LOTKA-VOLTERRA CUBICDIFFERENTIAL SYSTEMS WITH (1:-2)-SINGULARITY AND INVARIANT AFFINE STRAIGHT LINES OF TWO DIRECTIONS OF TOTAL ALGEBRAIC MULTIPLICITY SIX

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Abstract. The Lotka-Volterra cubicdifferential systems with (1:-2)-singularity possessing invariant straight lines of two direction and total multiplicity six are classified. There are obtained fifteen distinct classes modulo the affine transformations and time rescaling. The Darboux first integrals are constructed.
Keywords: differential cubic system, invariant straight line, Darboux integral.

1. Introduction

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \quad GCD(P,Q) = 1 \qquad (1.1)$$

and the vector field $X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to system (1.1). Here $GCD(P,Q) \in R[x;y]$ is the greatest common divisor of the polynomials P and Q.
Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ $(n = 3)$, then the system (1.1) is called quadratic (cubic).
A curve $f(x,y) = 0,\ f \in C[x,y]$ (a function $f = \exp(g/h);\ g,h \in C[x,y]$) is said to be an invariant algebraic curve (an invariant exponential function) of (1.1) if there exists a polynomial $K_f(x,y) \in C[x,y], \deg(K_f) \le n-1$, such that the identity

$$\frac{\partial f(x,y)}{\partial x} \cdot P(x,y) + \frac{\partial f(x,y)}{\partial y} \cdot Q(x,y) \equiv f(x,y) \cdot K_f(x,y)$$

holds. If (1.1) has an invariant curve f of degree one, i.e. $f(x,y) = \alpha x + \beta y + \gamma,\ |\alpha| + |\beta| \ne 0,$ then the curve f is called the invariant straight line of the system (1.1).
Definition 1. An invariant algebraic curve $f = 0$ of degree d of system (1.1) is called of algebraic multiplicity m, if m is the greatest positive integer such that the f^m divide $E_d(X)$, where

$$E_d(X) = \det\begin{pmatrix} v_1 & v_2 & \dots & v_l \\ X(v_1) & X(v_2) & \dots & X(v_l) \\ \dots & \dots & \dots & \dots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \dots & X^{l-1}(v_l) \end{pmatrix}, \qquad (1.2)$$

and the system $v_1, v_2, \dots, v_l$ is a basis of $C_d[x, y]$ (see [1]).

In the case of the invariant straight lines ($d = 1$) we can take $v_1 = 1, v_2 = x, v_3 = y$ and the polynomial $E_1(X)$ has the form:

$$E_1(X) = P \cdot X(Q) - Q \cdot X(P).$$

Definition 2. Let D be a domain in R^2 and $F \in C^1(D, R)$ $\left(\mu \in C^1(D, R)\right)$.

A function $F(x, y)$ ($\mu(x, y)$) is called a first integral (an integrating factor) of the system (1.1) if the following identity

$$P(x, y)\frac{\partial F}{\partial x} + Q(x, y)\frac{\partial F}{\partial y} \equiv 0$$

$$\left(P(x, y)\frac{\partial \mu}{\partial x} + Q(x, y)\frac{\partial \mu}{\partial y} \equiv -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mu(x, y)\right)$$

holds in D.

Let the system (1.1) have algebraic invariant curves $f_j(x, y) = 0,\ j = 1, \dots, s,$ i.e. there exist the polynomials $K_j(x, y),\ j = 1, \dots, s$, such that the following identities hold:

$$\frac{\partial f_j(x, y)}{\partial x} \cdot P(x, y) + \frac{\partial f_j(x, y)}{\partial y} \cdot Q(x, y) \equiv f_j(x, y) \cdot K_j(x, y).$$

When system (1.1) has a first integral (an integrating factor) of the form

$$F(x, y) = \prod_{j=1}^{s} f_j^{a_j}(x, y) \left(\mu(x, y) = \prod_{j=1}^{s} f_j^{a_j}(x, y)\right),$$

then we say that the system is Darboux integrable. Darboux proved that if the system (1.1) has at least $s \geq n(n+1)/2$ distinct algebraic invariant curves, then this system is Darboux integrable [6].

These last years, a great number of works are dedicated to the investigation of the problem of integrabilityfor polynomial differential systems and, in particular, of Lotka-Volterra systems, with resonant singular points (see [1], [3], [5]-[13]).This problem was completely solved in [7] for quadratic system with $1:-2$ resonant singular points, but for cubic systems is still open. In this paper, the cubic systems with $1:-2$rezonant singularity, having invariant straight lines of two direction of total multiplicity six, are classified.

We consider the cubic system

$$\begin{cases} \dot{x} = x(1 + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} = y(-2 + b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2), \end{cases} \tag{1.3}$$

where the variables x, y and coefficients a_{ij}, b_i are real.

Our main results are expressed in the following theorem.

Main Theorem. *The cubic system (1.3) has affine invariant straight lines of two directions of total algebraic multiplicity six if and only if it has one of the following fifteen forms:*

1) $\begin{cases} \dot{x} = x(x-1)(ax-1), a \neq 0, a \neq 1, \\ \dot{y} = y(y-1)(by+2), b \neq 0,\ b \neq -2; \end{cases}$ 2) $\begin{cases} \dot{x} = x(x-1)(ax-1),\ a \neq 0, \\ \dot{y} = -2y(y-1)^2, a \neq 1; \end{cases}$

3) $\begin{cases} \dot{x} = x(x-1)^2, \\ \dot{y} = y(y-1)(cy+2),\ c \neq 0, c \neq -2; \end{cases}$ 4) $\begin{cases} \dot{x} = x(x-1)^2, \\ \dot{y} = -2y(y-1)^2; \end{cases}$

5) $\begin{cases} \dot{x} = \frac{1}{2}x((2-c)y + 3cy^2 + 2), c \neq 0, \\ \dot{y} = y(y-1)(cy+2), c \neq -2, \end{cases}$ 6) $\begin{cases} \dot{x} = x(x-1)(ax-1),\ a \neq 1, \\ \dot{y} = y(x + ax - 2), a \neq 0; \end{cases}$

7) $\begin{cases} \dot{x} = x((1+d)y^2 + 1), \\ \dot{y} = y(y-1)(dy+2), d \neq 0, d \neq -2; \end{cases}$ 8) $\begin{cases} \dot{x} = x(x-1)(ax-1), a \neq 0, a \neq 1, \\ \dot{y} = y(-2 + 3x + (a-2)x^2); \end{cases}$

9) $\begin{cases} \dot{x} = x(x-1)(ax-1),\ a \neq 0, a \neq 1, \\ \dot{y} = y(py^2 + qy - 2), q^2 + 8p < 0; \end{cases}$ 10) $\begin{cases} \dot{x} = x(px^2 + qx + 1), q^2 - 4p < 0, \\ \dot{y} = y(y-1)(cy+2), c \neq 0, c \neq -2; \end{cases}$

11) $\begin{cases} \dot{x} = x(px^2 + qx + 1),\ q^2 - 4p < 0, \\ \dot{y} = y(ry^2 + sy - 2),\ s^2 + 8r < 0; \end{cases}$ 12) $\begin{cases} \dot{x} = x(x-1)^2, \\ \dot{y} = y(py^2 + qy - 2), q^2 + 8p < 0; \end{cases}$

13) $\begin{cases} \dot{x} = x(px^2 + qx + 1), q^2 - 4p < 0, \\ \dot{y} = -2y(y-1)^2; \end{cases}$ 14) $\begin{cases} \dot{x} = \frac{1}{2}x(3py^2 + qy + 2), \\ \dot{y} = y(py^2 + qy - 2), q^2 + 8p < 0; \end{cases}$

15) $\begin{cases} \dot{x} = x(px^2 + qx + 1),\ q^2 - 4p < 0, \\ \dot{y} = -y(2 + qx), \end{cases}$

where $a, b, c, d, q, p, r, s \in R$. *The systems 1) – 15) are Darboux integrable and have the following first integrals, respectively:*

1) $F(x,y) = (x-1)^{\frac{2(2+b)}{(1-a)b}} x^{-\frac{2(2+b)}{b}} (ax-1)^{\frac{2a(2+b)}{(a-1)b}} (y-1)^{\frac{2}{b}} y^{-\frac{2+b}{b}} (by+2)$;

2) $F(x,y) = e^{\frac{a-1}{y-1}} x^{2-2a} (ax-1)^{2a} (y-1)^{a-1} y^{1-a} (x-1)^{-2}$;

3) $F(x,y) = e^{\frac{2c+4}{x-1}} x^{-4-2c} (x-1)^{4+2c} (y-1)^2 y^{-2-c} (cy+2)^c$;

4) $F(x,y)=e^{\frac{2}{1-x}}\cdot e^{\frac{1}{1-y}}x^2(x-1)^{-2}(y-1)^{-1}y$;

5) $F(x,y)=e^{\frac{1}{2+cy}}\cdot e^{-\frac{5c+2c^2}{4(y-1)}}x^2(y-1)^{\frac{c^2+5c-4}{2}}(cy+2)y$;

6) $F(x,y)=e^{\frac{1}{ax-1}}\cdot e^{\frac{3a-4a^2}{x-1}}x^2(x-1)^{4a^2-6a-1}(ax-1)y$;

7) $F(x,y)=x^2y(dy+2)^{-\frac{2+d}{d}}(y-1)^{-2}$;

8) $F(x,y)=x^2y(ax-1)^{\frac{2-2a}{a}}(x-1)^{-1}$;

9) $F(x,y)=(x-1)^{\frac{-1}{a}}x^{\frac{1-a}{a}}(ax-1)y^{\frac{1-a}{2a}}(y+\frac{q+\sqrt{\gamma}}{2p})^{(a-1)(\sqrt{\gamma}-q)\delta}(y+\frac{q-\sqrt{\gamma}}{2p})^{(a-1)(\sqrt{\gamma}+q)\beta}$,

where $\gamma=q^2+8p$, $\delta=\frac{128p^3-q^3\gamma\sqrt{\gamma}+q^3\sqrt[3]{\gamma}}{512ap^3\sqrt{\gamma}}$, $\beta=\frac{128p^3+q^3\gamma\sqrt{\gamma}-q^3\sqrt[3]{\gamma}}{512ap^3\sqrt{\gamma}}$;

10) $F(x,y)=(y-1)^{\frac{2}{c}}x^{\frac{-4-2c}{c}}(cy+2)y^{\frac{-2-c}{c}}(x+\frac{q+\sqrt{\lambda}}{2p})^{-(c+2)(\sqrt{\lambda}-q)\beta}(x+\frac{q-\sqrt{\lambda}}{2p})^{(2+c)(\sqrt{\lambda}+q)\gamma}$,

where

$\lambda=q^2-4p$, $\beta=\frac{-16p^3+4pq^3\sqrt{\lambda}-q^5\sqrt{\lambda}+q^3\sqrt[3]{\lambda}}{16cp^3\sqrt{\lambda}}$, $\gamma=\frac{16p^3+4pq^3\sqrt{\lambda}-q^5\sqrt{\lambda}+q^3\sqrt[3]{\lambda}}{16cp^3\sqrt{\lambda}}$;

11) $F(x,y)=\frac{1}{4pr}\cdot x^{-32p^3\lambda\delta}\cdot(x+\frac{q+\lambda}{2p})^{-(q-\lambda)(16p^3-\gamma)\delta}\cdot(2px+q-\lambda)\cdot$

$\cdot y^{-16p^3\lambda\delta}\cdot(y+\frac{s-\beta}{2r})^{-16p^3s(q-\lambda)\delta+\frac{\beta}{s+\beta}}\cdot(2ry+s+\beta)$,

Where $\lambda=\sqrt{q^2-4p}$, $\beta=\sqrt{8r+s^2}$, $\gamma=4pq^3\lambda-q^5\lambda+q^3\lambda^3$,

$\delta=\frac{1}{(q+\lambda)(16p^3+\gamma)}$;

12) $F(x,y)=e^{\frac{\beta}{x-1}}\cdot(x-1)^{\beta}\cdot x^{-\beta}\cdot y^{-q\lambda+8p+q^2}\cdot\left(y+\frac{q+\lambda}{2p}\right)^{q\lambda-4p-q^2}\cdot\left(y+\frac{q-\lambda}{2p}\right)^{-4p}$,

where $\lambda=\sqrt{8p+q^2}$, $\beta=2q\lambda-16p-2q^2$;

13) $F(x,y)=e^{\frac{\beta}{y-1}}\cdot(y-1)^{\beta}\cdot x^{-2\beta}\cdot y^{-\beta}\cdot\left(x+\frac{q+\lambda}{2p}\right)^{q\lambda+\beta-q^2}\cdot\left(x+\frac{q-\lambda}{2p}\right)^{4p}$;

where $\lambda=\sqrt{q^2-4p}$, $\beta=q\lambda-\lambda^2$;

14) $F(x,y)=\dfrac{1}{16}p^{-2}(py^2+qy-2)^{-2}x^2y$;

15) $F(x,y)=(px^2+qx+1)^{-1}x^2y$.

2.Algebraic maximal multiplicity of the invariant straight lines $x=a, y=a, a\in C$

2.1.Multiplicity of the line $x=0(y=0)$

In this subsection, we compute the maximal algebraic multiplicity of the invariant straight line $x=0(y=0)$ of the system (1.3). For this purpose, we calculate the determinant $E_1(X)$. It has the form $E_1(X)=xy(6+\omega(x,y))$, where $\omega(x,y)$ is a polynomial of degree six and $\omega(0,0)\neq 0$. It is evident that the invariant straight line $x=0$ $(y=0)$ has the algebraic multiplicity one. In this way we have proved the following lemma.

Lemma 1. *For system(1.3) the algebraic multiplicity of the invariant straight line* $x=0$ $(y=0)$ *is one.*

2.2.Multiplicity of the line $x=\alpha,\ \alpha\in R^*$

Without loss of generality, we consider $\alpha=1$. Then system (1.3) can be written in the following form

$$\begin{cases}\dot{x}=x(x-1)(ax+by-1)\equiv P(x,y),\\ \dot{y}=-y(2+cx+dy+mx^2+nxy+sy^2)\equiv Q(x,y).\end{cases} \tag{2.1}$$

For (1.3) the polynomial $E_1(X)$ looks as

$$E_1(X)=xy(x-1)(A_2(y)+A_3(y)(x-1)+A_4(y)(x-1)^2+$$
$$A_5(y)(x-1)^3+A_6(y)(x-1)^4+A_7(y)(x-1)^5),$$

where $A_j(y), j=2,\dots,7$ are polynomials in y. For example, $A_2(y)=A_{21}(y)A_{22}(y)$, where $A_{21}(y)=2+c+m+dy+ny+sy^2$ and $A_{22}(y)=-1+a^2-c+ac-m+am+$

$$2(-b+ab-d+ad-n+an)y+(b^2+bd+bn-3s+3as)y^2+2bsy^3.$$

The algebraic multiplicity of the invariant straight line $x=1$ is at least two if the identity $A_2(y)\equiv 0$ holds. Let $A_2(y)\equiv 0$. As $GCD(P,Q)=1$, then the polynomial $A_{21}(y)$ is not identically zero and therefore $A_{22}(y)$ is identically zero. The identity $A_{22}(y)\equiv 0$ holds if one of the following three sets of conditions is satisfied:

2.1) $b=0,\ a=1$;

2.2) $a=1,\ s=0,\ n=-b-d$;

2.3) $s=0,\ n=-b-d,\ m=-1-a-c$.

In each of the cases 2.1), 2.2) and 2.3) the algebraic multiplicity of the line $x=1$ is at least three if the polynomial $A_3(y)$ is identicaly zero. Indeed,

In Case 2.1) we obtain $A_3(y)=A_{21}(y)A_{31}(y)$, where $A_{31}(y)=2+c+m+2dy+2ny+3sy^2$. If $A_{31}(y)\equiv 0$, then $\deg(GCD(P,Q))>0$. Therefore, $A_{31}(y)$ is not identicaly zero. Thus, in this case the multiplicity of $x=1$ is two.

In Case 2.2) we get

$\{A_3(y)=4+4c+c^2+4m+2cm+m^2+(b^2-2bd-bcd-b^2m-bdm)y^2+2b^2dy^3\equiv 0,$
$GCD(P,Q)=1\}\Rightarrow c=-3, d=0,\ m=1\Rightarrow$

$$A_4(y)=-4b^2y^2\not\equiv 0.$$

The system (1.3) has the form

$$\dot{x}=x(x-1)(x+by-1),\ \ \dot{y}=y(-2+3x-x^2+bxy), b\neq 0. \quad (2.2)$$

In Case 2.3) we have

$\{A_3(y)=(-1+a+by)(-6+6a-2c+2ac+(3b+bc-3d+3ad)y+2bdy^2)\equiv 0,$
$GCD(P,Q)=1\}\Rightarrow\{d=0,\ c=-3, GCD(P,Q)=1\}\Rightarrow$

$$A_4(y)=-2(-1+a+by)(-3+3a+2by)\not\equiv 0.$$

The system(1.3) looks as

$$\dot{x}=x(x-1)(ax+by-1),\ \ \dot{y}=y(-2+3x+(a-2)x^2+bxy), (a,b)\neq(1,0) \quad (2.3)$$

and has the invariant straight lines: $l_1\equiv x=0,\ l_{2,3,4}\equiv x-1=0, l_5\equiv y=0.$

Note that the system (2.2) is a particular case of the system (2.3).

Lemma 2. *The maximal algebraic multiplicity of the real invariant straight line $l\equiv x-\alpha=0,\ \alpha\in R^*$ in system (1.3) is three. If l=0 has the algebraic multiplicity three for (1.3), then via an affine transformation of coordinates and time rescaling the system (1.3) can be writing in the form (2.3) and $l\equiv x-1=0$ is the unique invariant straight line parallel to $x=0$.*

2.3. Multiplicity of the line $y=1$

The straight line $y=1$ is the invariant straight line for system (1.3) if and only if (1.3) looks as

$$\begin{cases}\dot{x}=x(kx^2+jxy+ry^2+ax+by+1)\equiv P(x,y),\\ \dot{y}=y(y-1)(cx+dy+2)\equiv Q(x,y).\end{cases} \quad (2.5)$$

To determine the maximal algebraic multiplicity of the line $y=1$ for (2.5), we write $E_1(X)$ in the form

$$E_1(X) = xy(y-1)(B_2(x) + B_3(x)(y-1) + B_4(x)(y-1)^2 +$$
$$B_5(x)(y-1)^3 + B_6(x)(y-1)^4 + B_7(x)(y-1)^5).$$

Taking into acountthat $GCD(P,Q)=1$, the polynomial $B_2(x) = B_{21}(x)B_{22}(x)$, where $B_{21}(x) = 1+b+r+ax+jx+kx^2$ and $B_{22}(x) = -2+2b-3d+bd-d^2+2r+dr+$
$$+2(2a-2c+ad-cd+2j+dj)x+(ac-c^2+6k+3dk+cj)x^2+2ckx^3,$$
is identically zero if one of the following three sets of conditions is satisfied:

2.4) $c=0,\ d=-2$;

2.5) $d=-2,\ k=0,\ c=a+j$;

2.6) $k=0,\ c=a+j,\ r=1-b+d$.

In Case 2.4) the condition $GCD(P,Q)=1$ gives
$$B_3(x) = 2x^2(a+j+kx)(2a+2j+3kx) \not\equiv 0.$$
In Case 2.5) we have $\{B_3(x) = (a+j)^2(b+2ax)x^2 \equiv 0,\ GCD(P,Q)=1\} \Rightarrow$ $\{a=b=0,\ GCP(P,Q)=1\} \Rightarrow B_4(x) = 2j^2x^2 \not\equiv 0$. The system (2.5) has the form
$$\dot{x} = x(jxy-y^2+1),\quad \dot{y} = y(y-1)(2+jx-2y),\ j\neq 0. \qquad (2.6)$$
In Case 2.6) we find that $B_3(x) = B_{31}(x)B_{32}(x)$, where $B_{31}(x) = 2+d+ax+jx$ and $B_{32}(x) = 4b+2bd+(6a+ab+3ad+bj)x+(2a^2+2aj)x^2$. Assume $B_3(x)\equiv 0$. As $GCD(P,Q)=1$ the polynomial $B_{31}(x)$ is not identicaly zero and therefore $B_{32}(x)$ is identically zero. $\{a=b=0,\ GCD(P,Q)=1\} \Rightarrow B_4(x) = (2+d+jx)(6+3d+2jx) \not\equiv 0$. The system (2.5) looks as
$$\dot{x} = x(jxy+(1+d)y^2+1),\quad \dot{y} = y(y-1)(2+jx+dy),\ (d,j)\neq(-2,0) \qquad (2.7)$$
which possesses the invariant straight lines: $l_1 \equiv x=0,\ l_2 \equiv y=0,\ l_{3,4,5} \equiv y-1=0.$

Note that (2.6) is a particular case of the system (2.7).

Lemma 3. *The maximal algebraic multiplicity of the real invariant straightline $l \equiv y-\alpha=0,\ \alpha \in R^*$ in system (1.3) is three. If l=0 has algebraic multiplicity threefor (1.3), then via an affine transformation of coordinates and time rescaling, the system (1.3) can be brought tothe form (2.7) and $l \equiv y-1=0$ is the unique invariant straight line parallel to $y=0$.*

2.4. Multiplicity of the line $x=\alpha,\ \alpha\in C\setminus R$

The straight line $x=\alpha,\ \alpha \in C\setminus R$ is the invariant straight line for system (1.3) if and only if (1.3) looks as

$$\begin{cases}\dot{x}=x(px^2+qx+1)\equiv P(x,y), \quad q^2-4p<0,\\ \dot{y}=-y(2+cx+dy+mx^2+nxy+sy^2)\equiv Q(x,y).\end{cases} \tag{2.8}$$

To determine the maximal algebraic multiplicity of the line $x=\alpha,\ \alpha\in C\setminus R$, we write $E_1(X)$ in the form:

$$E_1(X)=xy(px^2+qx+1)(A_2(x,y)+A_3(x,y)(px^2+qx+1)).$$

where

$$A_2(x,y)=\frac{1}{p^3}(m^2p-c^2p^2-2mp^2+2cmpq+cp^2q-m^2q^2-mpq^2+$$

$$(-2cmp^2+2cp^3+2m^2pq-c^2p^2q+mp^2q-2p^3q+2cmpq^2+cp^2q^2-m^2q^3-mpq^3)x+$$

$$(-3dmp^2-3cnp^2+4dp^3+3mnpq+np^2q)y+(-2n^2p^2+2d^2p^3-4mp^2s+6sp^3)y^2+$$

$$(-3mnp^2+3cdp^3+4np^3-3dmp^2q-3cnp^2q-dp^3q+3mnpq^2+np^2q^2)xy+$$

$$(4dnp^3-2n^2p^2q+4cp^3s-4mp^2qs-p^3qs)xy^2+5p^3s(d+n)xy^3+3p^3s^2y^4).$$

The algebraic multiplicity of the invariant straight line $x=\alpha,\ \alpha\in C/R$ is at least two if the identity $A_2(x,y)\equiv 0$ holds. Taking into account that the system's coefficients are real numbers and $GCD(P,Q)=1,$ the last identity yields the following set of conditions: $d=n=m=s=0,\ c=q$.

In this case we obtain the system 15) from the main theoremwith the invariant straight lines:

$$l_1\equiv x=0,\ l_{2,3}\equiv x+\frac{q+\sqrt{q^2-4p}}{2p}=0,\ l_{4,5}\equiv x+\frac{q-\sqrt{q^2-4p}}{2p}=0,\ l_6\equiv y=0.$$

The polinomial $E_1(X)$ has the form $E_1(X)=2xy(3+qx)(1+qx+px^2)^2$ and it does not divide the polynomial $(px^2+qx+1)^3$.

Lemma 4. *The maximal algebraic multiplicity of the non-real invariant straight line $x=\alpha,\ \alpha\in C\setminus R$ insystem (1.3) is two.*

2.5. Multiplicity of the line $y=\alpha,\ \alpha\in C\setminus R$

The straight line $y=\alpha,\ \alpha\in C\setminus R$ is the invariant straight line for system (1.3) if and only if (1.3) look as

$$\begin{cases}\dot{x}=x(kx^2+jxy+ry^2+ax+by+1)\equiv P(x,y),\\ \dot{y}=y(py^2+qy-2)\equiv Q(x,y),\ q^2+8p<0.\end{cases} \tag{2.9}$$

We write $E_1(X)$ in the form

$$E_1(X)=xy(py^2+qy-2)(A_2(x,y)+A_3(x,y)(py^2+qy-2)).$$

The polynomial $A_2(x,y)$ looks as

$$A_2(x,y)=\frac{1}{p^3}(2b^2p^2-3p^3+2bp^2q-4p^2r-4bpqr-2pq^2r+4pr^2+2q^2r^2+$$

$$(6bjp^2-ap^3+2jp^2q+6ap^2r-6jpqr)x+2p^2(2j^2+a^2p+4kr)x^2+5akp^3x^3+3k^2p^3x^4+$$

$$(-2bp^3-b^2p^2q+p^3q-bp^2q^2+4bp^2r+4p^2qr+2bpq^2r+pq^3r-4pqr^2-q^3r^2)y+$$

$$(3abp^3-jp^3-3bjp^2q+ap^3q-jp^2q^2+6jp^2r-3ap^2qr+3jpq^2r)xy+5kjp^3x^3y+$$

$$(4bkp^3+4ajp^3-2j^2p^2q+kp^3q-4kp^2qr)x^2y.$$

The algebraic multiplicity of the invariant straight line $y=\alpha,\ \alpha\in C\setminus R$ is at least two if the identity $A_2(x,y)\equiv 0$ holds. Under condition $GCD(P,Q)=1$ and taking into acount that the coefficients of thedifferentialsystem are real, the identity $A_2(x,y)\equiv 0$ holds if the following conditions are satisfied: $a=k=j=0,\ b=\frac{q}{2},\ r=\frac{3p}{2}$.In this case we get the system 14) of the main theorem possessing the invariant straight lines:

$$l_1\equiv x=0,\ l_2\equiv y=0,\ l_{3,4}\equiv y+\frac{q+\sqrt{q^2+8p}}{2p}=0,\ l_{5,6}\equiv y+\frac{q-\sqrt{q^2+8p}}{2p}=0.$$

The multiplicity of the lines $l_{5,6}$ is exactly two as $(py^2+qy-2)^3$ does not divide the polynomial $E_1(X)=-\frac{1}{4}xy(py^2+qy-2)^2(-6-qy+3py^2)$.

Lemma 5. *In the class of systems (1.3) the maximal algebraic multiplicity of the non-real invariant straight line* $y=\alpha,\ \alpha\in C\setminus R$ *is two.*

3. Configurations of the invariant straight lines

Taking into acount Lemmas 1-5 we have for system (1.3) the following twelve configurations of six invariant real straight lines of two directions:

A1) (*3**r**;3**r***);
A2) (*3**r** ;3(2)**r**);*
A3) (*3(2)**r*** ; *3**r***);
A4) (*2**r** ;4(1,2)**r***);
A5) (*4(1,2)**r*** ; *2**r***);
A6) (*3(2)**r*** ; *3(2)**r***);
A7)(*1**r***; *5(2,2)**r***);
A8) (*5(2,2)**r*** ; *1**r***);
A9) (*2**r**;4(3)**r***);
A10) (*4(3)**r***; *2**r***);
A11) (*1**r** ;5(1,3)**r***);
A12) (*5(1,3)**r***; *1**r***),

And the following seven configurations of six invariant straight lines, two of wich are non-real:

B1) (*3**r**; 1**r**+2**c***);
B2) (*1**r**+2**c**; 3**r***);
B3) (*1**r**+2**c** ;1**r**+2**c***);
B4) (*3(2)**r**; 1**r**+2**c***);
B5) (*1**r**+2**c** ; 3(2)**r***);
B6) (*1**r** ;1**r**+4(2,2)**c***);
B7) (*1**r**+4(2,2)**c**; 1**r***).

We denote by *3(2)**r***$(4(3)\boldsymbol{r})$ two parallel and distinct real straight lines one of which is counted twice (thrice) and we say that it has multiplicity equal to two (three). By*5(1,3)**r***we denote three parallel and distinct real straight lines one of which has multiplicity three and by *1**r**+2**c*** is denoted a triplet of distinct and parallel straight lines, one of which is real and two are complex (non-real).

We denote by (3r ; 1r+2c) the configuration consisting of six affine straight lines of two directions: a triplet of real distinct parallel straight lines in one direction; a real straight line and a pair of parallel complex (non-real) lines in the second direction. By (1r;5(2,2)r) is denoted the configuration of six straight lines of two directions consisting of: a real straight line (in one direction) and five real parallel straight lines two of which have multiplicity equal to two (in the second direction)

3.1. Unrealisable configurations

In this subsection we show that in the class of cubic systems of the form (1.3) the configurations **A4), A5), A9)**and **A10)** are not realisable.

Configuration A4) $(\boldsymbol{2r};\boldsymbol{4(1,2)r})$. Via an affine transformations of coordinates and time rescaling, the cubic system (1.3) with two real invariant straight lines in the direction of the axe Oy and three real invariant straight lines in the direction Ox, can be writen in the form

$$\begin{cases}\dot{x}=x(x-1)(ax+by-1), b\neq 0, a^2+((b-1)(b-c))^2\neq 0,\\ \dot{y}=y(y-1)(cy+2), c\neq 0,\ c\neq -2.\end{cases} \tag{3.1}$$

Without loss of generality, we consider that the invariant straight $y=1$ has the multiplicity two. For system (3.1) the polynomial $E_1(X)$ look as

$$\begin{aligned}E_1(X)=(1-x)x(y-1)y(2&\\ +cy)[(4-b^2+c-bc-4ax-4bx-3abx+2b^2x-acx+5abx^2&\\ -2by-b^2y+3cy-bcy+2b^2xy-3acxy-2bcy^2)(y-1)+\varphi(x)],&\end{aligned}$$

where $\varphi(x)=(1-b)(1+b+c)-(a(c+3b-1)-2(1-b)^2)x-a\big(2a+5(1-b)\big)x^2+3a^2x^3$. As$\varphi(x)$ is not identically zero, the polynomial $(y-1)^2$does not divide $E_1(X)$and therefore the invariant straight line $y=1$ does not have the multiplicity two. In this way we proved that the configurations**A4)**is not realisable.

Configuration A5) (*4(1,2)r* ; *2r*). The system (1.3) can by writen in the form

$$\begin{cases} \dot{x} = x(x-1)(ax-1), a \neq 0, a \neq 1 \\ \dot{y} = y(y-1)(bx+cy+2), (b+2)^2(b+2a)^2 + c^2 \neq 0. \end{cases} \tag{3.2}$$

Suppose that the invariant straight line $x = 1$ has the multiplicity two for (3.2). In this assumption the polynomial $E_1(X)$ looks as

$$E_1(X) = -xy(x-1)(ax-1)(y-1)[(-4+2a-5b+ab-b^2+(6a-b+ab-b^2)x+2abx^2+(8b+$$
$$+2b^2-2c+ac-3bc)y+(2b^2+3ac)xy+5bcy^2)(x-1)+\phi(y)],$$

where $\phi(y) = (2+2b+cy)(-3+a-b+4y+2by-2cy+3cy^2)$.

This implies the polynomial $\phi(y)$ to be not identically zero for the system (3.2), which contradicts the assumption that the invariant straight line $x = 1$ has the multiplicity two. We have proved that the configurations **A5)** is not realisable.

Configuration A9) $(2\boldsymbol{r};\ 4(3)\boldsymbol{r})$ **(Configuration A10)** $(4(3)\boldsymbol{r};\ 2\boldsymbol{r})$**).** According to Lemma 3 (Lemma 2) we consider the system (2.7) ((2.3)). This system does not have any invariant straight line described by an equation of the form $x = \alpha$ (y= α), $\alpha \in R^*$. Therefore, the configuration **A9)** (**A10)**) is not realisable.

4. Classification of cubic systems (1.3) with invariant straight lines of total multiplicity six and of two directions

Configuration A1) (*3**r***;*3**r***). Via an affine transformation of coordinates and time rescaling each system which realise this configuration can be written in the form 1) of the Main Theorem.The system 1) has the invariant straight lines:

$$l_1 \equiv x = 0,\ l_2 \equiv x-1 = 0,\ l_3 \equiv ax-1 = 0,\ l_4 \equiv y = 0,\ l_5 \equiv y-1 = 0,\ l_6 \equiv by+2 = 0$$

and is Darboux integrable.

Configuration A2) (*3**r*** ;*3(2)**r***). Any cubic system with a triplet of distinct parallel invariant straight lines and a couple of distinct parallel invariant straight lines modulo an affine transformation of coordinates can be writen in the form

$$\begin{cases} \dot{x} = x(x-1)(ax-1) \equiv P(x,y), a \neq 0, a \neq 1, \\ \dot{y} = y(y-1)(bx+cy+2) \equiv Q(x,y). \end{cases} \tag{4.1}$$

For this system we have $E_1(X) = xy(y-1)(x-1)(ax-1)(A_2(x)+A_3(x)(y-1))$,

$$A_2(x) = -2-3c-c^2-(4+4a+4b+2c+2ac+2bc)x+(6a-b-ab-b^2+3ac)x^2+2abx^3,$$

where and $A_3(x)$ is a polynomial in x. In order the straight line $y-1=0$ to have multiplicity two, we require the $A_2(x) \equiv 0$ and this yields $b=0,\ c=-2$. In these conditions the system (4.1) becomes the system 2) of the Main Theorem. It has the invariant straight lines:

$$l_1 \equiv x=0,\ l_2 \equiv x-1=0,\ l_3 \equiv ax-1=0,\ l_4 \equiv y=0,\ l_{5,6} \equiv y-1=0$$

and it is Darboux integrable.

Configuration A3) (*3(2)**r*** ; *3**r***). In this case the system (1.3) can be written as

$$\begin{cases} \dot{x} = x(x-1)(ax+by-1) \equiv P(x,y), \\ \dot{y} = y(y-1)(cy+2) \equiv Q(x,y),\ c \neq 0, c \neq -2. \end{cases} \tag{4.2}$$

For (4.2) the polynomial $E_1(X)$ looks as

$$E_1(X) = xy(y-1)(x-1)(cy+2)(A_2(y)+A_3(y)(x-1)),$$

where

$$A_2(y) = -1+a^2+(4-4a-2b+2ab-2c+2ac)y+(b^2-2b+3c-3ac+bc)y^2-2bcy^3.$$

The identity $A_2(y)\equiv 0$ holds if and only if $a=1,\ b=0$. Under these conditions the system (4.2) is of the form 3) of the Main Theorem andit has the invariant straight lines

$$l_1 \equiv x=0,\ l_{2,3} \equiv x-1=0,\ l_4 \equiv y=0,\ l_5 \equiv y-1=0,\ l_6 \equiv cy+2=0.$$

Configuration A6) (*3(2)**r*** ; *3(2)**r***). We write the system (1.3) in the form

$$\begin{cases} \dot{x} = x(x-1)(ax+by-1) \equiv P(x,y), \\ \dot{y} = y(y-1)(cx+dy+2) \equiv Q(x,y). \end{cases}$$

For this system the polynomial $E_1(X)$is

$$E_1(X) = xy(y-1)(x-1)(A_2(y)+A_3(x,y)(x-1))$$

where

$$A_2(y) = (2+c+dy)((1-a)(1+a+c-4y+2by-2cy+2dy-3dy^2)+ by^2(2-b+c-d+2dy)$$

and

$A_3(x,y) = 4+2a^2-3c+2ac+a^2c-c^2+ac^2-10ax+2a^2x+cx+2acx+ a^2cx-c^2x+ac^2x+6a^2x^2-3acx^2+a^2cx^2+ac^2x^2+2a^2cx^3-8ay-8by+ 4aby+8cy-8acy-2bcy+2abcy+2c^2y-2ac^2y+2dy+4ady+a^2dy- 3cdy+3acdy+10abxy-8acxy-2bcxy+2abcxy+2c^2xy-2ac^2xy- 5adxy+a^2dxy+3acdxy+3abcx^2y-2ac^2x^2y+3a^2dx^2y+4b^2y^2-4bcy^2+ b^2cy^2-bc^2y^2-10ady^2-4bdy^2+2abdy^2+5cdy^2-5acdy^2+bcdy^2+ 2ad^2y^2+b^2cxy^2-bc^2xy^2+5abdxy^2-5acdxy^2+2b^2dy^3-3bcdy^3-3ad^2y^3$

The straight line $x-1=0$ will have the multiplicity two, only if $A_2(y)$ isidenticaly zero: $\{A_2(y)\equiv 0, GCD(P,Q)=1\} \Rightarrow a=1,\ b=0$ or $a=1,\ c=b-2,\ d=0,\ b\neq 0$.

If $a=1,\ b=0$, thenwe have the system

$$\begin{cases}\dot{x}=x(x-1)^2,\\ \dot{y}=y(y-1)(cx+dy+2),\end{cases} \tag{4.3}$$

For(4.3)the polynomial $E_1(X)$ has the form

$$E_1(X)=xy(y-1)(x-1)^2(B_2(x)+B_3(x)(y-1)),$$

where $B_2(x)=2+3d+d^2+(8+4c+4d+2cd)x+(2c-6+c^2-3d)x^2-2cx^3$.

The straight line $y-1=0$ has the multiplicity twoif $B_2(x)\equiv 0$ and this is realizedwhen $c=0,\ d=-2$. Thus, the system (4.3) will have the form 4) of the Main Theorem. This system has the invariant straight lines:

$$l_1\equiv x=0,\ l_{2,3}\equiv x-1=0,\ l_4\equiv y=0,\ l_{5,6}\equiv y-1=0.$$

When $a=1,\ c=b-2,\ d=0$, the polynomial $B_2(x)$ looks

$$B_2(x)=2-2b^2-(2b+2b^2)x+(4b-6-2b^2)x^2+(4-2b)x^3.$$

Obviously, $B_2(x)$ is not identicaly zero. So, in this case, the multiplicity of the invariant straight line $y-1=0$ can not be equal to two.

Configuration A7) (*1**r***; *5(2,2)**r***) .We rewrite the system (1.3) into the form

$$\begin{cases}\dot{x}=x(fx^2+gxy+hy^2+ax+by+1)\equiv P(x,y),\\ \dot{y}=y(y-1)(cy+2)\equiv Q(x,y),\ c\neq 0,\ c\neq -2.\end{cases} \tag{4.4}$$

For (4.4) we have $E_1(X)=xy(y-1)(cy+2)(A_2(x)+A_3(x)(y-1))$, where

$$A_2(x)=-A_{21}(x)\cdot A_{22}(x),$$

$A_{21}(x)=1+b+h+ax+gx+fx^2$ and $A_{22}(x)=-1+b-c+h+2ax+2gx+3fx^2$.

Taking into consideration the condition $GCD(P,Q)=1$, we obtain $A_{21}(x)\not\equiv 0$, and therefore $A_{22}(x)$ is identicallyzero. This is realized when $f=0,\ g=-a,\ h=1-b+c$. In these conditions, the system (4.4) looks as

$$\begin{cases}\dot{x}=x(-axy+(1-b+c)y^2+ax+by+1),\\ \dot{y}=y(y-1)(cy+2),\ c\neq 0, c\neq -2.\end{cases} \tag{4.5}$$

For (4.5) we have $E_1(X)=xy(y-1)^2(cy+2)(B_2(x)+B_3(x)(cy+2))$, where

$$B_2(x)=\frac{1}{c^3}((2+c)(2-2b+c+acx)(2-2b-c+2acx).$$

The implication $\{B_2(x)\equiv 0, GCD(P,Q)=1\}\Rightarrow\left\{a=0,\, b=\frac{2-c}{2}\right\}$ reduces (4.5) to the system 5) of the Main Theorem, which has the following invariant straight lines:

$$l_1\equiv x=0,\; l_2\equiv y=0,\, l_{3,4}\equiv y-1=0,\, l_{5,6}\equiv cy+2=0.$$

Configuration A8) (*5(2,2)**r*** ; *1**r***).We consider the system

$$\begin{cases}\dot{x}=x(x-1)(ax-1)\equiv P(x,y),\; a\neq 0, a\neq 1,\\ \dot{y}=y(fy^2+gxy+hx^2+bx+cy-2)\equiv Q(x,y).\end{cases}\tag{4.6}$$

In this case

$$E_1(X)=xy(y-1)(x-1)(A_2(y)+A_3(y)(x-1)),$$

where $A_2(y)=-A_{21}(y)\cdot A_{22}(y)$, $A_{21}(y)=1+a-b-h-2cy-2gy-3fy^2$ and $A_{22}(y)=-2+b+h+cy+gy+fy^2$. The straight line $x-1=0$ has the multiplicitytwo, only if the polinomial $A_2(y)$ is identically zero. We have $\{A_2(y)\equiv 0, GCD(P,Q)=1\}\Rightarrow A_{21}\equiv 0\Rightarrow\{f=0,\; g=-c,\; h=1+a-b\}$. The system (4.6) obtains the form

$$\begin{cases}\dot{x}=x(x-1)(ax-1),\; a\neq 0, a\neq 1,\\ \dot{y}=y(-cxy+(1+a-b)x^2+bx+cy-2).\end{cases}\tag{4.7}$$

We will require for system (4.7) to have the invariant straight line $ax-1=0$ ofthe multiplicity two. We computethe polynomial $E_1(X)$ for (4.7)

$$E(X)=xy(x-1)^2(ax-1)(B_2(y)+B_3(y)(ax-1)),$$

where $B_2(y)=\frac{1}{a^3}(a-1)(-1-2a+b+acy)(-1-a+b+2acy)$ .Taking into account that $GCD(P,Q)=1$, the polynomial $B_2(y)$ will be identical zero if and only if $c=0,\, b=a+1$. In these conditions (4.7) becomes the system 6) of the Main Theorem wich has the following invariant straight lines

$$l_1\equiv x=0,\; l_{2,3}\equiv x-1=0,\, l_{4,5}\equiv ax-1=0,\, l_6\equiv y=0.$$

Configuration A11) (*1**r*** ;*5(1,3)**r***). For realisation of this configuration it is sufficient to put $l=0$ in (2.7). In this way, we obtain the system 7) of the Main Theorem which has the invariant straight lines:

$$l_1\equiv x=0,\; l_2\equiv y=0,\, l_{3,4,5}\equiv y-1=0,\, l_6\equiv dy+2=0.$$

Configuration A12) (*5(1,3)****r****; 1****r***). In this case we put $b=0$, $a\neq 0$ in (2.3) and obtain the system 8) of the Main Theorem which possesses the invariant straight lines:

$$l_1 \equiv x=0,\ l_{2,3,4} \equiv x-1=0,\ l_5 \equiv ax-1=0,\ l_6 \equiv y=0.$$

Configuration B1) (*3****r****; 1****r****+2****c***). By an affine transformation of coordinates the system (1.3) can be brought to the form 9) of the Main Theorem, which has the following invariant straight lines:

$$l_1 \equiv x=0,\ l_2 \equiv x-1=0,\ l_3 \equiv ax-1=0,\ l_4 \equiv y=0,$$

$$l_5 \equiv y+\frac{q+\sqrt{q^2+8p}}{2p}=0,\ l_6 \equiv y+\frac{q-\sqrt{q^2+8p}}{2p}=0,\ \text{where } q^2+8p<0.$$

Configuration B2) (*1****r****+2****c****; 3****r***). In this casethe system (1.3) can be brought to the system10) of the Main Theorem and it admits the following invariant straight lines:

$$l_1 \equiv x=0,\ l_2 \equiv x+\frac{q+\sqrt{q^2-4p}}{2p}=0,\ l_3 \equiv x+\frac{q-\sqrt{q^2-4p}}{2p}=0,$$

$$l_4 \equiv y=0,\ l_5 \equiv y-1=0,\ l_6 \equiv cy+2=0,\ \text{where } q^2-4p<0.$$

Configuration B3) (*1****r****+2****c**** ;1****r****+2****c***). The system (1.3) can be brought to the form 11) of the Main Theorem and it has the fallowing invariant straight lines:

$$l_1 \equiv x=0,\ l_2 \equiv x+\frac{q+\sqrt{q^2-4p}}{2p}=0,\ l_3 \equiv x+\frac{q-\sqrt{q^2-4p}}{2p}=0,\ l_4 \equiv y=0,$$

$$l_5 \equiv y+\frac{s+\sqrt{s^2+8r}}{2r}=0,\ l_6 \equiv y+\frac{s-\sqrt{s^2+8p}}{2r}=0,\ \text{where } q^2-4p<0,\ s^2+8r<0$$

.

Configuration B4) (*3(2)****r****; 1****r****+2****c***). In this case the system (1.3) can be written:

$$\begin{cases} \dot{x}=x(x-1)(ax+by-1), \\ \dot{y}=y(py^2+qy-2),\ q^2+8p<0. \end{cases} \tag{4.8}$$

The polynomial $E(X)$ has the form

$$E(X)=xy(x-1)(py^2+qy-2)(A_2(y)+A_3(y)(x-1)),$$

where

$$A_2(y)=1-a^2+(2b-2ab-2q+2aq)y+(3ap+bq-b^2-3p)y^2+2bpy^3.$$

The straight line $x-1=0$ will have the multiplicity two only if $A_2(y)$ is identicaly zero, and this will be realized when $b=0,\ a=1$. In this condition, the system (4.8) has the form12) of the Main Theorem having the following invariant straight lines:

$$l_1 \equiv x=0,\ l_{2,3} \equiv x-1=0,\ \ l_5 \equiv y=0,\ \ l_5 \equiv y+\frac{q+\sqrt{q^2+8p}}{2p}=0,\ l_6 \equiv y+\frac{q-\sqrt{q^2+8p}}{2p}=0.$$

Configuration B5) (*1**r**+2**c*** ; *3(2)**r***). We write the system (1.3) as follows:

$$\begin{cases} \dot{x} = x(px^2+qx+1),\ \ q^2-4p<0, \\ \dot{y} = y(y-1)(cx+dy+2). \end{cases} \tag{4.9}$$

The polynomial $E(X)$ for (4.9) has the form

$$E(X)=xy(y-1)(px^2+qx+1)(A_2(x)+A_3(x)(y-1)),$$

where

$$A_2(x)=2+3d+d^2+(4c+2cd-4q-2dq)x+(c^2-6p-3dp-cq)x^2-2cpx^3.$$

The expresion $A_2(x)$ will be identicaly zero if the condition $c=0,\ d=-2$ is fulfilled. Thus, the system (4.9) has the form 13) from the Main Theorem. The system has the following invariant straight lines:

$$l_1 \equiv x=0,\ \ l_2 \equiv x+\frac{q+\sqrt{q^2-4p}}{2p}=0,\ l_3 \equiv x+\frac{q-\sqrt{q^2-4p}}{2p}\equiv 0,\ \ l_4 \equiv y=0,\ l_{5,6} \equiv y-1=0.$$

Configuration B6) (*1**r*** ;*1**r**+4(2,2)**c***). The system (1.3), which admits the configuration **B6),** is the system 14) from the Main Theorem, obtained in Section **2.5**.

Configuration B7) (*1**r**+4(2,2)**c***; *1**r***). The system having the configuration of the invariant straight lines **B7)** was determined in Section**2.4**. It represents the system 15) from the Main Theorem.

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