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The ordinary Hilbert series for Sibirsky graded algebras of the differential system $s(5,7)$

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Abstract. In this paper are examined the construction of ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants in autonomous polynomial differential systems. These series are highly significant for addressing specific problems within the qualitative theory of differential systems. However, constructing them for more complex systems typically presents insurmountable computational difficulties, particularly for generalized Hilbert series, from which ordinary Hilbert series are derived. To overcome these massive calculation barriers, this study shows how adapting Molien's formula can effectively solve the problem. Using this approach, the ordinary Hilbert series for the Sibirsky graded algebras of comitants and invariants was obtained for the differential system $s(5,7)$.

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Seriile Hilbert obişnuite pentru algebrele graduate Sibirschi ale sistemului diferenţial $s(5,7)$

Rezumat. În această lucrare se examinează construcția seriilor Hilbert obişnuite pentru algebrele graduate Sibirschi ale comitanţilor şi invarianţilor în sisteme diferenţiale polinomiale autonome. Aceste serii sunt extrem de importante pentru abordarea unor probleme specifice din cadrul teoriei calitative a sistemelor diferenţiale. Cu toate acestea, construirea lor pentru sisteme mai complexe prezintă de obicei dificultăţi de calcul enorme, în special pentru seriile Hilbert generalizate, din care se obţin seriile Hilbert obişnuite. Pentru a depăşi aceste bariere masive de calcul, acest studiu arată cum adaptarea formulei lui Molien poate rezolva eficient problema. Folosind această abordare, seriile Hilbert obişnuite pentru algebrele graduate Sibirschi a comitanţilor şi invarianţilor au fost obţinute pentru sistemul diferenţial $s(5,7)$.

Cuvinte-cheie: sisteme diferenţiale, algebre graduate Sibirschi, serii Hilbert, dimensiunea Krull.

1. Introduction

Consider a general two-dimensional autonomous polynomial system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = P_5(x,y) + P_7(x,y), \\ \dfrac{dy}{dt} = Q_5(x,y) + Q_7(x,y), \end{cases} \tag{1}$$

where $P_5(x,y)$ $(P_7(x,y))$ and $Q_5(x,y)$ $(Q_7(x,y))$ are homogeneous polynomials of degree 5 (7) in real variables x and y with real coefficients.

Hereafter for the system (1) we will use the notation $s(5,7)$ where $(5,7)$ are degree of homogeneities of the polynomials on the right side with respect to the phase variables x and y.

One of the method to study the differential systems of the form (1) is "*The method of algebraic invariants in the theory of differential equations*", which is developed in the works of Academician K. S. Sibirsky [1], [2], [3] and his disciples. This method generated applications of Lie groups and algebras, graded algebras of invariants and comitants, generating functions and Hilbert series to the study of the system (1) (see, for example [4], [5]).

While Sylvester's generalized method [4] is a standard technique for evaluating both generalized and ordinary Hilbert series for differential systems, its application to systems with high-degree polynomial right-hand sides requires immense computational power, often demanding supercomputers. Conversely, the residue method offers a more efficient alternative and has yielded ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants across various differential systems [6, 7, 8, 9, 12]. Building on these successes, extending this approach to derive an ordinary Hilbert series for additional unexamined system such as the $s(5,7)$ is a valuable and highly desirable direction.

2. Computational methods for ordinary and generalized Hilbert series of Sibirsky graded algebras

In [4] it is shown that the set of centro-affine comitants generate a finite-determined graded algebra of comitants (invariants) with respect to the unimodular group $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$. This algebras in [5] were named Sibirsky graded algebra of comitants (invariants), respectively.

A central challenge in analyzing these algebras lies in identifying the generating set, determining both how many generators are needed and their explicit forms. This requires a systematic investigation of the specific types of comitants and invariants that constitute these generators. As highlighted in Modern Algebra [10], generating functions and Hilbert series serve as crucial tools in tackling this problem.

Computing the generalized Hilbert series for differential systems featuring high-degree polynomial right-hand sides demands considerable computational power, making supercomputing resources indispensable in practice. This highlights the necessity of relying on the ordinary Hilbert series. As established in [4] and [5], analyzing the structure of these Sibirsky graded algebras for the system (1) can be achieved via both generalized and ordinary Hilbert series. Ultimately, the Hilbert series provides crucial information regarding the upper bound on the degrees of the algebra's generators.

The generalized Sylvester's method for computing ordinary and generalized Hilbert series is established in [4]. This approach has yielded the Hilbert series for the Sibirsky graded algebras associated with some simple differential systems [4], [5]. For increasingly complicated differential systems, this approach encounters insurmountable computational barriers, underscores the necessity for alternative methodologies.

The Hilbert series serves as a powerful instrument for computing algebraic invariants, encoding substantial structural information about the underlying ring. For instance, crucial data, including the dimension and other geometric invariants, can be extracted directly from the coefficients and properties of its Hilbert series.

Theorem 2.1. (Molien's formula [11]) *Let G be a finite group acting on a finite dimensional vector space V over a field K of characteristic not dividing $|G|$. Then,*

$$H(K[V]^G, t) = \frac{1}{|G|}\sum_{\sigma\in G}\frac{1}{det_V^0(1-t\sigma)}.$$

If K has characteristic 0, then $det_V^0(1-t\sigma)$ can be taken as $det_V(1-t\sigma)$.

In complex analysis, the Residue Theorem offers an effective method for computing the Hilbert series of invariant rings [11].

Theorem 2.2. (The Residue Theorem [11]) *Suppose that D is a connected, simply connected compact region in $\mathbb{C}$, whose border is ∂D, and $\gamma : [0,1] \to \mathbb{C}$ is a smooth curve such that $\gamma([0,1]) = \partial D$, $\gamma(0) = \gamma(1)$ and circles around D exactly once in counterclockwise direction. Assume that f is a meromorphic function on $\mathbb{C}$ with no poles in ∂D. Then we have*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{a\in D} Res(f,a).$$

There are only finitely many points in the compact region D such that f has non-zero residue there.

Theorem 2.3. [11]

$$H(K[V]^G, t) = \frac{1}{2\pi i}\int\limits_{S^1} \frac{1}{det(I - t_{\rho_V}(z))}\frac{dz}{z}$$

where $S^1 \subset \mathbb{C}$ *is the unit circle* $\{z : |z| = 1\}$.

To compute the ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems, this formula was adapted using the Residue Theorem and the corresponding generating function [4] as follows:

Theorem 2.4. [5]

$$H_{SI_\Gamma}(t) = \frac{1}{2\pi i}\int\limits_{S^1} \frac{\varphi_\Gamma^{(0)}(z)}{z}dz$$

where $S^1 \subset \mathbb{C}$ *is the unit circle* $\{z : |z| = 1\}$, $\varphi_\Gamma^{(0)}(z)$ *is the corresponding generating function* [4].

3. Computation of the ordinary Hilbert series for Sibirsky graded algebras of differential system $s(5,7)$

Using Theorem 2.4, we obtain

Theorem 3.1. *For the differential system having homogeneities of the fifth and seventh degrees* $s(5,7)$*, the ordinary Hilbert series for the Sibirsky graded algebras of comitants* $H_{S_{5,7}}$ *and invariants* $H_{SI_{5,7}}$ *were determined*

$$H_{S_{5,7}}(t) = \frac{1}{(1-t)^7(1+t)^{13}(1+t^2)^5(1-t^5)^7(1-t^3)^8(1-t^7)^4(1-t^9)}(1+$$
$$+6t + 18t^2 + 42t^3 + 151t^4 + 779t^5 + 3749t^6 + 15671t^7 + 58716t^8+$$
$$+202287t^9 + 649005t^{10} + 1950772t^{11} + 5517001t^{12} + 14740457t^{13}+$$
$$+37355380t^{14} + 90115809t^{15} + 207604880t^{16} + 458003340t^{17} + 969964807t^{18}+$$
$$+1976267574t^{19} + 3881399657t^{20} + 7361236339t^{21} + 13502867975t^{22}+$$
$$+23990630351t^{23} + 41339698826t^{24} + 69170571861t^{25}+$$
$$+112506237844t^{26} + 178059336344t^{27} + 274460863325t^{28}+$$
$$+412367580387t^{29} + 604375976373t^{30} + 864670341406t^{31}+$$
$$+1208345021692t^{32} + 1650368490795t^{33} + 2204199741711t^{34}+$$
$$+2880122495419t^{35} + 3683423607355t^{36} + 4612599259248t^{37}+$$

$$+5657815206829t^{38} + 6799864449186t^{39} + 8009849118621t^{40}+$$
$$+9249759010353t^{41} + 10474029634184t^{42} + 11632046713335t^{43}+$$
$$+12671437176360t^{44} + 13541867560076t^{45} + 14198979547101t^{46}+$$
$$+14608045390919t^{47} + 14746934237052t^{48} + 14608045390919t^{49}+$$
$$+14198979547101t^{50} + 13541867560076t^{51} + 12671437176360t^{52}+$$
$$+11632046713335t^{53} + 10474029634184t^{54} + 9249759010353t^{55}+$$
$$+8009849118621t^{56} + 6799864449186t^{57} + 5657815206829t^{58}+$$
$$+4612599259248t^{59} + 3683423607355t^{60} + 2880122495419t^{61}+$$
$$+2204199741711t^{62} + 1650368490795t^{63} + 1208345021692t^{64}+$$
$$+864670341406t^{65} + 604375976373t^{66} + 412367580387t^{67}+$$
$$+274460863325t^{68} + 178059336344t^{69} + 112506237844t^{70}+$$
$$+69170571861t^{71} + 41339698826t^{72} + 23990630351t^{73}+$$
$$+13502867975t^{74} + 7361236339t^{75} + 3881399657t^{76}+$$
$$+1976267574t^{77} + 969964807t^{78} + 458003340t^{79} + 207604880t^{80}+$$
$$+90115809t^{81} + 37355380t^{82} + 14740457t^{83} + 5517001t^{84}+$$
$$+1950772t^{85} + 649005t^{86} + 202287t^{87} + 58716t^{88} + 15671t^{89} + 3749t^{90}+$$
$$+779t^{91} + 151t^{92} + 42t^{93} + 18t^{94} + 6t^{95} + t^{96}),$$

$$H_{SI_{5,7}}(t) = \frac{1}{(1-t)^8(1+t)^{13}(1+t^2)^6(1-t^3)^8(1-t^5)^6(1-t^7)^3}(1 + 5t+$$
$$+13t^2 + 29t^3 + 116t^4 + 595t^5 + 2706t^6 + 10635t^7 + 37540t^8 + 121812t^9+$$
$$+367576t^{10} + 1036904t^{11} + 2745806t^{12} + 6853126t^{13} + 16186016t^{14}+$$
$$+36308766t^{15} + 77607412t^{16} + 158496385t^{17} + 310035338t^{18} + 582111660t^{19}+$$
$$+1051077807t^{20} + 1828274311t^{21} + 3068278763t^{22} + 4975025978t^{23}+$$
$$+7803367474t^{24} + 11853300638t^{25} + 17454318136t^{26} + 24938165691t^{27}+$$
$$+34599970227t^{28} + 46649935356t^{29} + 61160481876t^{30} + 78016076271t^{31}+$$
$$+96874664089t^{32} + 117149725965t^{33} + 138020463248t^{34} + 158474105409t^{35}+$$
$$+177379606952t^{36} + 193586406849t^{37} + 206036981612t^{38} + 213878266987t^{39}+$$
$$+216556075990t^{40} + 213878266987t^{41} + 206036981612t^{42} + 193586406849t^{43}+$$
$$+ + 177379606952t^{44} + 158474105409t^{45} + 138020463248t^{46} + 117149725965t^{47}+$$

$$+968746664089t^{48} + 78016076271t^{49} + 61160481876t^{50} + 46649935356t^{51}+$$
$$+34599970227t^{52} + 24938165691t^{53} + 17454318136t^{54} + 11853300638t^{55}+$$
$$+7803367474t^{56} + 4975025978t^{57} + 3068278763t^{58} + 1828274311t^{59}+$$
$$+1051077807t^{60} + 582111660t^{61} + 310035338t^{62} + 158496385t^{63}+$$
$$+77607412t^{64} + 36308766t^{65} + 16186016t^{66} + 6853126t^{67}+$$
$$+2745806t^{68} + 1036904t^{69} + 367576t^{70} + 121812t^{71} + 37540t^{72}+$$
$$+10635t^{73} + 2706t^{74} + 595t^{75} + 116t^{76} + 29t^{77} + 13t^{78} + 5t^{79} + t^{80}).$$

Remark 3.1. The Sibirsky graded algebras $S_{5,7}$ and $SI_{5,7}$ possess Krull dimensions of 27 and 25, respectively.

Remark 3.2. The Krull dimension of the Sibirsky graded algebras $S_{5,7}$ and $SI_{5,7}$ sets the maximal number of algebraically independent comitants and invariants for the differential system $s(5,7)$.

To compute the ordinary Hilbert series of invariant rings in [11] it was introduced a residue-based method that was later applied to Sibirsky graded algebras of comitants and invariants of differential systems. Crucially, this approach offers significantly greater computational efficiency than the generalized Sylvester's method.

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