

From (2) we have

$$\varphi(\varphi(y) + \psi(x)) + \psi(\varphi(y) + \psi(z)) = \varphi(z) + \psi(x), \tag{10}$$

$$\varphi^2(y) + \varphi\psi(x) + \psi\varphi(y) + \psi^2(z) = \varphi(z) + \psi(x). \tag{11}$$

If we substitute $x = z = 0$ in equality (11), then we obtain

$$\varphi^2(y) + \psi\varphi(y) = 0. \tag{12}$$

Similarly, if we substitute $z = y = 0$ in equality (11), then we obtain

$$\varphi(\psi(x)) = \psi(x). \tag{13}$$

If in equality (11) $x = y = 0$, then we get

$$\psi^2(z) = \varphi(z). \tag{14}$$

Converse. Substituting the expression $x \cdot y = \varphi(x) + \psi(y)$ in identity (8), we get (11). Substituting equalities (12), (13) and (14) in (11), we obtain that the identity (11) is true. In this way, in this case, identity (8) is true. The proof is complete. □

Remark 3.1. *We can rewrite Theorem 3.3 in the next form.*

Theorem 3.3. *Let G be a T-quasigroup. Then G is Schweitzer quasigroup if and only if for any of its T-$forms$ $(G, +, \varphi, \psi)$, $\psi^2 = \varepsilon$, $\varphi = \varepsilon$, $\varepsilon + \psi(\varepsilon) = 0$.*

Example 3.6. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_7, +)$, according to Theorem 3.3, $\varphi(x) = x$, $\psi(x) = -x$. Then $x \cdot y = x - y$ and $\varphi(x) = x = -\psi(x)$, $x^2 = x \cdot x = 0$. Hence, $(G, \cdot) = g(G, +, \varphi, \psi)$ is an Schweitzer* *quasigroup. Below we have constructed the Cayley table for Schweitzer quasigroup $(G,\cdot)$. This quasigroup is also AD-quasigroup.*

$(\cdot)$	0	1	2	3	4	5	6
0	0	6	5	4	3	2	1
1	1	0	6	5	4	3	2
2	2	1	0	6	5	4	3
3	3	2	1	0	6	5	4
4	4	3	2	1	0	6	5
5	5	4	3	2	1	0	6
6	6	5	4	3	2	1	0

Remark 3.2. *In* [14], *N. Diduric and V. Shcherbacov proved that any Neumann quasigroup $(Q,\cdot)$ can be presented in the form $x \cdot y = x - y$, where $(Q,+)$ is an abelian group. Automorphism group of Neumann quasigroup coincides with the group $Aut(Q,+)$. Any Schweizer quasigroup is a Neumann quasigroup and vice versa.*

4. On a method of constructing medial, semimedial and AD topological quasigroups

In Section 4 we examined:

Problem 4.1. *Let $(G,+)$ be a commutative topological group. Under which conditions on the set $G\times G$ can one define a binary operation $(\circ)$ such that, $(G\times G,\circ)$ is a nonassociative topological quasigroups obeying certain laws?*

We prove a new method of constructing non-associative medial, semimedial, non-paramedial and AD topological quasigroup.

Theorem 4.1. *Let $(G,+,\tau)$ be a commutative topological group where G is not a singleton. For (x_1,y_1) and (x_2,y_2) in $G\times G$ define*

$$(x_1,y_1)\circ(x_2,y_2) = (x_2+y_2+y_1, -x_1-y_1-x_2)$$

Then $(G\times G,\circ,\tau_G)$, relative to the product topology τ_G, is a medial, Moufang and AD non-associative topological quasigroup. Moreover, if (G,τ) is T_i-space, then $(G\times G,\tau_G)$ is T_i-space too, where $i = 1,2,3,3.5$.

Proof. **1.** We will prove that $(G\times G,\circ)$ is a quasigroup. To this end, we will show that the equations $y\circ a = b$ and $a\circ x = b$ have unique solutions in $(G\times G,\circ)$. Let $y = (y_1,y_2)$,

$x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $y \circ a = b$ we have

$$(y_1, y_2) \circ (a_1, a_2) = (b_1, b_2). \tag{15}$$

According to the conditions of Theorem 4.1

$$(y_1, y_2) \circ (a_1, a_2) = (a_1 + a_2 + y_2, -y_1 - y_2 - a_1). \tag{16}$$

From (15) and (16) we get

$$a_1 + a_2 + y_2 = b_1 \tag{17}$$

and

$$-y_1 - a_1 - y_2 = b_2. \tag{18}$$

From (17) and (18) we obtain

$$y_2 = b_1 - a_1 - a_2. \tag{19}$$

and

$$y_1 = a_2 - b_1 - b_2. \tag{20}$$

Hence, $y_1 = a_2 - b_1 - b_2$ and $y_2 = b_1 - a_1 - a_2$ are solutions of the equation $y \circ a = b$. It is easy to show that any other solutions of that equation coincide with y_1 and y_2.

In this case

$$l((a_1, a_2), (b_1, b_2)) = (a_2 - b_1 - b_2, b_1 - a_1 - a_2)$$

and

$$l((a_1, a_2), (b_1, b_2)) \circ (a_1, a_2) = (b_1, b_2).$$

We will show that the equation $a \circ x = b$ have unique solutions in $(G \times G, \circ)$. Let $x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $a \circ x = b$ we have

$$(a_1, a_2) \circ (x_1, x_2) = (b_1, b_2). \tag{21}$$

According to the conditions of Theorem 4.1

$$(a_1, a_2) \circ (x_1, x_2) = (x_1 + x_2 + a_2, -a_1 - a_2 - x_1). \tag{22}$$

From (21) and (22) we get

$$x_1 + x_2 + a_2 = b_1 \tag{23}$$

and

$$-a_1 - a_2 - x_1 = b_2. \tag{24}$$

From (23) and (24) we obtain

$$x_2 = b_1 + b_2 + a_1. \tag{25}$$

and

$$x_1 = -a_1 - a_2 - b_2. \tag{26}$$

Hence, $x_1 = -a_1 - a_2 - b_2$ and $x_2 = b_1 + b_2 + a_1$ are solutions of the equation $a \circ x = b$. It is easy to show that any other solutions of that equation coincide with x_1 and x_2. Then $x_1 = -a_1 - b_2 - a_2$ and $x_2 = b_1 + b_2 + a_1$ are unique solutions.

In this case

$$r((a_1, a_2), (b_1, b_2)) = (-a_1 - b_2 - a_2, b_1 + b_2 + a_1)$$

and $(a_1, a_2) \circ r((a_1, a_2), (b_1, b_2)) = (b_1, b_2)$. Thus $(G \times G, \circ)$ is a quasigroup.

2. We will prove that associativity

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) \tag{27}$$

does not hold in $(G \times G, \circ)$.

Indeed, for the first side of the law (27), we obtain

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_2 + y_2 + y_1, -x_1 - y_1 - x_2) \circ (x_3, y_3) =$$

$$= (x_3 + y_3 - x_1 - y_1 - x_2, -x_2 - y_2 - y_1 + x_1 + y_1 + x_2 - x_3). \tag{28}$$

$$= (x_3 + y_3 - x_1 - y_1 - x_2, -y_2 + x_1 - x_3). \tag{29}$$

Similarly, for the second side of the law (27), we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_1, y_1) \circ (x_3 + y_3 + y_2, -x_2 - y_2 - x_3) =$$

$$= (y_3 + y_1 - x_2, -x_1 - y_1 - y_2 - y_3 - x_3). \tag{30}$$

From (29) and (30), it is clear that associativity does not hold in $(G \times G, \circ)$.

3. We will show that $(G \times G, \circ)$ is medial quasigroup, that is, the property $xy \cdot zt = xz \cdot yt$ holds.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3), t = (x_4, y_4)$, then

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) = ((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_7)). \tag{31}$$

According to the Theorem, for the first side of the law (31), we have

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) =$$

$$= ((x_2 + y_2 + y_1, -x_1 - y_1 - x_2)) \circ ((x_4 + y_4 + y_3, -x_3 - y_3 - x_4)) =$$

$$= (y_4 - x_3 - x_2 - x_1 - y_1, -y_2 + x_1 - x_4 - y_4 - y_3). \tag{32}$$

Similarly, for the other side of the law (31), we get

$$((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_4)) =$$

$$= (x_3 + y_3 + y_1, -x_1 - y_1 - x_3) \circ (x_4 + y_4 + y_2, -x_2 - y_2 - x_4) =$$

$$= (y_4 - x_3 - x_2 - x_1 - y_1, -y_2 + x_1 - x_4 - y_4 - y_3). \qquad (33)$$

From (32) and (33), we obtain that both sides are equal and $(G \times G, \circ)$ is a medial quasigroup.

Similarly, it is shown that $(G \times G, \circ)$ is a semimedial(commutative Moufang), non-paramedial quasigroup.

4. Similarly, we will show that AD identity $a \cdot bc = c \cdot ba$ is fulfilled in $(G \times G, \circ)$.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3)$. Then

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_3, y_3) \circ ((x_2, y_2)) \circ (x_1, y_1)). \qquad (34)$$

Indeed, according to the Theorem for the first side of the law (34), we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) =$$

$$(x_1, y_1) \circ ((x_3 + y_3 + y_2, -x_2 - y_2 - x_3)) =$$

$$(y_3 - x_2 + y_1, -x_1 - y_1 - x_3 - y_3 - y_2). \qquad (35)$$

Similarly, for the other side of the law (34) we get

$$(x_3, y_3) \circ ((x_2, y_2) \circ (x_1, y_1)) =$$

$$(x_3, y_3) \circ (x_1 + y_1 + y_2, -x_2 - y_2 - x_1) =$$

$$(y_3 - x_2 + y_1, -x_1 - y_1 - x_3 - y_3 - y_2). \qquad (36)$$

From (35) and (36), we obtain that both sides are equal and $(G \times G, \circ)$ is an AD quasigroup.

Multiplication $(\circ)$ and divisions $l(a, b)$ and $r(a, b)$ are jointly continuous relative to the product topology.

Consequently, $(G \times G, \circ, \tau_G)$ is a non-associative, medial, semimedial and AD topological quasigroup.

If (G, τ) is T_i-space, then according to Theorem 2.3.11 in [6], a product of T_i-spaces is a T_i-spaces, where $i = 1, 2, 3, 3.5$. The proof is complete. □

In [7] was proved the following Theorem.

Theorem 4.2. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a medial, semimedial, paramedial, bicommutative, Manin, Cote and GA non-associative topological quasigroup. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

The following Theorem was proved in [2].

Theorem 4.3. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (-x_1 - y_1 + y_2, -x_2 - y_2 + x_1).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a non-associative, medial, AG and AD-topological quasigroup. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

Example 4.1. *Let $G = \{0, 1, 2\}$. We define the binary operation " + ".*

(+)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(G, +)$ is a commutative group. Define a binary operation $(\circ)$ on the set $G \times G$ by $(x_1, y_1) \circ (x_2, y_2)$=$(x_2 + y_2 + y_1, -x_1 - y_1 - x_2)$ for all $x_1, y_1, x_2, y_2 \in G \times G$. If we label the elements as follows $(0, 0) \leftrightarrow 0$, $(0, 1) \leftrightarrow 1$, $(0, 2) \leftrightarrow 2$, $(1, 0) \leftrightarrow 3$, $(1, 1) \leftrightarrow 4$, $(1, 2) \leftrightarrow 5$, $(2, 0) \leftrightarrow 6$, $(2, 1) \leftrightarrow 7$, $(2, 2) \leftrightarrow 8$, then, we obtain:

(∘)	0	1	2	3	4	5	6	7	8
0	0	3	6	5	8	2	7	1	4
1	5	8	2	7	1	4	0	3	6
2	7	1	4	0	3	6	5	8	2
3	2	5	8	4	7	1	6	0	3
4	4	7	1	6	0	3	2	5	8
5	6	0	3	2	5	8	4	7	1
6	1	4	7	3	6	0	8	2	5
7	3	6	0	8	2	5	1	4	7
8	8	2	5	1	4	7	3	6	0

Then $(G \times G, \circ)$ is a medial, semimedial, non-paramedial, AD non-associative quasigroup.

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