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SO$(2, \mathbb{R})$-invariant polynomials of a differential system which are involved in solving the stability problem of unperturbed motion in the case of a center and a focus

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Abstract. We consider a two-dimensional polynomial differential system $s(1, m_1, ..., m_l)$ $(l < \infty)$, for which the origin is a singular point of the second group (center or focus). Using the Lie operator of the linear representation of the rotation group $SO(2, \mathbb{R})$ in the space of phase variables and coefficients of the system $s(1, m_1, ..., m_l)$ $(l < \infty)$, an upper bound is determined for the number of functionally independent $SO(2, \mathbb{R})$-invariant polynomials involved in solving the stability problem of the unperturbed motion described by this system.
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Polinoamele SO$(2, \mathbb{R})$-invariante ale sistemului diferenţial, care participă la rezolvarea problemei stabilităţii mişcării neperturbate în cazul centrului şi al focarului

Rezumat. Se examinează sistemul diferenţial bidimensional $s(1, m_1, \ldots, m_l)$ $(l < \infty)$, ce conţine în originea de coordonate punct singular de speţa a doua (centru sau focar). Cu ajutorul operatorului Lie de reprezentare liniară a grupului de rotaţie $SO(2, \mathbb{R})$ în spaţiul variabilelor de fază şi al coeficienţilor sistemului $s(1, m_1, \ldots, m_l)$ $(l < \infty)$ a fost determinat hotarul de sus al numărului de polinoame $SO(2, \mathbb{R})$-invariante şi funcţional independente, care participă la rezolvarea problemei de stabilitate a mişcării neperturbate, guvernate de acest sistem.
Cuvinte-cheie: sistem diferenţial, grup de rotaţie, operatorul lui Lie, invarianţi şi comitanţi, mărimi focale, stabilitatea mişcării neperturbate.

1. Introduction

Suppose that the differential equations of perturbed motion [4, §1.2] on the plane have the form

$$\frac{dx}{dt} = \sum_{i=0}^{l} P_{m_i}(x, y), \quad \frac{dy}{dt} = \sum_{i=0}^{l} Q_{m_i}(x, y) \quad (l < \infty), \tag{1}$$

where P_{m_i} and Q_{m_i} are homogeneous polynomials of degree $m_i \geq 1$ with respect to the phase variables x, y, and $m_0 = 1$. The set $\{1, m_1, m_2, ..., m_l\}$ consists of a finite number ($l < \infty$) of distinct natural numbers. The coefficients and variables of the polynomials of the system (1) take values from the field of real numbers $\mathbb{R}$. We will denote the system (1) by $s(m_0, m_1, ..., m_l)$, where $x = x(t)$, $y = y(t)$.

In the works of A.M. Lyapunov and his followers (see, for example, [1, 2, 3, 4, 5]), polynomial differential equations of perturbed motion were considered not only on the plane but also in the space. However, here and what follows, we restrict our investigation only to two-dimensional differential systems of the form (1) with the notation $s(1, m_1, m_2, ..., m_l)$.

In the present paper, we study the $SO(2, \mathbb{R})$-invariant stability conditions of unperturbed motion governed by the system $s(1, m_1, ..., m_l)$ from (1), on a $GL(2, \mathbb{R})$-invariant surface, when the characteristic equation of the specified system contains two purely imaginary roots, i.e., the origin of this system is a singular point of the second group (center or focus).

Note that in this case, the conditions for the system (1) to have a singular point of the second group (center or focus) are related to the existence of a certain sign-definite Lyapunov function. Along the trajectories of the equations (1), infinite sequences of polynomials (focus quantities) are constructed from the coefficients of these equations, which solve the problem of distinguishing a center from a focus. It is known from the works [12], [6], that these focus quantities are $SO(2, \mathbb{R})$-invariant.

Using these quantities and Lyapunov's theorems relating to Lyapunov's second method on the stability of unperturbed motion, $SO(2, \mathbb{R})$-invariant stability conditions for the unperturbed motion of a system with quadratic nonlinearities $s(1, 2)$ in the case of a center and a focus are established. Also, using the Lie algebra corresponding to the rotation group $SO(2, \mathbb{R})$, an upper bound for the number of functionally independent $SO(2, \mathbb{R})$-invariant polynomials involved in solving the stability problem of unperturbed motion in the case of a center and a focus is determined for the system (1).

The invariant approach mentioned above to the problems of stability of unperturbed motion originates in the works [1], [5] and in other publications of the same authors. Such results are noteworthy because they allow specialists who are not familiar with the intricacies of the theory of stability of unperturbed motion to immediately verify, using the given system of differential equations, the stability conditions of the motion under investigation.

2. Definition of stability of unperturbed motion. Lyapunov functions and Lyapunov's theorems relating to Lyapunov's second method on stability and instability of unperturbed motion

In [4, §1.1 - §1.2], it is shown that the differential equations of perturbed motion (1) are obtained as a result of the linearization of certain differential equations characterizing various processes in mechanics, electromechanics, and other fields. Consequently, *the unperturbed motion in the equations* (1) *corresponds to zero values of the variables x and y, i.e., the unperturbed motion $x = y = 0$ corresponds to a fixed point – the origin.*

Lyapunov's Definition. [4, §1.1]. *If for any positive number ϵ, no matter how small, one can find a positive number δ such that for any perturbations $x_0 = x(t_0)$, $y_0 = y(t_0)$ satisfying the condition*

$$x_0^2 + y_0^2 \le \delta \tag{2}$$

and for any $t \ge t_0$, the inequality

$$x^2(t) + y^2(t) < \epsilon$$

is satisfied, then the unperturbed motion is stable; otherwise, it is unstable.

If the unperturbed motion is stable and, moreover, any perturbed motion under sufficiently small initial perturbations (2) tends to the unperturbed motion, i.e.,

$$\lim_{t\to\infty}[x^2(t) + y^2(t)] = 0,$$

then the unperturbed motion is called *asymptotically stable*.

Consider some functions $V(x, y)$ of the phase variables x, y of the system (1), defined in a certain neighborhood of the origin. Assume that these functions are single-valued, vanish at $x = y = 0$, and possess continuous partial derivatives.

Definition 2.1. *Following* [3, §6]*), we say that a function $V(x, y)$ is called sign-definite (positive-definite or negative-definite) if under*

$$|x| < h, \quad |y| < h, \tag{3}$$

where h is a sufficiently small positive number, it can take values of only one definite sign and vanishes only at $x = y = 0$.

Example 2.1. The simplest examples of such functions can be written as

$$V(x, y) = g(x^2 + y^2)^k \quad (k = 1, 2, \dots), \tag{4}$$

where g is a non-zero constant for each k.

For $g > 0$, the function (4) is positive-definite on the entire plane, and for $g < 0$, it is negative-definite on the entire plane as well.

Example 2.2. Consider the functions

$$V(x,y) = g(x^2+y^2)^k + f_{2k+1}(x,y) + f_{2k+2}(x,y) + f_{2k+3}(x,y) + \dots \\ (k = 1, 2, 3, \dots), \tag{5}$$

where g is a non-zero constant, and $f_{2k+1}(x,y)$, $f_{2k+2}(x,y)$, $f_{2k+3}(x,y), \dots$ are certain forms of degree $2k+1, 2k+2, 2k+3, \dots$ respectively, with respect to the variables x and y. It is clear (see [3, §6])) that for a fixed $k = 1, 2, 3, \dots$ and a sufficiently small positive h from (3), the function $V(x,y)$ from (5) is sign-definite, with a sign that coincides with the sign of the constant g.

Definition 2.2. *Following* [3, §6], *we say that a function* $V(x,y)$ *is called semi-definite (positive semi-definite or negative semi-definite) if it, in the region* (3), *can take values of only one definite sign, but can vanish also when*

$$x^2 + y^2 \neq 0.$$

The specified functions in Definitions 2.1 and 2.2 will be called *Lyapunov functions*.

Following [3, §9, §10]), as well as [4, §2.4]), we formulate Lyapunov's theorems relating to his second method for equations (1).

Theorem A (on the stability of motion). *If for the differential equations of perturbed motion* (1) *it is possible to find a sign-definite function* $V(x,y)$ *whose total time derivative* t *along the trajectories of these equations is a semi-definite function of a sign opposite to that of* V*, or vanishes identically, then the unperturbed motion is stable.*

Theorem B (on the asymptotic stability of motion). *If for the differential equations of perturbed motion* (1) *it is possible to find a sign-definite function* $V(x,y)$ *whose total time derivative with respect to time* t*, composed along the trajectories of these equations, is also a sign-definite function of a sign opposite to that of* V*, then the unperturbed motion is asymptotically stable.*

Theorem C (on the instability of motion). *If the differential equations of perturbed motion* (1) *allow one to find a function* $V(x,y)$ *which, along the trajectories of these equations, possesses a sign-definite derivative* $\frac{dV}{dt}$ *and can take values of the same sign as* $\frac{dV}{dt}$ *in the neighborhood of zero* (3)*, then the unperturbed motion is unstable.*

3. Functional bases of $SO(2,\mathbb{R})$-comitants ($SO(2,\mathbb{R})$-invariants) for the differential system $s(1, m_1, \ldots, m_l)$ in the case of a center and a focus

Let us write the system $s(1, m_1, \ldots, m_l)$ from (1), in the following expanded form:

$$\begin{aligned} \frac{dx}{dt} &= a_1^1 x + a_2^1 y + \sum_{i=1}^{l} P_{m_i}(x,y), \\ \frac{dy}{dt} &= a_1^2 x + a_2^2 y + \sum_{i=1}^{l} Q_{m_i}(x,y) \quad (m_i \geq 2), \end{aligned} \tag{6}$$

where

$$\begin{aligned} P_{m_i}(x,y) &= \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}_k^1 x^{m_i-k} y^k, \\ Q_{m_i}(x,y) &= \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}_k^2 x^{m_i-k} y^k. \end{aligned} \tag{7}$$

In [12, §19]), it is shown that on the $GL(2,\mathbb{R})$-invariant surface

$$a_1^1 + a_2^2 = 0, \quad (a_1^1)^2 + 2a_2^1 a_1^2 + (a_2^2)^2 < 0 \tag{8}$$

by a centroaffine transformation and a linear time rescaling t, the system (6)–(7) can be brought to the form

$$\begin{aligned} \frac{dx}{dt} &= y + \sum_{i=1}^{l} \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}_k^1 x^{m_i-k} y^k, \\ \frac{dy}{dt} &= -x + \sum_{i=1}^{l} \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}_k^2 x^{m_i-k} y^k \quad (m_i \geq 2), \end{aligned} \tag{9}$$

for which the origin is a singular point of center or focus type. In the system (9), the same notations for variables and coefficients as in the system (6)–(7) are preserved.

We investigate the action of the rotation group $SO(2,\mathbb{R})$ given by the formulas

$$\bar{x} = x\cos\varphi + y\sin\varphi, \quad \bar{y} = -x\sin\varphi + y\cos\varphi \quad (0 \leq \varphi < \pi) \tag{10}$$

on the system (9).

As a result of the transformation (10), in the system (9) we obtain

$$\begin{aligned} \bar{x} &= \bar{y} + \sum_{i=1}^{l} \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{\bar{a}}{}_k^1 \bar{x}^{m_i-k} \bar{y}^k, \\ \bar{y} &= -\bar{x} + \sum_{i=1}^{l} \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{\bar{a}}{}_k^2 \bar{x}^{m_i-k} \bar{y}^k \quad (m_i \geq 2), \end{aligned} \tag{11}$$

meaning that it preserves the form of system (9).

Note that the coefficients $\overset{i}{\bar{a}}{}^j_k$ $(j = 1, 2)$ are linear functions of the coefficients of system (9) and contain $\sin\varphi$ and $\cos\varphi$ in each term to a power of at least two.

Let us denote by

$$N = 2\sum_{i=1}^{l}(m_i + 1) \tag{12}$$

the maximum number of all possible non-zero coefficients of the non-linear right-hand side of system (9), and by a the set of these coefficients.

In [11, §7.1]), the following theorem is proven:

Theorem 3.1. *The Lie operator of the representation of the group* $SO(2,\mathbb{R})$ *in the space* $E^{N+2}(x, y, a)$ *of system* (9) *has the form*

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + D, \tag{13}$$

where

$$\begin{aligned} D = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\Bigg\{&\left[(m_i - k)\overset{i}{a}{}^1_{k+1} - k\overset{i}{a}{}^1_{k-1} + \overset{i}{a}{}^2_k\right]\frac{\partial}{\partial \overset{i}{a}{}^1_k} + \\ &+\left[-\overset{i}{a}{}^1_k + (m_i - k)\overset{i}{a}{}^2_{k+1} - k\overset{i}{a}{}^2_{k-1}\right]\frac{\partial}{\partial \overset{i}{a}{}^2_k}\Bigg\}, \end{aligned} \tag{14}$$

and N *is given by* (12).

Corollary 3.1. *The Lie operator of the representation of the group* $SO(2,\mathbb{R})$ *in the coefficient space* $E^N(a)$ *of system* (9) *has the form* (14)*, where* N *is given by* (12).

Note 3.1. *Theorem 3.1 and corollary 3.1 are known from* [9, 10] *for the general form of the system* $s(1, m_1, m_2, \ldots, m_l)$ *from* (6).

Let $\bar{a}$ be the set of coefficients of the non-linear right-hand side of system (11).

Following [11, §7.2]), we present the following definitions:

Definition 3.1. *A polynomial* $K(x, y, a)$ *in the coefficients of the system* (9) *and the phase variables* x, y *is called an algebraic comitant of this system under the rotation group* $SO(2,\mathbb{R})$ *from* (10) *if the identity*

$$K(\bar{x}, \bar{y}, \bar{a}) = K(x, y, a) \tag{15}$$

holds for all admissible a, x, y*, and* φ*, where* $a(\bar{a})$ *is the set of coefficients of system* (9) ((11))*, and* (x, y) *(*$(\bar{x}, \bar{y})$*) are the phase variables of this system.*

If the comitant K of the system (9) does not depend on the phase variables x, y, then, following [12], it is conventionally called an *invariant* of the specified system under the rotation group, and we denote it by $I(a)$.

Definition 3.2. *Comitants (invariants) of system* (9) *with respect to the rotation group will henceforth be called $SO(2,\mathbb{R})$-comitants ($SO(2,\mathbb{R})$-invariants) of this system.*

Definition 3.3. *A set of algebraic $SO(2,\mathbb{R})$-comitants of the system* (9) *with respect to the rotation group*

$$\{K_\alpha(x, y, a), \alpha \in N^+\} \tag{16}$$

is called a functional basis of $SO(2,\mathbb{R})$-comitants of the specified system if any $SO(2,\mathbb{R})$-comitant $K(x, y, a)$ of the system under consideration under the rotation group can be represented as a single-valued function of the comitants (16).

A functional basis of $SO(2,\mathbb{R})$-comitants ($SO(2,\mathbb{R})$-invariants) is called minimal if, upon the removal of any of its comitants (invariants), it ceases to be a functional basis.

Definition 3.4. [11, § 7.3]. *If an $SO(2,\mathbb{R})$-comitant K of the system* (9) *is given in the form*

$$K = A_0x^m + A_1x^{m-1}y + \cdots + A_{m-1}xy^{m-1} + A_my^m,$$

where A_i $(i = \overline{0, m})$ are polynomials depending on the coefficients of system (9)*, then the* ***leading coefficient*** *A_0 is called an $SO(2,\mathbb{R})$****-semi-invariant****.*

Remark 3.1. [11, § 7.3]. *If the $SO(2,\mathbb{R})$-comitants $K_1, K_2, \dots, K_r$ are functionally independent, then their semi-invariants can also be functionally independent.*

From the works [9, 10], it follows that

Theorem 3.2. *For a polynomial $K(x, y, a)$ to be a comitant of system* (9) *under the rotation group* (10)*, i.e., to satisfy the equality* (15)*, it is necessary and sufficient that it satisfies the equation*

$$X(K) = 0, \tag{17}$$

where X is the Lie operator from (13)–(14).

From Theorem 3.2, we obviously have

Corollary 3.2. *For a polynomial $I(a)$ to be an invariant of system* (9) *under the rotation group* (10)*, it is necessary and sufficient that it satisfies the equality*

$$D(I) = 0, \tag{18}$$

where D is the Lie operator (14).

Note 3.2. *From* (17) ((18)) *it follows that the* $SO(2,\mathbb{R})$*-comitants (*$SO(2,\mathbb{R})$*-invariants) of system* (9) *are solutions to a first-order linear homogeneous partial differential equation with the Lie operator* (13) ((14))*. Note that the number of variables involved in the operator* (13) ((14)) *is equal to* $N+2$ *(*N*), where* N *is given by* (12).

From the general theory of equations of the type (17) ((18)) (see, for example, [8]), it is known that *the maximum number of functionally independent solutions* is equal to $N+1$ ($N-1$), where N is given by (12).

Since in this case the solutions to equation (17) ((18)) are $SO(2,\mathbb{R})$-comitants ($SO(2,\mathbb{R})$-invariants) ofthe system (9), we obtain that the following holds:

Lemma 3.1. *The maximum number of* $SO(2,\mathbb{R})$*-comitants (*$SO(2,\mathbb{R})$*-invariants) of the system* (9) *contained in a functional basis of the set of* $SO(2,\mathbb{R})$*-comitants (*$SO(2,\mathbb{R})$*-invariants) of this system is equal to* $N+1$ *(*$N-1$*), where* N *is given by* (12).

4. On the $SO(2,\mathbb{R})$-invariance of focus quantities in the center-focus problem for the system (9)

It is known (see, for example, [11]) that the center-focus problem in its classical formulation is set as follows: *for an infinite system of polynomials*

$$\{(x^2+y^2)^k\}_{k=1}^{\infty}$$

there exists a function

$$V(x,y)=x^2+y^2+\sum_{k=3}^{\infty} f_k(x,y), \tag{19}$$

where $f_k(x,y)$ *are homogeneous polynomials of degree* k *with respect to the variables* x, y*, and constants (focus quantities)*

$$L_1, L_2, \dots, L_k, \dots, \tag{20}$$

such that the identity

$$\frac{dV}{dt}=\sum_{k=1}^{\infty} L_k(x^2+y^2)^{k+1} \tag{21}$$

(with respect to x *and* y*) holds along the trajectories of the system* (9).

Note 4.1. *The center-focus problem can be reformulated as follows:* ***What are the conditions for the system*** **(9)** ***that distinguish a center from a focus?*** *This problem was posed by H. Poincaré* [7]*. Fundamental results were obtained by A. Lyapunov* [2]*, who showed that the conditions for a center (the origin* $O(0,0)$ *of the system* (9) *is surrounded by closed trajectories) consist in the vanishing of an infinite sequence of polynomials* *(focus quantities)* (20) *in the coefficients of the right-hand side of system* (9). *If at least one of the quantities* (20) *is non-zero, then the origin for the system* (9) *is a focus, i.e., it is surrounded by spirals. These conditions are necessary and sufficient. It should be noted that in* [11], *a significant result in solving the center-focus problem was obtained.*

In [12, §16], and subsequently in [6], it is shown that the following lemma holds:

Lemma 4.1. *The focus quantities* (20) *for the system* (9) *are invariant with respect to the rotation group* $SO(2,\mathbb{R})$ *given by formulas* (10).

5. $SO(2,\mathbb{R})$-Invariant Focus Quantities for a Differential System with Quadratic Nonlinearities of the Form (9)

Consider the special case of the system (9) for $l = 1$ and $m_1 = 2$, which we write in the form

$$\begin{aligned} \frac{dx}{dt} &= y + gx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= -x + lx^2 + 2mxy + ny^2, \end{aligned} \tag{22}$$

where

$$g = \overset{1}{a}{}^1_0, \quad h = \overset{1}{a}{}^1_1, \quad k = \overset{1}{a}{}^1_2, \quad l = \overset{1}{a}{}^2_0, \quad m = \overset{1}{a}{}^2_1, \quad n = \overset{1}{a}{}^2_2. \tag{23}$$

Taking into account (23), for the Lie operators (13)–(14) of the system (22) we have

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + D, \tag{24}$$

where

$$\begin{aligned} D = (2h + l)\frac{\partial}{\partial g} + (-g + k + m)\frac{\partial}{\partial h} + (2h + n)\frac{\partial}{\partial k} + (-g + 2m)\frac{\partial}{\partial l} + \\ +(-h - l + n)\frac{\partial}{\partial m} - (k + 2m)\frac{\partial}{\partial n}. \end{aligned} \tag{25}$$

Note 5.1. [11, § 7.4]. *The focus quantities (20) for the system (9) are not uniquely constructed, since equation (21) with the function (19) contains arbitrary constants that can take different values.*

Following notes 4.1, 5.1 and according to [11, §7.4], we state the following result:

Lemma 5.1. *The first three focus quantities of the sequence* (20) *solve the center-focus problem for the system* (22), *i.e., when they vanish, they annihilate all focus quantities of this sequence for the system* (22), *provided they belong to the same set:*

$$a)\; L_1 = L_1 S,\; L_2 = L_2 S,\; L_3 = L_3 S, \tag{26}$$

$$b)\ L_1 = L_1C = L_1S,\ L_2 = L_2C,\ L_3 = L_3C, \tag{27}$$

where

$$L_1S = \frac{1}{2}(-gh - hk + gl + lm - kn + mn),$$

$$\begin{gathered}
L_2S = \frac{1}{24}(62g^3h - 2gh^3 + 119g^2hk - 2h^3k + 62ghk^2 + 5hk^3- \\
-62g^3l + 27gh^2l - 63g^2kl + 29h^2kl - 15gk^2l - 8ghl^2 + 15hkl^2- \\
-5gl^3 + 101g^2hm + 138ghkm + 37hk^2m - 175g^2lm - 6h^2lm- \\
-116gklm - 15k^2lm - 13hl^2m - 5l^3m + 54ghm^2 + 54hkm^2- \\
-159glm^2 - 53klm^2 - 46lm^3 + 6g^3n + 37gh^2n + 72g^2kn + 39h^2kn+ \\
+57gk^2n + 5k^3n + 14ghln + 68hkln - 33gl^2n + 15kl^2n - 72g^2mn- \\
-6h^2mn + 34gkmn + 32k^2mn - 42hlmn - 38l^2mn - 109ghm^2n- \\
-3km^2n - 46m^3n + 48ghn^2 + 79hkn^2 - 48gln^2 + 39kln^2- \\
-29hmn^2 - 71lmn^2 - 6gn^3 + 38kn^3 - 38mn^3),
\end{gathered}$$

$$\begin{gathered}
L_3S = \frac{1}{2304}(-44725g^5h + 11142g^3h^3 - 88gh^5 - 124537g^4hk+ \\
+30842g^2h^3k - 88h^5k - 121728g^3hk^2 + 27186gh^3k^2 - 45492g^2hk^3+ \\
+7486h^3k^3 - 2651ghk^4 + 925hk^5 + 44725g^5l - 51066g^3h^2l- \\
-1216gh^4l + 84372g^4kl - 95044g^2h^2kl - 1704h^4kl + 53320g^3k^2l- \\
-44602gh^2k^2l + 10096g^2k^3l - 2880h^2k^3l - 465gk^4l + 28368g^3hl^2- \\
-3362gh^3l^2 + 7708g^2hkl^2 - 5802h^3kl^2 - 13582ghk^2l^2 - 3650hk^3l^2+ \\
+2436g^3l^3 + 1858gh^2l^3 + 9332g^2kl^3 - 1700h^2kl^3 + 3650gk^2l^3- \\
-875ghl^4 + 465hkl^4 - 925gl^5 - 157320g^4hm + 7528g^2h^3m- \\
-356424g^3hkm + 19904gh^3km - 264096g^2hk^2m + 12376h^3k^2m- \\
-67288ghk^3m - 2296hk^4m + 211165g^4lm - 72038g^2h^2lm+ \\
+888h^4lm + 310816g^3klm - 109100gh^2klm + 144112g^2k^2lm- \\
-26630h^2k^2lm + 17812gk^3lm - 465k^4lm + 67344g^2hl^2m+ \\
+2776h^3l^2m + 12024ghkl^2m - 7552hk^2l^2m + 14808g^2l^3m+ \\
+978h^2l^3m + 17844gkl^3m + 3650k^2l^3m - 1800hl^4m - 925l^5m- \\
-214106g^3hm^2 - 4560gh^3m^2 - 214106g^2hkm^2 - 4560h^3km^2- \\
-29106ghk^2m^2 - 29106hk^3m^2 + 389610g^3lm^2 - 16864gh^2lm^2+
\end{gathered}$$

$$+249820g^2klm^2-32896h^2klm^2+131634gk^2lm^2+7716k^3lm^2+$$
$$+60798ghl^2m^2+9126hkl^2m^2+15478gl^3m^2+8512kl^3m^2-$$
$$-131528g^2hm^3-188448ghkm^3-56920hk^2m^3+350410g^2lm^3+$$
$$+7632h^2lm^3+83832gklm^3+40842k^2lm^3+15912hl^2m^3+$$
$$+3106l^3m^3-29432ghm^4-29432hkm^4+152800glm^4+$$
$$+60456klm^4+25560lm^5-4560g^5n-49528g^3h^2n-$$
$$-2168gh^4n-56129g^4kn-71382g^2h^2kn-2656h^4kn-$$
$$-86332g^3k^2n-8356gh^2k^2n-42376g^2k^3n+11242h^2k^3n-$$
$$-3576gk^4n+925k^5n-8288g^3hln-21396gh^3ln-$$
$$-107608g^2hkln-28148h^3kln-78028ghk^2ln-6580hk^3ln+$$
$$+41832g^3l^2n-3496gh^2l^2n+29628g^2kl^2n-28690h^2kl^2n-$$
$$-7552gk^2l^2n-3650k^3l^2n+7132ghl^3n-5780hkl^3n-$$
$$-520gl^4n+465kl^4n+44753g^4mn-87182g^2h^2mn+$$
$$+888h^4mn-78864g^3kmn-108972gh^2kmn-155168g^2k^2mn-$$
$$-11358h^2k^2mn-60956gk^3mn-3221k^4mn+51096g^2hlmn+$$
$$+9632h^3lmn-124544ghklmn-50104hk^2lmn+144264g^2l^2mn+$$
$$+16334h^2l^2mn+63884gkl^2mn-1522k^2l^2mn+5272hl^3mn-$$
$$-1445l^4mn+173116g^3m^2n-43264gh^2m^2n+17830g^2km^2n-$$
$$-59296h^2km^2n+82532gk^2m^2n-25890k^3m^2n+121900ghlm^2n-$$
$$-26836hklm^2n+146916gl^2m^2n+39066kl^2m^2n+222962g^2m^3n+$$
$$+7632h^2m^3n-103272gkm^3n-18814k^2m^3n+53184hlm^3n+$$
$$+38574l^2m^3n+125304gm^4n+32960km^4n+25560m^5n-$$
$$-56942g^3hn^2-23378gh^3n^2-141270g^2hkn^2-27690h^3kn^2-$$
$$-60328ghk^2n^2+6856hk^3n^2+48734g^3ln^2-38598gh^2ln^2-$$
$$-11808g^2kln^2-81368h^2kln^2-38236gk^2ln^2-3700k^3ln^2+$$
$$+22272ghl^2n^2-43344hkl^2n^2+12584gl^3n^2-4080kl^3n^2-$$
$$-57664g^2hmn^2+6856h^3mn^2-184664ghkmn^2-$$
$$-49232hk^2mn^2+202966g^2lmn^2+36790h^2lmn^2+$$
$$+13848gklmn^2-28284k^2lmn^2+43488hl^2mn^2+$$

$$+11604l^3mn^2+28654ghm^2n^2-68410hkm^2n^2+$$
$$+243030glm^2n^2+34596klm^2n^2+37272hm^3n^2+75942lm^3n^2-$$
$$-288g^3n^3-43772gh^2n^3-48542g^2kn^3-64906h^2kn^3-$$
$$-30696gk^2n^3+3100k^3n^3+4176ghln^3-85408hkln^3+$$
$$+31676gl^2n^3-20456kl^2n^3+55598g^2mn^3+21434h^2mn^3-$$
$$-58400gkmn^3-31408k^2mn^3+69384hlmn^3+39200l^2mn^3+$$
$$+100760gm^2n^3-6790km^2n^3+40474m^3n^3-18481ghn^4-$$
$$-56701hkn^4+25249gln^4-30484kln^4+32968hmn^4+$$
$$+43993lmn^4+3408gn^5-16917kn^5+16917mn^5),$$

$$L_2C=\frac{1}{24}(53g^3h+16gh^3+95g^2hk+16h^3k+47ghk^2+5hk^3-$$
$$-53g^3l+18gh^2l-48g^2kl+38h^2kl-15gk^2l-17ghl^2+15hkl^2-$$
$$-5gl^3+62g^2hm+84ghkm+22hk^2m-127g^2lm-24h^2lm-86gklm-$$
$$-15k^2lm-22hl^2m-5l^3m+24ghm^2+24hkm^2-90glm^2-38klm^2-$$
$$-16lm^3+6g^3n+70gh^2n+63g^2kn+90h^2kn+42gk^2n+5k^3n-$$
$$-10ghln+86hkln-42gl^2n+15kl^2n-63g^2mn-24h^2mn+10gkmn+$$
$$+17k^2mn-84hlmn-47l^2mn-70gm^2n-18km^2n-16m^3n+63ghn^2+$$
$$+127hkn^2-63gln^2+48kln^2-62hmn^2-95lmn^2-6gn^3+53kn^3-53mn^3)$$

$$L_3C=\frac{1}{2304}(-31393g^5h-17022g^3h^3-1144gh^5-77985g^4hk-$$
$$-27330g^2h^3k-1144h^5k-67264g^3hk^2-8666gh^3k^2-21292g^2hk^3+$$
$$+1642h^3k^3+305ghk^4+925hk^5+31393g^5l-27138g^3h^2l-10960gh^4l+$$
$$+50720g^4kl-70216g^2h^2kl-13080h^4kl+30636g^3k^2l-41898gh^2k^2l+$$
$$+5940g^2k^3l-5844h^2k^3l-465gk^4l+35076g^3hl^2-1706gh^3l^2+$$
$$+16672g^2hkl^2-14946h^3kl^2-6030ghk^2l^2-3650hk^3l^2+828g^3l^3+$$
$$+7270gh^2l^3+6664g^2kl^3-3560h^2kl^3+3650gk^2l^3+265ghl^4+465hkl^4-$$
$$-925gl^5-79756g^4hm-30792g^2h^3m-164120g^3hkm-37792gh^3km-$$
$$-106536g^2hk^2m-7000h^3k^2m-21512ghk^3m+660hk^4m+119405g^4lm+$$
$$+4582g^2h^2lm+3096h^4lm+157960g^3klm-48148gh^2klm+$$

$$+72848g^2k^2lm - 19738h^2k^2lm + 9500gk^3lm - 465k^4lm + 74424g^2hl^2m+$$
$$+12424h^3l^2m + 31600ghkl^2m + 6844g^2l^3m + 7530h^2l^3m + 12508gkl^3m+$$
$$+3650k^2l^3m - 660hl^4m - 925l^5m - 74114g^3hm^2 - 15376gh^3m^2-$$
$$-127550g^2hkm^2 - 15376h^3km^2 - 60966ghk^2m^2 - 7530hk^3m^2+$$
$$+168022g^3lm^2 + 41568gh^2lm^2 + 177140g^2klm^2 + 57158gk^2lm^2+$$
$$+3560k^3lm^2 + 52258ghl^2m^2 + 19738hkl^2m^2 + 4374gl^3m^2 + 5844kl^3m^2-$$
$$-29432g^2hm^3 - 41856ghkm^3 - 12424hk^2m^3 + 104770g^2lm^3+$$
$$+15376h^2lm^3 + 82980gklm^3 + 14946k^2lm^3 + 7000hl^2m^3 - 1642l^3m^3-$$
$$-3096ghm^4 - 3096hkm^4 + 25904glm^4 + 13080klm^4 + 1144lm^5-$$
$$-4128g^5n - 89564g^3h^2n - 23784gh^4n - 41357g^4kn - 183390g^2h^2kn-$$
$$-25904h^4kn - 51912g^3k^2n - 91176gh^2k^2n - 21132g^2k^3n - 4374h^2k^3n-$$
$$-620gk^4n + 925k^5n + 30248g^3hln - 43620gh^3ln - 64504g^2hkln-$$
$$-82980h^3kln - 67900ghk^2ln - 12508hk^3ln + 43332g^3l^2n + 17324gh^2l^2n+$$
$$+30812g^2kl^2n - 57158h^2kl^2n - 3650gk^2l^2n + 21092ghl^3n - 9500hkl^3n+$$
$$+620gl^4n + 465kl^4n + 32573g^4mn - 89970g^2h^2mn + 3096h^4mn-$$
$$-32600g^3kmn - 175220gh^2kmn - 61672g^2k^2mn - 52258h^2k^2mn-$$
$$-21092gk^3mn - 265k^4mn + 162352g^2hlmn + 41856h^3lmn-$$
$$-31600hk^2lmn + 119932g^2l^2mn + 60966h^2l^2mn + 67900gkl^2mn+$$
$$+6030k^2l^2mn + 21512hl^3mn - 305l^4mn + 93296g^3m^2n + 38762g^2km^2n-$$
$$-41568h^2km^2n - 17324gk^2m^2n - 7270k^3m^2n + 175220ghlm^2n+$$
$$+48148hklm^2n + 91176gl^2m^2n + 41898kl^2m^2n + 78650g^2m^3n+$$
$$+15376h^2m^3n + 43620gkm^3n + 1706k^2m^3n + 37792hlm^3n+$$
$$+8666l^2m^3n + 23784gm^4n + 10960km^4n + 1144m^5n-$$
$$-68550g^3hn^2 - 78650gh^3n^2 - 204974g^2hkn^2 - 104770h^3kn^2-$$
$$-119932ghk^2n^2 - 6844hk^3n^2 + 59766g^3ln^2 - 38762gh^2ln^2-$$
$$-177140g^2kln^2 - 30812gk^2ln^2 - 6664k^3ln^2 + 61672ghl^2n^2-$$
$$-72848hkl^2n^2 + 21132gl^3n^2 - 5940kl^3n^2 + 29432h^3mn^2-$$

$$
\begin{gathered}
-162352ghkmn^2 - 74424hk^2mn^2 + 204974g^2lmn^2+\\
+127550h^2lmn^2 + 64504gklmn^2 - 16672k^2lmn^2+\\
+106536hl^2mn^2 + 21292l^3mn^2 + 89970ghm^2n^2-\\
-4582hkm^2n^2 + 183390glm^2n^2 + 70216klm^2n^2+\\
+30792hm^3n^2 + 27330lm^3n^2 - 93296gh^2n^3-\\
+59766g^2kn^3 - 168022h^2kn^3 - 43332gk^2n^3 - 828k^3n^3+\\
+32600ghln^3 - 157960hkln^3 + 51912gl^2n^3 - 30636kl^2n^3+\\
+68550g^2mn^3 + 74114h^2mn^3 - 30248gkmn^3 - 35076k^2mn^3+\\
+164120hlmn^3 + 67264l^2mn^3 + 89564gm^2n^3 + 27138km^2n^3+\\
+17022m^3n^3 - 32573ghn^4 - 119405hkn^4 + 41357gln^4-\\
-50720kln^4 + 79756hmn^4 + 77985lmn^4 + 4128gn^5-\\
-31393kn^5 + 31393mn^5).
\end{gathered} \tag{28}
$$

Note 5.2. [11, § 7.4]**.** *For the focus quantities of the set* (26) *with expressions* (28), *using the Lie operator* (25) *we find*

$$D(L_1S) = 0,\ D(L_2S) \neq 0,\ D(L_3S) \neq 0,$$

which implies that L_1S is an $SO(2,\mathbb{R})$-invariant of the system (22), *whereas L_2S and L_3S are not.*

Taking into account the expressions (28) for L_2S and L_3S in [11, §7.4] and using the operator D from (25), the following corresponding $SO(2,\mathbb{R})$-comitants of the system (22) are constructed:

$$K(L_2S) = L_2Sx^4 + A_1x^3y + A_2x^2y^2 + A_3xy^3 + A_4y^4, \tag{29}$$

where

$$
\begin{gathered}
A_1 = D(L_2S), \quad A_2 = \frac{1}{2}\Big[D(A_1) + 4L_2S\Big], \quad A_3 = \frac{1}{3}\Big[D(A_2) + 3A_1\Big],\\
A_4 = \frac{1}{4}\Big[D(A_3) + 2A_2\Big],
\end{gathered} \tag{30}
$$

and

$$
\begin{gathered}
K(L_3S) = L_3Sx^{12} + A_1x^{11}y + A_2x^{10}y^2 + A_3x^9y^3 + A_4x^8y^4 + A_5x^7y^5+\\
+A_6x^6y^6 + A_7x^5y^7 + A_8x^4y^8 + A_9x^3y^9 + A_{10}x^2y^{10} + A_{11}xy^{11} + A_{12}y^{12},
\end{gathered} \tag{31}
$$

under

$$
\begin{gathered}
A_1 = D(L_3S), \quad A_2 = \frac{1}{2}\Big[D(A_1) + 12L_3S\Big], \quad A_3 = \frac{1}{3}\Big[D(A_2) + 11A_1\Big],\\
A_4 = \frac{1}{4}\Big[D(A_3) + 10A_2\Big],\ A_5 = \frac{1}{5}\Big[D(A_4) + 9A_3\Big],\ A_6 = \frac{1}{6}\Big[D(A_5) + 8A_4\Big],\\
A_7 = \frac{1}{7}\Big[D(A_6) + 7A_5\Big],\ A_8 = \frac{1}{8}\Big[D(A_7) + 6A_6\Big],\ A_9 = \frac{1}{9}\Big[D(A_8) + 5A_7\Big],\\
A_{10} = \frac{1}{10}\Big[D(A_9) + 4A_8\Big],\ A_{11} = \frac{1}{11}\Big[D(A_{10}) + 3A_9\Big],\\
A_{12} = \frac{1}{12}\Big[D(A_{11}) + 2A_{10}\Big].
\end{gathered}
\tag{32}
$$

In [11, §7.4]), it is shown that using the Lie operator (24)–(25) and the expressions (26), (28)–(32), we have

$$X[K(L_2S)] = X[K(L_3S)] = 0.$$

According to the Theorem 3.2 and these equalities, it follows that the following holds:

Lemma 5.2. [11, § 7.4]. *The focus quantity L_1S is an $SO(2,\mathbb{R})$-invariant, while the focus quantities L_2S and L_3S are $SO(2,\mathbb{R})$-semi-invariants in the comitants from* (29) *and* (31)*, respectively.*

Using the Lie operator (25) and the expressions of the focus quantities of the set (27) from (28), we find

$$D(L_1C) = D(L_2C) = D(L_3C) = 0.$$

According to the Corollary 3.2, we obtain that the following holds:

Lemma 5.3. [11, § 7.4]. *The focus quantities L_iC ($i = \overline{1,3}$) from* (28) *are $SO(2,\mathbb{R})$-invariants of system* (22).

6. $SO(2,\mathbb{R})$-invariant stability conditions of the unperturbed motion governed by the system (22)

Theorem 6.1. *The stability of the unperturbed motion governed by the differential system* (22) *with the set of focus quantities* (26) *is exhausted by the following seven possible cases:*

I. $L_1S > 0$, *the unperturbed motion is unstable;*

II. $L_1S < 0$, *the unperturbed motion is asymptotically stable;*

III. $L_1S = 0,\ K(L_2S)\big|_{y=0} > 0$, *the unperturbed motion is unstable;*

IV. $L_1S = 0,\ K(L_2S)\big|_{y=0} < 0$*, the unperturbed motion is asymptotically stable;*

V. $L_1S = K(L_2S)\big|_{y=0} = 0,\ K(L_3S)\big|_{y=0} > 0$*, the unperturbed motion is unstable;*

VI. $L_1S = K(L_2S)\big|_{y=0} = 0,\ K(L_3S)\big|_{y=0} < 0$*, the unperturbed motion is asymptotically stable;*

VII. $L_1S = K(L_2S)\big|_{y=0} = K(L_3S)\big|_{y=0} = 0$*, the unperturbed motion is stable.*

These conditions are $SO(2,\mathbb{R})$-invariant, where L_1S from (28) *is an $SO(2,\mathbb{R})$-invariant, and L_2S and L_3S are $SO(2,\mathbb{R})$-semi-invariants in the $SO(2,\mathbb{R})$-comitants* (29) *and* (31).

The **proof** is obtained using the identity (21) (with respect to x and y), written for the set (26) in the form

$$\frac{dV}{dt} = L_1S(x^2 + y^2) + L_2S(x^2 + y^2)^3 + L_3S(x^2 + y^2)^4 + \cdots \tag{33}$$

along the trajectories of the system (22), where V is the Lyapunov function (19). According to the Example 2.2, for a sufficiently small positive h from (3), we have that V from (19) is positive definite, and the sign of the right-hand side of the identity (33) for $L_1S \neq 0$ coincides with the sign of this quantity. Taking into account the Theorem C, for $L_1S > 0$, it follows that the unperturbed motion governed by system (22) is unstable in this case. For $L_1S < 0$, according to Theorem B, the specified unperturbed motion is asymptotically stable. Analogously, the validity of the cases III–VI is proven, taking into account that $L_2S = K(L_2S)\big|_{y=0}$ and $L_3S = K(L_3S)\big|_{y=0}$ are $SO(2,\mathbb{R})$-semi-invariants in the comitants (29)–(32).

In the case VII, applying the Lemma 3 for $L_1S = 0$, $L_2S = K(L_2S)\big|_{y=0} = 0$, $L_3S = K(L_3S)\big|_{y=0} = 0$ and the Theorem A, we find that the unperturbed motion is stable. Theorem 6.1 is proven.

Analogously, we prove the following:

Theorem 6.2. *The stability of the unperturbed motion governed by the differential system* (22) *with the set of focus quantities* (27) *is exhausted by the following seven possible cases:*

I. $L_1C > 0$*, the unperturbed motion is unstable;*

II. $L_1C < 0$*, the unperturbed motion is asymptotically stable;*

III. $L_1C = 0, L_2C > 0$*, the unperturbed motion is unstable;*

IV. $L_1C = 0, L_2C < 0$*, the unperturbed motion is asymptotically stable;*

V. $L_1C = L_2C = 0, L_3C > 0$*, the unperturbed motion is unstable;*

VI. $L_1C = L_2C = 0, L_3C < 0$*, the unperturbed motion is asymptotically stable;*

VII. $L_1C = L_2C = L_3C = 0$*, the unperturbed motion is stable.*

These conditions are $SO(2,\mathbb{R})$-invariant and are expressed through the $SO(2,\mathbb{R})$-invariants from (28).

Question 1. If the sequence of focus quantities

$$L_1, L_2, \ldots, L_k, \ldots$$

of the system (22) contains more than one $SO(2,\mathbb{R})$-semi-invariant ($SO(2,\mathbb{R})$-invariant) of this system, will the remaining elements of this sequence be $SO(2,\mathbb{R})$-semi-invariants ($SO(2,\mathbb{R})$-invariants) of the specified system?

Such a question can be formulated regarding the sequences of focus quantities (20) concerning the general system (9).

7. On the number of functionally independent $SO(2,\mathbb{R})$-invariant focus quantities of system (9) solving the stability problem of unperturbed motion in the case of a center and a focus

Since, as shown by the example in the Section 6, the focus quantities (20) of the system (9) can be both $SO(2,\mathbb{R})$-semi-invariants (see definition 3.4) and $SO(2,\mathbb{R})$-invariants, using the lemmas 3.1, 4.1 and remark 3.1, we obtain that the following holds:

Theorem 7.1. *The maximum number of functionally independent $SO(2,\mathbb{R})$-invariant focus quantities* (20) *of the system* (9) *contained in a functional basis of the set of $SO(2,\mathbb{R})$-invariant elements of this system does not exceed $N+1$, or cannot be greater than $N-1$, where N is given by* (12).

Corollary 7.1. *The maximum number of functionally independent $SO(2,\mathbb{R})$-invariant focus quantities of the system* (22) *contained in a functional basis of the set of $SO(2,\mathbb{R})$-invariant elements of this system does not exceed 7 or cannot be greater than 5.*

By constructing the Jacobi matrix on the $SO(2,\mathbb{R})$-semi-invariants L_iS ($i=\overline{1,3}$) from (28) – or, equivalently, on the $SO(2,\mathbb{R})$-comitants $L_1S, K(L_2S), K(L_3S)$ from (28)–(32) – and separately on the invariants L_iC ($i=\overline{1,3}$), we find that its rank in each case is equal to 3. Consequently, according to the Theorems 6.1-6.2, the number of functionally independent $SO(2,\mathbb{R})$-semi-invariants ($SO(2,\mathbb{R})$-comitants), or separately $SO(2,\mathbb{R})$-invariants, solving of the stability problem of unperturbed motion in the case of a center or a focus for system (22) is equal to 3, which does not exceed 5, and moreover, 7 (see corollary 7.1).

Since it naturally follows that the focus quantities solving the center-focus problem and participating in the solution of the stability problem of unperturbed motion governed by the system (9) must be functionally independent, according to the lemma 3.1, it can be reasonably assumed that the following holds:

Theorem 7.2. *The upper bound on the number of functionally independent $SO(2,\mathbb{R})$-semi-invariants ($SO(2,\mathbb{R})$-invariants) participating in the solution of the stability problem of unperturbed motion governed solving by the system* (9) *does not exceed $N+1$ $(N-1)$, where N is given by* (12).

8. Comments on the theorem 7.2

In the Sections 6–7, it is shown that the solution of the stability problem of unperturbed motion governed by system (9) using $SO(2,\mathbb{R})$-invariant elements is closely intertwined with the problem of the number of functionally independent focus quantities participating in the solution of the center-focus problem for the specified system. This conclusion was reached through the investigation of the stability of unperturbed motion governed by the system (22). However, it is known [13] that for a differential system of the type (9) with cubic nonlinearities, the maximum number of focus quantities solving the center-focus problem for this system does not exceed 5. At the same time, according to the lemma 3.1, the number of elements in the functional basis of $SO(2,\mathbb{R})$-semi-invariants ($SO(2,\mathbb{R})$-invariants) is equal to 9 (7), which exceeds the number 5 (of the specified focus quantities). There exist other special examples confirming the validity of the theorem 7.2.

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