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Maximal multiplicity of the line at infinity in a sextic polynomial differential system

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Abstract. We investigate planar polynomial differential systems of degree six having an invariant straight line at infinity with high algebraic multiplicity in the sense of the Poincaré compactification. By analyzing the determinant associated with the invariant line at infinity, we show that the maximal multiplicity attainable in this class is 16. The computations involved are extremely large, and only representative cases are presented. The obtained results support the conjecture that the bound $M_\infty(n) = 3n - 2$ is sharp for polynomial differential systems of degree $n \geq 2$.
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Keywords: planar polynomial differential systems, invariant straight line at infinity, algebraic multiplicity, Poincaré transformation.

Multiplicitatea maximală a dreptei de la infinit într-un sistem diferențial polinomial de gradul şase

Rezumat. În această lucrare investigăm sisteme diferențiale polinomiale plane de gradul șase care admit o dreaptă invariantă la infinit cu multiplicitate algebrică mare, în sensul compactificării lui Poincaré. Prin analiza determinantului asociat dreptei invariante de la infinit, arătăm că multiplicitatea maximă ce poate fi atinsă în această clasă este 16. Calculele implicate sunt extrem de voluminoase, fiind prezentate doar cazuri reprezentative. Rezultatele obținute susțin conjectura conform căreia limita $M_\infty(n) = 3n-2$ este optimă pentru sisteme diferențiale polinomiale de grad $n \geq 2$.
Cuvinte-cheie: sisteme diferențiale polinomiale, dreaptă invariantă la infinit, multiplicitate algebrică, transformare Poincaré.

1. Introduction

We consider the real planar polynomial differential system

$$\begin{cases} \dfrac{dx}{dt} = P(x,y), \\ \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, $\gcd(P, Q) = 1$, and let

$$\mathbb{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

be the associated polynomial vector field. The degree of system (1) is defined as $n = \max\{\deg P, \deg Q\}$.

Definition 1.1. *An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$, is called an invariant algebraic curve of system* (1) *if there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$ such that*

$$\mathbb{X}(f) = f(x, y)K(x, y). \tag{2}$$

Invariant algebraic curves play an important role in the qualitative theory of polynomial differential systems [1], [2], [3], [4]. The problem of determining the maximal number of invariant straight lines of a polynomial differential system has been studied in [5].

Polynomial differential systems having an invariant straight line of maximal multiplicity were studied in [6], [7], [8], and polynomial differential systems with a center-focus critical point having an invariant straight line of maximal multiplicity were investigated in [9], [10], [11], [12]. Invariant straight lines are also used in the construction of Darboux first integrals; see, for example, [13], [14], [15].

In this paper, we use the notion of algebraic multiplicity introduced in [16].

Definition 1.2. *Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \binom{n+m}{n}$. Let $v_1, v_2, \ldots, v_R$ be a base of $\mathbb{C}_m[x]$. If k is the greatest positive integer such that the k-th power of f divides* $\det M_R$*, where*

$$M_R = \begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \ldots & \mathbb{X}(v_R) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \ldots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic curve f of degree m of the vector field $\mathbb{X}$ has algebraic multiplicity k.

In [7], it was proved that for planar polynomial differential systems of degree $n \geq 2$, the maximal algebraic multiplicity $M_a(n)$ of an affine invariant straight line satisfies

$$3n - 2 \leq M_a(n) \leq 3n - 1.$$

Moreover, for cubic systems the maximal multiplicity of a real straight line, including the line at infinity, is equal to seven. Following the notation introduced in [7], we denote by $M_\infty(n)$ the maximal multiplicity of the invariant straight line at infinity.

The aim of this paper is to study planar polynomial differential systems of degree six of the form

$$\begin{cases} \dot{x} = \displaystyle\sum_{n=0}^{6}\sum_{i=0}^{n} a_{i,n-i}x^i y^{n-i}, \\ \dot{y} = \displaystyle\sum_{n=0}^{6}\sum_{i=0}^{n} b_{i,n-i}x^i y^{n-i}, \end{cases} \tag{3}$$

where $a_{i,j}, b_{i,j} \in \mathbb{R}$.

The main contribution of this paper is the identification and classification, up to orbital equivalence, of planar polynomial differential systems of degree six whose invariant straight line at infinity has maximal algebraic multiplicity.

2. Obtaining systems with maximal multiplicity

For system (3), we apply the Poincaré transformation $x \mapsto \frac{1}{z}$, $y \mapsto \frac{y}{z}$, followed by the substitution $y \mapsto x$ and $z \mapsto y$. In this way, we obtain the system

$$\begin{cases} \dot{x} = \displaystyle\sum_{n=0}^{6}\sum_{i=0}^{n} \left(a_{i,n-i}\, x^{n-i+1} y^{6-n} + b_{i,n-i}\, x^{n-i} y^{6-n}\right), \\ \dot{y} = y\displaystyle\sum_{n=0}^{6}\sum_{i=0}^{n} a_{i,n-i}\, x^{n-i} y^{6-n}. \end{cases} \tag{4}$$

where the coefficients coincide with those of system (3). The invariant straight line $y = 0$ of system (4) represents the invariant straight line located at infinity of system (3).

Therefore, in order to obtain an invariant line at infinity of maximal multiplicity, it is necessary to impose that the polynomial $\det M_R$ admits y as a factor of maximal possible power. In this case, taking $v_1 = 1$, $v_2 = x$, and $v_3 = y$, we obtain the matrix M_R from Definition 1.2:

$$M_R = \begin{pmatrix} 1 & x & y \\ \mathbb{X}(1) & \mathbb{X}(x) & \mathbb{X}(y) \\ \mathbb{X}^2(1) & \mathbb{X}^2(x) & \mathbb{X}^2(y) \end{pmatrix} = \begin{pmatrix} 1 & x & y \\ 0 & P & Q \\ 0 & P\frac{\partial P}{\partial x} + Q\frac{\partial P}{\partial y} & P\frac{\partial Q}{\partial x} + Q\frac{\partial Q}{\partial y} \end{pmatrix}.$$

Computing the determinant of this matrix, we obtain

$$\det M_R = P\left(P\frac{\partial Q}{\partial x} + Q\frac{\partial Q}{\partial y}\right) - Q\left(P\frac{\partial P}{\partial x} + Q\frac{\partial P}{\partial y}\right) = y\sum_{i=1}^{17} A_i(x) y^i. \tag{5}$$

Thus, y is a factor of the polynomial (5), which shows that the line at infinity is invariant. The highest power of y appearing in (5) is 18, and therefore we can conclude that

$$M_\infty(6) \le 18.$$

In order for the line at infinity of system (3) to have maximal multiplicity, we require that all the polynomials $A_i(x)$ vanish, that is, all coefficients of these polynomials with respect to x are equal to zero. From the condition $A_1(x) = 0$, we obtain that the multiplicity of the line at infinity is equal to 2, and so on.

Despite extensive computational efforts, some cases could not be fully resolved due to their extreme algebraic complexity. In particular, the corresponding polynomial systems require up to 4.5 GB of memory, and even on a workstation equipped with an Intel Core i7-14700K processor and 32 GB of RAM, the computations could not be completed after more than 100 hours of continuous runtime. Consequently, several cases have been omitted.

Despite these limitations, three polynomial differential systems attaining maximal multiplicity were identified. The first of these systems is presented below.

$$
\begin{cases}
\dot{x} = \dfrac{a_{6,0}^{11}\left(a_{6,0}b_{1,0} - a_{1,0}b_{6,0}\right)^4}{432b_{6,0}^5}\Big(432x^6a_{6,0}^4b_{6,0}^6 - 2592x^5ya_{6,0}^5b_{6,0}^5 + 432x^5a_{5,0}a_{6,0}^3b_{6,0}^6 + 6480x^4y^2a_{6,0}^6b_{6,0}^4 - \\
-2160x^4ya_{5,0}a_{6,0}^4b_{6,0}^5 + 180x^4a_{5,0}^2a_{6,0}^2b_{6,0}^6 - 8640x^3y^3a_{6,0}^7b_{6,0}^3 + 4320x^3y^2a_{5,0}a_{6,0}^5b_{6,0}^4 - 720x^3ya_{5,0}^2a_{6,0}^3b_{6,0}^5 + \\
+40x^3a_{5,0}^3a_{6,0}b_{6,0}^6 + 6480x^2y^4a_{6,0}^8b_{6,0}^2 - 4320x^2y^3a_{5,0}a_{6,0}^6b_{6,0}^3 + 1080x^2y^2a_{5,0}^2a_{6,0}^4b_{6,0}^4 - 120x^2ya_{5,0}^3a_{6,0}^2b_{6,0}^5 + \\
+5x^2a_{5,0}^4b_{6,0}^6 - 2592xy^5a_{6,0}^9b_{6,0} + 2160xy^4a_{5,0}a_{6,0}^7b_{6,0}^2 - 720xy^3a_{5,0}^2a_{6,0}^5b_{6,0}^3 + 120xy^2a_{5,0}^3a_{6,0}^3b_{6,0}^4 - \\
-10xya_{5,0}^4a_{6,0}b_{6,0}^5 + 432xa_{1,0}a_{6,0}^3b_{6,0}^6 - 432y^5a_{5,0}a_{6,0}^8b_{6,0} + 180y^4a_{5,0}^2a_{6,0}^6b_{6,0}^2 - 40y^3a_{5,0}^3a_{6,0}^4b_{6,0}^3 + \\
+5y^2a_{5,0}^4a_{6,0}^2b_{6,0}^4 + 2160ya_{6,0}^5b_{1,0}b_{6,0}^4 - 2592ya_{1,0}a_{6,0}^4b_{6,0}^5 + 432a_{6,0}^4b_{0,0}b_{6,0}^5 - 72a_{5,0}a_{6,0}^3b_{1,0}b_{6,0}^5 + \\
+72a_{1,0}a_{5,0}a_{6,0}^2b_{6,0}^6 + 432y^6a_{6,0}^{10}\Big), \\
\dot{y} = \dfrac{a_{6,0}^{10}\left(a_{6,0}b_{1,0} - a_{1,0}b_{6,0}\right)^4}{432b_{6,0}^4}\Big(432x^6a_{6,0}^4b_{6,0}^6 - 2592x^5ya_{6,0}^5b_{6,0}^5 + 432x^5a_{5,0}a_{6,0}^3b_{6,0}^6 + 6480x^4y^2a_{6,0}^6b_{6,0}^4 - \\
-2160x^4ya_{5,0}a_{6,0}^4b_{6,0}^5 + 180x^4a_{5,0}^2a_{6,0}^2b_{6,0}^6 - 8640x^3y^3a_{6,0}^7b_{6,0}^3 + 4320x^3y^2a_{5,0}a_{6,0}^5b_{6,0}^4 - 720x^3ya_{5,0}^2a_{6,0}^3b_{6,0}^5 + \\
+40x^3a_{5,0}^3a_{6,0}b_{6,0}^6 + 6480x^2y^4a_{6,0}^8b_{6,0}^2 - 4320x^2y^3a_{5,0}a_{6,0}^6b_{6,0}^3 + 1080x^2y^2a_{5,0}^2a_{6,0}^4b_{6,0}^4 - 120x^2ya_{5,0}^3a_{6,0}^2b_{6,0}^5 + \\
5x^2a_{5,0}^4b_{6,0}^6 - 2592xy^5a_{6,0}^9b_{6,0} + 2160xy^4a_{5,0}a_{6,0}^7b_{6,0}^2 - 720xy^3a_{5,0}^2a_{6,0}^5b_{6,0}^3 + 120xy^2a_{5,0}^3a_{6,0}^3b_{6,0}^4 - \\
-10xya_{5,0}^4a_{6,0}b_{6,0}^5 + 432xa_{6,0}^4b_{1,0}b_{6,0}^5 - 432y^5a_{5,0}a_{6,0}^8b_{6,0} + 180y^4a_{5,0}^2a_{6,0}^6b_{6,0}^2 - 40y^3a_{5,0}^3a_{6,0}^4b_{6,0}^3 + \\
+5y^2a_{5,0}^4a_{6,0}^2b_{6,0}^4 + 1728ya_{6,0}^5b_{1,0}b_{6,0}^4 - 2160ya_{1,0}a_{6,0}^4b_{6,0}^5 + 432a_{6,0}^4b_{0,0}b_{6,0}^5 + 432y^6a_{6,0}^{10}\Big),
\end{cases}
$$

where the nondegeneracy conditions $a_{6,0}b_{6,0}\left(a_{6,0}b_{1,0} - a_{1,0}b_{6,0}\right) \neq 0$ arise naturally in the process of solving the corresponding system of algebraic equations. After performing the time rescaling

$$
t = \frac{432b_{6,0}^5\tau}{a_{6,0}^{10}\left(a_{6,0}b_{1,0} - a_{1,0}b_{6,0}\right)^4}
$$

and the change of variables

$$\begin{cases} X = -6a_{6,0}b_{6,0}x + 6a_{6,0}^2 y + \dfrac{a_{5,0}^6 b_{6,0}^7}{108a_{6,0}^2} + \dfrac{72}{5}a_{6,0}^3 b_{6,0}^6\left(6a_{6,0}b_{0,0} - a_{5,0}b_{1,0}\right) + a_{5,0}b_{6,0}, \\ Y = -\dfrac{b_{6,0}^6\left(1296a_{6,0}^5 b_{1,0} - a_{5,0}^5 b_{6,0}\right)}{18a_{6,0}}x - \dfrac{1}{18}b_{6,0}^5\left(a_{5,0}^5 b_{6,0} + 6480a_{6,0}^5 b_{1,0} - \right. \\ \left. -7776a_{1,0}a_{6,0}^4 b_{6,0}\right) y + 12a_{5,0}a_{6,0}^3 b_{1,0}b_{6,0}^6 - \dfrac{a_{5,0}^6 b_{6,0}^7}{108a_{6,0}^2}, \end{cases}$$

the system is transformed into

$$\begin{cases} \dot{X} = X, \\ \dot{Y} = -5Y + \frac{b_{6,0}}{108a_{6,0}^2}X^6. \end{cases}$$

By rescaling the OY-axis using the transformation $X = x$, $Y = \dfrac{b_{6,0}}{108a_{6,0}^2}y$, we obtain the system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -5y + x^6. \end{cases} \tag{6}$$

The other two systems are omitted, as they are equivalent, up to affine transformations and time rescaling, to system (6). Consequently, up to this point, we have obtained a single system whose invariant straight line at infinity has multiplicity 16. Since some cases could not be fully resolved, we formulate the following conjecture:

Conjecture 2.1. *Let $\mathbb{X}$ be a real planar polynomial differential system of degree six. Then the maximal algebraic multiplicity of the invariant straight line at infinity is*

$$M_\infty(6) = 16.$$

Moreover, up to affine changes of coordinates and time rescaling, there exists a unique system attaining this maximal multiplicity, which can be reduced to the normal form (6).

3. The general case

From system (6) and from the polynomial systems with maximal multiplicity at infinity studied in [17], we observe that some of these systems have the form

$$\begin{cases} \dot{x} = x, \\ \dot{y} = (1-n)y + x^n, \end{cases} \tag{7}$$

where n is the degree of the system.

Applying the Poincaré transformation $x \mapsto \frac{1}{z}$, $y \mapsto \frac{y}{z}$, followed by the substitution $y \mapsto x$ and $z \mapsto y$, system (7) is transformed into

$$\begin{cases} \dot{x} = nxy^{n-1} - 1, \\ \dot{y} = y^n. \end{cases} \tag{8}$$

The polynomial $\det M_R$ associated with system (8), which determines the multiplicity of the invariant straight line at infinity for system (7), is given by

$$\det M_R = n(n-1)xy^{3n-2}.$$

This implies that any planar polynomial differential system may admit the invariant line at infinity with multiplicity at least equal to $3n - 2$. Therefore, in general, we obtain that

$$M_\infty(n) \geq 3n - 2.$$

As shown in [17] for the cases $n \in \{2, 3, 4, 5\}$, this value indeed represents the maximal multiplicity, that is,

$$M_\infty(n) = 3n - 2, \quad n \in \{2, 3, 4, 5\}.$$

This suggests that the bound $M_\infty(n) = 3n - 2$ is sharp for all $n \geq 2$, although a complete proof for arbitrary degree remains open.

Conjecture 3.1. *For any planar polynomial differential system of degree $n \geq 2$, the maximal algebraic multiplicity of the invariant straight line at infinity, in the sense of the Poincaré compactification, is given by*

$$M_\infty(n) = 3n - 2.$$

4. Conclusions

In this paper, we studied planar polynomial differential systems of degree six admitting an invariant straight line at infinity of maximal algebraic multiplicity. Using the Poincaré compactification and the associated determinant criterion, we provide strong evidence that the maximal multiplicity of the invariant straight line at infinity for sextic systems is equal to 16 and formulate a conjecture asserting the uniqueness of the normal form attaining this value. Moreover, by considering a general family of polynomial systems, we showed that the multiplicity $3n - 2$ of the invariant straight line at infinity can be attained for any degree $n \geq 2$. This result confirms and extends previous classifications for quadratic, cubic, quartic, and quintic systems, and places the sextic case into a unified theoretical framework.

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References

[1] Popa, M., Sibirskii, K. Conditions for the existence of a homogeneous linear partial integral of a differential system. *Differencial'nye Uravnenia*, 1987, no. 23, pp. 1324–1331 (in Russian).

[2] Sibirskii, K. Conditions of the existence of an invariant straight line of the quadratic system in the case of centre or focus. *Mat. Issled.* Kishinev, 1989, vol. 106, pp. 114–118 (in Russian).

[3] Kooij, R. Cubic systems with four real line invariants. *Math. Proc. Camb. Phil. Soc.*, 1995, vol. 118, no. 1, pp. 7–19.

[4] Llibre, J., Vulpe, N. Planar cubic polynomial differential systems with the maximum number of invariant straight lines. *Rocky Mountain J. Math.*, 2006, vol. 36, no. 4, pp. 1301–1373.

[5] Artes, J., Grunbaum, B., Llibre, J. On the number of invariant straight lines for polynomial differential systems. *Pacific Journal of Mathematics*, 1998, vol. 184, no. 2, pp. 207–230.

[6] Şubă, A. Quadratic differential systems with the line at infinity of maximal multiplicity. În: *Materialele conferinţei ştiinţifice internaţionale "Ştiinţă şi Educaţie: Noi Abordări şi Perspective"*, 24-25 martie 2023. Chişinau, UPSC, 2023, pp. 306–310.

[7] Şubă, A., Vacaraş, O. Cubic differential systems with an invariant straight line of maximal multiplicity. *Annals of the University of Craiova. Mathematics and Computer Science Series*, 2015, vol. 42 no. 2, pp. 427–449.

[8] Şubă, A., Vacaraş, O. Quartic differential systems with an invariant straight line of maximal multiplicity. *Bulletin of Academy of Sciences of the Republic of Moldova, Mathematics*, 2018, vol. 86, no. 1, pp. 76–91.

[9] Şubă, A. Centers of cubic differential systems with the line at infinity of maximal multiplicity. *Acta et Commentationes. Exact and Natural Sciences*, 2022, vol. 14, no. 2, pp. 38–46.

[10] Şubă, A., Vacaraş, O. Quartic differential systems with a non-degenerate monodromic critical point and multiple line at infinity. *Acta et Commentationes, Exact and Natural Sciences*, 2023, vol. 16, no. 2, pp. 25–34.

[11] Vacaraş, O. Center problem for quartic differential systems with an affine invariant straight line of maximal multiplicity. *Acta et Commentationes, Exact and Natural Sciences*, 2024, vol. 18, no. 2, pp. 38–54.

[12] Vacaraş, O., Cozma, D. The problem of the center for quartic differential systems with an affine invariant straight line and the line at infinity of maximal multiplicity. *Annals of the University of Craiova, Mathematics and Computer Science Series*, 2025, vol. 52, no. 2, pp. 582–593.

[13] Llibre, J., Xiang, Z. On the Darboux integrability of polynomial differential systems. *Qual. Theory Dyn. Syst.*, 2012, vol. 11, pp. 129–144.

[14] Cozma, D. Darboux integrability of a cubic differential system with two parallel invariant straight lines. *Carpathian J. Math.*, 2022, vol. 38, no. 1, pp. 129–137.

[15] Cozma, D. *Integrability of cubic systems with invariant straight lines and invariant conics*, Ştiinţa, Chişinău, 2013.

[16] Christopher, C., Llibre, J., Pereira, J. Multiplicity of invariant algebraic curves in polynomial vector fields. *Pacific Journal of Mathematics*, 2007, vol. 329, no. 1, pp. 63–117.

[17] Repeşco, V. Qualitative analysis of polynomial differential systems with the line at infinity of maximal multiplicity: exploring linear, quadratic, cubic, quartic, and quintic cases. *Acta et Commentationes, Exact and Natural Sciences*, 2023, vol. 16, no. 2, pp. 111–117.

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