AVERAGING IN MULTIFREQUENCY SYSTEMS WITH LINEARLY TRANSFORMED ARGUMENTS AND WITH POINT AND INTEGRAL CONDITIONS

Yaroslav BIHUN, Doctor of Science, Professor
Applied Mathematics and Informational Technologies Department
Yuriy Fedkovych Chernivtsi National University, Chernivtsi, Ukraine
Roman PETRYSHYN, Doctor of Science, Professor
Yuriy Fedkovych Chernivtsi National University, Chernivtsi, Ukraine
Inessa KRASNOKUTSKA, PhD, Associate Professor
Applied Mathematics and Informational Technologies Department
Yuriy Fedkovych Chernivtsi National University, Chernivtsi, Ukraine

Abstract. The review of the results of solvability research of multifrequency differential systems with linearly transformed arguments and multipoint and integral conditions is shown in this paper. The oscillation resonance condition, which depends on delay in fast variables, is introduced. The question of existence and uniqueness of the solution is considered, and the justification of averaging method on fast variables is investigated. The finest estimates of averaging method, which obviously depend on small parameter, are obtained.

Keywords: averaging method, multifrequency systems, linearly transformed argument, boundary conditions, Noether problem.
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MEDIEREA ÎN SISTEMELE DE MULTIFRECVENȚĂ CU ARGUMENTE LINIAR TRANSFORMATE ȘI CU PUNCT ȘI CONDIȚIILE DE INTEGRARE

Rezumat. În această lucrare se prezintă o sinteză a rezultatelor ce țin de solvabilitatea sistemelor diferențiale de multifrecvență cu argumente liniar transformate și multipunct și condițiile de integrare. Este introdusă condiția de rezonanță a oscilației, care depinde de întârzierea în variabilele rapide. Se consideră problema de existență și unicitate a soluției și se justifică metoda de mediere pe variabile rapide. Sunt obținute cele mai bune estimări ale metodei de mediere, care, evident, depind de un parametru mic.

Cuvinte-cheie: metoda de mediere, sisteme de multifrecvență, argument liniar transformat, condiții de frontieră, problema Noether.

Introduction

Numerous oscillation processes in mechanics, physics, ecology, etc. are described with multifrequency nonlinear systems in the form [1]

$$\frac{da}{d\tau} = X(\tau, a, \varphi, \varepsilon), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau, a)}{\varepsilon} + Y(\tau, a, \varphi, \varepsilon), \quad 0 \le \tau \le L, \tag{1}$$

where a and φ are $n-$ and $m-$dimensional vectors, respectively, $\tau = \varepsilon t$ is slow time, $0 < \varepsilon$ – small parameter, X, Y and vector of frequency ω belong to certain classes of smooth functions $2\pi-$periodic in φ.

As the system of equation (1) is complex both for research and for solution finding, then, in the times of Lagrange and Laplace, the procedure of averaging over fast variables φ is used. Much simpler system of equation is obtained

$$\frac{d\bar{a}}{d\tau} = X_0(\tau, \bar{a}, \varepsilon), \qquad \frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau, \bar{a})}{\varepsilon} + Y_0(\tau, \bar{a}, \varepsilon), \tag{2}$$

where

$$[X_0, Y_0] = (2\pi)^{-m} \int_0^{2\pi} [X(\tau, \bar{a}, \varphi, \varepsilon), Y(\tau, \bar{a}, \varphi, \varepsilon)] d\varphi. \tag{3}$$

The main problem in investigation of system (1), where $m \geq 2$, is the problem of resonances. Here, the resonance case is understood as the case where the scalar product of the vector $\omega(a, \tau)$ and a nonzero vector with integer-valued coordinates turns into zero or becomes close to zero for certain values of a and τ.

The system (1) could remain in the neighborhood of the resonance quite long, and then deviation of the solutions could be

$$\| a(L, \varepsilon) - \bar{a}(L, \varepsilon) \| = O(1) \quad \text{for} \quad a(0, \varepsilon) = \bar{a}(0, \varepsilon).$$

For two-frequency system ($m = 2$) when $\omega = \omega(a)$ averaging method was justified in the work of V. Arnold [2] and estimate $\| a(t, \varepsilon) - \bar{a}(t, \varepsilon) \| \leq c\sqrt{\varepsilon} \ln^2 \varepsilon$ was obtained for $0 < \varepsilon \leq \varepsilon_0 << 1$ and $0 \leq t \leq L\varepsilon^{-1}$.

Multifrequency systems (1) were investigated by E. Grebenikov [3], M. Khapaev [4], A. Neishtadt [5] and others.

Significant progress in investigation of multifrequency systems is achieved in the works of A. Samoilenko and R. Petryshyn. Such systems both with initial and with multipoint and integral conditions are investigated in [1].

The works of Ya. Bihun [6] and others are devoted to multifrequency systems with constant and variable delay. In particular, systems with integral conditions are investigated in [7]. Multifrequency systems with Noether boundary conditions are investigated by I. Krasnokutska [8]. Some new results for multifrequency systems with many linearly transformed arguments and with multipoint and/or integral conditions are also shown in [9].

Methods and materials used

Oscillation integrals, suggested in [1, 10], are used for averaging method justification. For system (1), when $\omega = \omega(\tau)$ the oscillation integral takes a form

$$I_k(t, \bar{t}, \tau, \varepsilon) = \int_t^{t+\tau} f(y) \exp\left\{ \frac{i}{\varepsilon} \int_{\bar{t}}^{y} (k, \omega(z)) dz \right\} dy, \tag{4}$$

where $\tau \in [0, L]$, $t, \bar{t} \in \mathrm{R}$, $k \in Z^m$, $\|k\| \neq 0$, $(k, \omega) = k_1\omega_1 + ... + k_m\omega_m$.

The proving of existence and uniqueness of the solution is based on the Banach fixed-point theorem [11].

Obtained results and discussion

1. Multifrequency system of ODE

By $W_p(t)$ and $W_p^T(t)$ we denote the matrix $\left(\omega_\nu^{(j-1)}(t)\right)_{\nu,j=1}^{m,p}$ and its transpose, respectively.

Theorem 1 [1]. Let $\left\|(W_p^T(t)W_p(t))^{-1}W_p^T(t)\right\|$ be uniformly bounded and let the functions $\omega_\nu^{(j-1)}(t)$, $\nu=1,...,m$, $j=1,...,p$ be uniformly continuous for $t\in R$. Then one can indicate constants $\varepsilon_1>0$ and $c_1>0$ independent of $k,t,\bar{t},\tau,\varepsilon$ and such that the following estimate holds for all $k\neq 0$, $t\in R$, $\bar{t}\in R$, $\tau\in[0,L]$, and $\varepsilon\in(0,\varepsilon_1]$:

$$\left\|I_k(t,\bar{t},\tau,\varepsilon)\right\|\leq c_1\varepsilon^{1/p}\left[\max_{[t,t+L]}\left\|f(y)\right\|+\frac{1}{\|k\|}\max_{[t,t+L]}\left\|f^{(1)}(y)\right\|\right].$$

Remark 1. If $p=m$ then $\det(W_m^T(t)W_m(t))=(\det W_m(t))^2$. Therefore, in this case, the condition that the Wronskian determinant of the functions $\omega_1(\tau),...\omega_m(\tau)$ is nonzero on $[0,L]$ is a sufficient condition for finding an efficient estimate for the oscillation integral $I_k(\tau,\varepsilon)$.

Let us consider the nonlinear multifrequency system (1), where $\omega=\omega(\tau)$, $\tau\in[0,L]$. Let $\omega\in C^l[0,L]$, $l\geq m+1$, $F:=[X,Y]$, $\frac{\partial F}{\partial\tau},\frac{\partial F}{\partial a}\in C^l(G)$, $\frac{\partial F}{\partial\varphi}\in C^{l+1}(G)$.

Theorem 2 [1]. Let us suppose that the following conditions are satisfied:

1) $\det\left(W_p^T(\tau)W_p(\tau)\right)\neq 0\ \ \forall\tau\in[0,L]$ for certain minimal $m\leq p\leq l+1$;

2) X, Y and ω belong to certain classes of smooth functions;

3) for all $\tau\in[0,L]$, $y\in D_1\subset D$ and $\varepsilon\in(0,\varepsilon_0]$ the curve $\bar{a}=\bar{a}(\tau,y,\varepsilon)$, $\bar{a}(0,y,\varepsilon)=y$, lies in D together with its $\rho-$neighborhood.

Then one can find the constant $c_2>0$ independent on ε and such that, for sufficiently small $\varepsilon_2>0$ and for every $\tau\in[0,L]$, $y\in D_1$, $y\in D_1\subset D$ and $\psi\in R^m$, and $\varepsilon\in(0,\varepsilon_2]$ the following estimate holds:

$$\left\|a(\tau,y,\psi,\varepsilon)-\bar{a}(\tau,y,\varepsilon)\right\|+\left\|\varphi(\tau,y,\psi,\varepsilon)=\bar{\varphi}(\tau,y,\psi,\varepsilon)\right\|\leq c_2\varepsilon^{1/p}, \qquad (5)$$

where $a(0,y,\psi,\varepsilon)=\bar{a}(0,y,\varepsilon)=y$, $\varphi(0,y,\psi,\varepsilon)=\bar{\varphi}(0,y,\psi,\varepsilon)=\psi$.

Theorem 2 is generalized for multifrequency systems with oscillation vector $\omega=\omega(\tau,a)$ and higher approximation systems. Result of theorem 2 is applied for the problem of existence of the solution and justification of averaging method for system (1) with boundary conditions of the form [1]

$$F\left(a\big|_{\tau=0},\varphi\big|_{\tau=0},a\big|_{\tau=L},\varphi\big|_{\tau=L},\varepsilon\right)=0 \qquad (6)$$

and multipoint conditions, boundary-value problems with parameters.

2. Multifrequency Systems of Equations with Linearly Transformed Arguments

Let us suppose that λ_i and θ_j are numbers from semi-interval $(0,1]$,

$$0<\lambda_1<...<\lambda_r\le 1,\ 0<\theta_1<...<\theta_r\le 1,\ a_{\lambda_i}(\tau)=a(\lambda_i\tau),\ \varphi_{\theta_j}(\tau)=\varphi(\theta_j\tau),$$

$$a_\Lambda=(a_{\lambda_1},...,a_{\lambda_r}),\ \varphi_\Theta=(\varphi_{\theta_1},...,\varphi_{\theta_r}).$$

The system of equations is considered

$$\frac{da}{d\tau}=X(\tau,a_\Lambda,\varphi_\Theta),\qquad \frac{d\varphi}{d\tau}=\frac{\omega(\tau)}{\varepsilon}+Y(\tau,a_\Lambda,\varphi_\Theta), \qquad (7)$$

where $a\in D\subset R^n$, $\varphi\in R^m$, $m\ge 1$, $\tau\in[0,L]$, $\varepsilon\in(0,\varepsilon_0]$.

In [12] the problem of existence of the solution of the system of equations (7), which satisfies integral conditions

$$\int_0^L f(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau))d\tau=d_{1,}$$
$$\int_0^L\left[\sum_{j=1}^s b_j(\tau,a_\Lambda(\tau))\varphi_{\theta_j}(\tau)+g(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau))\right]d\tau=d_2, \qquad (8)$$

is solved. Here vector-functions f,g,X and Y are $2\pi-$periodic in variables φ_{θ_j}, $d_1\in R^n$, $d_2\in R^m$.

In the problem (7), (8) both system (7) and vector-functions f and g in conditions (8) are averaged over fast variables. The averaged system takes the form

$$\frac{d\bar a}{d\tau}=X_0(\tau,\bar a_\Lambda),\qquad \frac{d\bar\varphi}{d\tau}=\frac{\omega(\tau)}{\varepsilon}+Y_0(\tau,\bar a_\Lambda),$$
$$\int_0^L f_0(\tau,\bar a_\Lambda(\tau))d\tau=d_{1,}\qquad \int_0^L\left[\sum_{j=1}^s b_j(\tau,\bar a_\Lambda(\tau))\bar\varphi_{\theta_j}(\tau)+g_0(\tau,\bar a_\Lambda(\tau))\right]d\tau=d_2. \qquad (9)$$

The oscillation resonance condition in point τ, which depends on delay in fast variables in contradistinction to condition $(k,\omega)=0$ [13, 14], and takes the form

$$\sum_{j=1}^s\theta_j(k_j,\omega(\theta_j\tau))=0,\qquad k_j\in Z^m,\qquad \sum_{j=1}^s\|k_j\|\ne 0, \qquad (10)$$

is found.

The existence of the solution of the problem (7), (8), is proved and the estimate of error of averaging method for slow variables is obtained

$$\|a(\tau,\bar y+\mu,\bar\psi+\xi,\varepsilon)-\bar a(\tau,\bar y,\varepsilon)\|\le c_3\varepsilon^\alpha,$$

where $0<\alpha\le(ms)^{-1}$, $\bar a(0,\bar y,\varepsilon)=\bar y$, $\|\mu\|\le c_4\varepsilon^\alpha$, $\|\eta\|\le c_5\varepsilon^{\alpha-1}$.

If $b_j = b_j(\tau)$, $j = 1,...,s$, then the solution of the problem (7), (8) exists and is unique.

In the work [9] there is investigated the system of equation (7), when for slow variables (amplitudes) the value

$$a(\tau_0) = a_0, \quad 0 \le \tau_0 \le L,$$

or linear combination of values, is set, and integral conditions have the form

$$\int_{\tau_1}^{\tau_2} \left[\sum_{j=1}^{s} b_j(\tau, a_\Lambda(\tau))\varphi_{\theta_j}(\tau) + g(\tau, a_\Lambda(\tau), \varphi_\Theta(\tau)) \right] d\tau = d_1, \quad 0 \le \tau_1 < \tau_2 \le L.$$

Let us denote

$$S(\tau_1, \tau_2) := \sum_{j=1}^{s} \int_{\tau_1}^{\tau_2} b_j(\tau, \bar{a}_\Lambda(\tau, \bar{y}))d\tau,$$

$$S(\tau_0) := I - \sum_{j=1}^{r} \int_{0}^{\tau_0} \frac{\partial X_0((\tau, \bar{a}_\Lambda(\tau, \bar{y}))}{\partial \bar{a}_{\lambda_j}} \frac{\partial \bar{a}_{\lambda_j}(\tau, \bar{y})}{\partial \bar{y}} d\tau.$$

Theorem 3. Let us suppose that the following conditions are satisfied:

1) vector-fuctions X, Y, ω, f, g and matrix functions b_j belong to certain classes of smooth functions;

2) the Wronskian determinant of ms order of the functions $\{\omega(\theta_1\tau),..., \omega(\theta_s\tau)\}$ is not zero for $\tau \in [0, L]$;

3) the unique solution of averaged problem (9) for slow variables, which lies in D together with its $\rho-$neighborhood, exists;

4) the matrixes $S(\tau_1, \tau_2)$ and $S(\tau_0)$ are non-degenerate.

Then for sufficiently small $\varepsilon_3 > 0$ the unique solution of the problem (7), (8) exists and for every $\tau \in [0, L]$ and $\varepsilon \in (0, \varepsilon_3]$ the following estimate holds:

$$\|a(\tau, \bar{y} + \mu, \bar{\psi} + \xi, \varepsilon) - \bar{a}(\tau, \bar{y})\| + \|\varphi(\tau, \bar{y} + \mu, \bar{\psi} + \xi, \varepsilon) - \bar{\varphi}(\tau, \bar{y}, \bar{\psi}, \varepsilon) - \eta(\varepsilon)\| \le c_6 \varepsilon^\alpha,$$

where $\alpha = (ms)^{-1}$, $\|\eta(\varepsilon)\| \le c_7 \varepsilon^{\alpha - 1}$.

Remark 2. The asymptotic of estimates in theorems 1–3 under the imposed conditions is the finest.

Example 1. Let us consider the problem:

$$\frac{da}{d\tau} = 1 + \cos(\varphi - 2\varphi_\theta), \quad \theta = 0.5, \quad a(\tau_0) = a_0, \quad 0 < \tau_0 \le 1;$$

$$\frac{d\varphi}{d\tau} = \frac{1 + 2\tau}{\varepsilon}, \quad \tau \in [0,1], \quad \int_{\tau_1}^{\tau_2} \varphi(\tau) = d, \quad 0 \le \tau_1 < \tau_2 \le 1.$$

There is resonance $\omega(\tau) - 2\theta\omega(\theta\tau) = \tau$ in the point $\tau = 0$. The Wronskian determinant equals to -1. The estimate of error for slow variable is

$$\left|a(\tau,\varepsilon)-\bar{a}(\tau)\right|=\left|\int_{\tau_0}^{\tau}\cos\left(\frac{\tau^2}{\varepsilon}+\psi\right)d\tau\right|\le c_8\sqrt{\varepsilon}\,.$$

3. Averaging of Multifrequency System with Noether Boundary Conditions

Let us consider the system (7) with boundary conditions

$$A_0a\big|_{\tau=0}+A_1a\big|_{\tau=L}+\int_0^L f(s,a_\Lambda,\varphi_\Theta(s))ds=d, \qquad (11)$$

$$B_0\varphi\big|_{\tau=0}+B_1\varphi\big|_{\tau=L}+\int_0^L B(s)\varphi(s)ds=g_0a\big|_{\tau=0}+g_1a\big|_{\tau=L}+g_2\int_0^L a(s)ds, \qquad (12)$$

where f – preset $n-$measurable function $2\pi-$periodic in components φ_Θ, A_0,A_1 are constant $(n\times n)-$matrixes, B_0,B_1 – constant $(q\times m)-$matrixes, and B is vector-function of the same extension, d – preset n vector, g_0,g_1,g_2 – constant $(q\times n)-$matrixes.

Under the solution of problem (7), (11), (12) we will understand vector-function $\{a(\tau),\varphi(\tau)\}$, which satisfies the system of equations (7) and boundary condition (11) in classical understanding, and boundary condition (12) as pseudo solution [15], i.e. by substitution $\varphi=\varphi(\tau,y,\psi,\varepsilon)$, $\varphi(0,y,\psi,\varepsilon)=\psi$ in the condition (12), the initial value ψ is found as vector, which minimizes euclidean norm of discrepancy and the norm of which is the least under the conditions.

The oscillation resonance condition is condition (10).

Theorem 4. Let us suppose, that:

1) conditions 1), 2) of Theorem 3 are true;
2) the unique solution of averaged Noether problem for slow variables, which lies in D together with its $\rho-$neighborhood, exists;
3) matrix $M_1=A_0+A_1\dfrac{\partial\bar{a}(L,\bar{y})}{\partial\bar{y}}+\int_0^L f_0(s,\bar{a}(s,\bar{y}))\dfrac{\partial\bar{a}(s,\bar{y})}{\partial\bar{y}}ds$ is invertible, and $M_2=B_0+B_1+\int_0^L B(s)ds$ is $(q\times m)$ full rank matrix, $q\ge m$.

Then there will be found constants $c_9>0,\varepsilon_4>0$ such that for every $\varepsilon\in(0,\varepsilon_2]$ the unique solution of the boundary problem (7), (11), (12) exists, moreover, for fast variables φ as pseudo solution, and for all $\tau\in[0,L]$ and $\varepsilon\in(0,\varepsilon_4]$, estimate performs

$$\left\|a(\tau,y,\psi,\varepsilon)-\bar{a}(\tau,y)\right\|+\left\|\phi(\tau,y,\psi,\varepsilon)-\bar{\varphi}(\tau,y,\psi,\varepsilon)-\eta(\varepsilon)\right\|\le c_9\varepsilon^\alpha.$$

Example 2. Let us consider the problem:

$$\frac{da}{d\tau} = 1 + \cos 4\varphi_\theta, \quad \theta = 0.5, \quad a(0) = a_0,$$
$$\frac{d\varphi}{d\tau} = \frac{2\tau}{\varepsilon}, \quad \varphi(0) + \varphi(1) = 1, \quad 3\varphi(0) - 2\varphi(0) = -2.$$

There is resonance in the point $\tau = 0$ because $\omega_1(\tau) = 2\tau$. The pseudo solution $\varphi(0) = 0$ is found from boundary conditions. The estimate of error for slow variable for $\tau = 1$ is $|a(1,\varepsilon) - \bar{a}(1)| \le c_{10}\sqrt{\varepsilon}$.

The Case of the Classic Solution. Let us write boundary condition (12) in the form

$$\Omega\varphi = g_0 a\big|_{\tau=0} + g_1 a\big|_{\tau=L} + g_2 \int_0^L a(s)ds,$$

where $\Omega\varphi := B_0\varphi\big|_{\tau=0} + B_1\varphi\big|_{\tau=L} + \int_0^L B(s)\varphi(s)ds$ is linear bounded Noether operator.

Condition, which provides the existence of solving the system (7), which would satisfy the condition (12) in classic understanding was received in [15] and was written as

$$P_{\Omega^*}(g_0 a\big|_{\tau=0} + g_1 a\big|_{\tau=L} + g_2 \int_0^L a(s)ds) = 0,$$

where P_{Ω^*} is orthoprojector on the core $\ker\Omega^*$ of operator Ω^*, conjugated to Ω.

Conclusion

The results of research of multifrequency systems with linearly transformed arguments, with in the process of evolution pass through the resonances, are shown. The existence and uniqueness of solution of the boundary problems with multipoint and integral conditions are proved and the averaging method on fast variables is justified.

The obtained results are the basis for further investigation of new classes of systems, especially systems with frequencies depending on slow variables, and systems of higher approximation, and systems with transformed arguments.

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