CENTER CONDITIONS FOR A CUBIC DIFFERENTIAL SYSTEM WITH TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT ELLIPTIC CURVE

Dumitru COZMA, associate professor, dr. hab.
Angela MATEI, lecturer, dr.
Tiraspol State University

Abstract. For a cubic differential system with a singular point of a center or a focus type having two invariant straight lines one invariant elliptic cubic curve it was proved that a singular point is a center if and only if the first two Lyapunov quantities at this point vanish. There were obtained five sets of conditions for a singular point to be a center.
Keywods: cubic differential system, invariant algebraic curves, the problem of the center.

CONDIȚII DE CENTRU PENTRU UN SISTEM DIFERENȚIAL CUBIC CU DOUĂ DREPTE INVARIANTE ŞI O CURBĂ INVARIANTĂ DE TIP ELIPTIC

Rezumat. Pentru un sistem diferențial cubic cu punct singular de tip centru sau focar, care posedă două drepte invariante şi o cubică invariantă de tip eliptic, s-a demonstrat că punctul singular este de tip centru dacă şi numai dacă primele două mărimi Lyapunov în acest punct se anulează. Au fost obținute cinci serii de condiții ca punctul singular să fie de tip centru.
Cuvinte-cheie: sistem diferențial cubic, curbe invariante algebrice, problema centrului.

1. Introduction

We consider the cubic system of differential equations of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \tag{1.1}$$

where the coefficients are real numbers, $x = x(t)$ and $y = y(t)$ are real variables. The origin $O(0,0)$ is a singular point of a center or a focus type (a weak focus) for (1.1). The problem arises of finding the coefficient conditions under which $O(0,0)$ is a center.

The derivation of necessary conditions for a singular point $O(0,0)$ to be a center involves use of computer algebra and we obtain them by calculating the focus quantities, which are polynomials in the coefficients of the system [1, 2, 4].

The necessary conditions are shown to be sufficient by a variety of methods [3, 4]. It is known that a singular point $O(0,0)$ is a center for (1.1) if and only if the system has an analytic first integral of the form $F(x, y) = C$ in some neighborhood of $O(0,0)$. Also, $O(0,0)$ is a center if and only if the system (1.1) has an analytic integrating factor of the form $\mu(x,y) = 1 + \sum \mu_k(x,y)$ in some neighborhood of $O(0,0)$.

There exists a power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x, y)$ along trajectories of (1.1) is a linear combination of polynomials $\{(x^2 + y^2)\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2 + y^2)^j.$$

Quantities L_j are polynomials with respect to the coefficients of system (1.1) called the *Lyapunov quantities (the focus quantities)* [1, 4]. The order of the weak focus $O(0,0)$ is r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ and $L_r \neq 0$. The origin $O(0,0)$ is a center for (1.1) if and only if $L_j = 0,\ j = 1,2,\ldots$.

2. Invariant algebraic curves and Darboux integrability

We shall study the problem of the center for cubic system (1.1) assuming that the system has invariant algebraic curves.

Definition 2.1. An algebraic curve $\Phi(x,y) = 0$ in $\mathbf{C}^2$ with $\Phi(x,y) \in \mathbf{C}[x,y]$ is an *invariant algebraic curve* of a differential system (1.1) if

$$\frac{d\Phi}{dt} = P(x,y)\frac{\partial\Phi}{\partial x} + Q(x,y)\frac{\partial\Phi}{\partial y} \equiv K(x,y)\cdot\Phi(x,y) \tag{2.1}$$

for some polynomial $K(x,y) \in \mathbf{C}[x,y]$ called the *cofactor* of the curve $\Phi(x,y) = 0$.

Let the cubic system (1.1) have sufficiently many invariant algebraic curves $\Phi_j(x,y) = 0,\ j = 1,\ldots,q$ with cofactors $K_j(x,y)$. Then in most cases a first integral (an integrating factor) can be constructed in the Darboux form [4, p.26]

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\ldots\Phi_q^{\alpha_q} = C \qquad \left(\mu = \Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\ldots\Phi_q^{\alpha_q}\right) \tag{2.2}$$

and we say that the cubic system (1.1) is *Darboux integrable*. The function (2.2), with $\alpha_j \in \mathbf{C}$ not all zero, is a first integral (an integrating factor) for (1.1) if and only if

$$\sum_{j=1}^{q}\alpha_j K_j(x,y) \equiv 0 \qquad \left(\sum_{j=1}^{q}\alpha_j K_j(x,y) \equiv -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right). \tag{2.3}$$

The method of Darboux is very useful and elegant one to prove integrability for some classes of differential systems depending on parameters [5].

By Definition 2.1, a straight line $C + Ax + By = 0,\ A,B,C \in \mathbf{C},\ (A,B) \neq 0$ is an *invariant straight line* for (1.1) if and only if there exists a polynomial $K(x,y)$ such that the following identity holds

$$A\cdot P(x,y) + B\cdot Q(x,y) \equiv (C + Ax + By)\cdot K(x,y)\,.$$

If the cubic system (1.1) has complex invariant straight lines then obviously they occur in complex conjugated pairs $l_1 \equiv C + Ax + By = 0$ and $l_2 \equiv \overline{C} + \overline{A}x + \overline{B}y = 0$.

Let the cubic ststem (1.1) have two distinct invariant straight lines l_1, l_2 that are real or l_1, l_2 are complex ($l_2 \equiv \overline{l_1}$). The conditions for the existence of two distinct invariant straight lines for cubic system (1) where obtained in [6]. It was proved

Theorem 2.1. *The cubic differential system (1.1) has two distinct invariant straight lines if and only if one of the following sets of conditions holds:*

(I) $a = f = k = p = r = 0,\ \ m(c^2 - 4m) \neq 0.$

The invariant straight lines and their cofactors are

$$l_{1,2} \equiv 2 + (c \pm \sqrt{c^2 - 4m}\,)x = 0,\; K_{1,2}(x,y) = [y(c + 2mx \pm \sqrt{c^2 - 4m}\,)]/2;$$

(II) $g = b + c,\; f = a + d,\; q = p + l - k,\; s = m + n - r.$

The invariant straight lines and their cofactors are $l_{1,2} \equiv x \mp iy = 0$,

$$K_1(x,y) = -i + (a - ib - ic)x - (b + ia + id)y + (k - im - in + ir)x^2 + \\ +(r - n - il - ip)xy - (l + ir)y^2,\; K_2 = \overline{K_1};$$

(III) $a = 1, k = g, l = -b, q = [(d + n + 1)(c\gamma + b - c - p) - g\gamma^2]/\gamma^2,$

$m = [(\gamma(d+1) + c^2)(\gamma - 1) - (b - p)(c(\gamma - 2) + b - p) - n\gamma]/\gamma^2,$

$r = 1 - \gamma, s = 0, \gamma = f + 2, \gamma[(b - c - p)^2 + 4\gamma(d + n + 1)] \neq 0.$

The invariant straight lines are $l_{1,2} \equiv 1 + A_{1,2}x - y = 0$, *where* A_1, A_2 *are distinct solutions of the equation* $\gamma A^2 + (b - c - p)A - d - n - 1 = 0$. *The cofactors are*

$$K_{1,2}(x,y) = x + A_{1,2}y + gx^2 + (1 + d - A_{1,2}^2 + cA_{1,2})xy + ((\gamma - 1)A_{1,2} + b)y^2;$$

(IV) $p = [(b - c)h + (k - g)\gamma]/h, q = [h(cs - gh) + s(g - k)]/h^2,$

$l = -b, m = [(d - h + 1)h^2 + h(c(k - g) - s) - (k - g)^2]/h^2,$

$r = 1 - \gamma, n = [s\gamma - (1 + d)h]/h, h = a - 1, h((g - k)^2 + 4sh) \neq 0.$

The invariant straight lines are $l_{1,2} \equiv 1 + A_{1,2}x - y = 0$, *where* A_1, A_2 *are distinct solutions of the equation* $hA^2 + (g - k)A - s = 0$. *The cofactors are*

$$K_{1,2}(x,y) = x + A_{1,2}y + (g + hA_{1,2})x^2 + (1 + d + cA_{1,2} - A_{1,2}^2)xy + (b + (\gamma - 1)A_{1,2})y^2.$$

The sufficient conditions for a singular point $O(0,0)$ to be a center in system (1.1) with two invariant straight lines were determined in [6]. The presence of a center was proved by using the method of Darboux integrability and the rational reversibility.

The problem of the center was solved for cubic system (1.1) with two invariant straight lines and one invariant irreducible conic in [4]; with two parallel invariant straight lines and one invariant cubic $x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0$ in [7].

The problem of the center was solved in [8] for a nine-parametric cubic system that can be reduced to a Liénard type system.

3. Conditions for the existence of one elliptic cubic curve

In this paper assuming that one set of conditions (I) - (IV) holds, we shall determine the conditions under which the cubic system (1.1) has an elliptic cubic curve of the form

$$\Phi(x,y) \equiv a_{30}x^3 + a_{20}x^2 + a_{10}x + a_{00} - y^2 = 0, \tag{3.1}$$

with $a_{30}, a_{20}, a_{10}, a_{00} \in \mathbf{R}$ and $a_{00}a_{30} \neq 0$.

By Definition 2.1, an algebraic curve (3.1) is an *invariant cubic curve* for (1.1) if and only if there exists a polynomial $K_{\Phi}(x,y)=c_{20}x^2+c_{11}xy+c_{02}y^2+c_{10}x+c_{01}y$, with real coefficients, such that the following identity holds

$$(3a_{30}x^2+2a_{20}x+a_{10})\cdot P(x,y)-2y\cdot Q(x,y)\equiv\Phi(x,y)\cdot K_{\Phi}(x,y). \qquad (3.2)$$

Identifying the coefficients of the monomials $x^i y^j$ in (3.2), we reduce this identity to a system of 18 equations $\{F_{ij}=0\}$ for the unknowns $a_{30}, a_{20}, a_{10}, a_{00}, c_{ij}$. We find that

$$c_{10}=0,\ c_{01}=-2b,\ c_{20}=3k,\ c_{11}=3m,\ c_{02}=3p,\ a_{10}=-2ba_{00},$$
$$r=0,\ 2l+3p=0,\ 2n+3m=0,\ 3k+2ab=0,\ l-bf=0 \qquad (3.3)$$

and a_{00}, a_{20}, a_{30} are the solutions of the following algebraic system:

$$\begin{cases} F_{40}\equiv 9aa_{30}+2aba_{20}=0,\\ F_{31}\equiv(2b+3c)a_{30}-ma_{20}+2s=0,\\ F_{22}\equiv 9fa_{30}+2bfa_{20}-6ab+6q=0,\\ F_{30}\equiv 6aa_{20}-8ab^2a_{00}=0,\\ F_{21}\equiv 3a_{30}+2(b+c)a_{20}+4bma_{00}+2g=0,\\ F_{12}\equiv 3fa_{20}-4b^2fa_{00}+3d=0,\\ F_{11}\equiv 2a_{20}-(4b^2+2bc+3m)a_{00}+2=0. \end{cases} \qquad (3.4)$$

We solve the system (3.4) assuming that one set of conditions (I) - (IV) holds. In this way we determine the conditions for the existence of two invariant straight lines and one invariant elliptic cubic curve of the form (3.1).

Theorem 3.1. *The cubic system (1.1) has two distinct invariant straight lines* $l_1=0, l_2=0$ *and one invariant cubic* $\Phi=0$ *of the form (3.1) if and only if one the following eight sets of conditions holds:*

(i) $a=d=f=k=l=p=q=r=0, n=3(2b+c)(b+c),$
$m=-2(2b+c)(b+c), s=(4b^2+4bc+2bg+3cg)/3;$

(ii) $a=d=f=k=l=p=q=r=0, n=b(2b+3c)/3,$
$m=-2b(2b+3c)/9, s=(3g-3c-2b)(2b+3c)/9;$

(iii) $a=d=f=k=l=p=q=r=0, n=-3m/2;$

(iv) $a=k=r=0, f=d, g=b+c,\ m=-2b(2b+3c)/9, q=bd/3,$
$l=bd, n=b(2b+3c)/3, p=(-2bd)/3, s=b(2b+3c)/9;$

(v) $d=r=0, c=(-7b)/3, f=a, g=(-4b)/3, l=q=ab, m=(-8b^2)/9,$
$n=(4b^2)/3, k=p=(-2ab)/3, s=(4b^2)/9;$

(vi) $a=k=r=0, d=f=-1,\ g=(3c-b)/3, m=-2(bc+2)/3,$
$l=-b, n=bc+2, p=(2b)/3, q=b, s=-bc-2, b^2=3;$

(vii) $a=k=r=0, g=(3c-b)/3, d=b(b^2-3bc-24)/(21b+9c),$

$q=(5b^3c-2b^4+3b^2c^2+27b^2-6bc-9c^2)/(21b+9c),$

$m=2(b^2-3bc-9)/9, n=(9+3bc-b^2)/3, p=(2b)/3,$

$l=-b, s=(2b^3-5b^2c-3bc^2-20b-12c)/(7b+3c),$

$f=-1,\ 19b^3-33b^2c-63bc^2-252b-27c^3-108c=0;$

(viii) $d=r=0, a=s=2/3,\ c=(-7b)/3, f=-1, g=(-4b)/3,\ k=(-4b)/9,$

$l=-b, m=2, n=-3, p=q=(2b)/3, b^2=3/2.$

Proof. To prove Theorem 3.1, we solve the system (3.4) assuming that one set of conditions (I) - (IV) holds.

In Case (I) the equations of (3.4) yield $d=q=0$ and the system (3.4) becomes

$$\begin{cases} F_{31}\equiv(2b+3c)a_{30}-ma_{20}+2s=0, \\ F_{21}\equiv 3a_{30}+2(b+c)a_{20}+4bma_{00}+2g=0, \\ F_{11}\equiv 2a_{20}-(4b^2+2bc+3m)a_{00}+2=0. \end{cases}$$

We find a_{30} from $F_{21}=0$ and a_{20} from $F_{11}=0$, then $F_{31}=0$ looks

$$F_{31}\equiv 4(2b+3c)(b+c-g)+6m+12s-f_1f_2a_{00}=0, \quad (3.5)$$

where $f_1=4b^2+6bc+2c^2+m,\ f_2=4b^2+6bc+9m.$

If $f_1=0,$ then (3.5) implies

$$m=-2(2b+c)(b+c),\ s=(4b^2+4bc+2bg+3cg)/3$$

and we obtain the set of conditions (i) of Theorem 3.1. The invariant curves are

$$l_1\equiv 2(b+c)x+1=0, l_2=(2b+c)x-1=0,$$

$$\Phi\equiv 2(g-c-b)x^3+3(x^2+y^2)-3(2bx+2cx+1)(2bx+cx-1)^2a_{00}=0$$

and have the cofactors

$$K_1=-2(b+c)(2bx+cx-1)y,\ K_2=-(2b+c)(2bx+2cx+1)y,$$

$$K_\Phi=-2by-6(2b+c)(b+c)xy.$$

If $f_1\neq 0, f_2=0$, then (3.5) implies

$$m=-2b(2b+3c)/9,\ s=(3g-3c-2b)(2b+3c)/9.$$

In this case we get the set of conditions (ii) of Theorem 3.1. The invariant curves are

$$l_1\equiv(2b+3c)x+3=0, l_2=2bx-3=0,$$

$$\Phi\equiv 18(g-c-b)x^3+27(x^2+y^2)+(2bx-3)^3a_{00}=0$$

and have the cofactors

$$K_1=y(3-2bx)(2b+3c)/9, K_2=-2by(3(cx+1)+2bx)/9,$$

$$K_\Phi=-2by((2b+3c)x+3)/3.$$

Let $f_1\neq 0, f_2\neq 0$. In this case the equation (3.5) yields

$$a_{00}=[4(2b+3c)(b+c-g)+6m+12s]/(f_1f_2)$$

and we obtain the set of conditions (iii). The invariant curves are

$$l_{1,2}\equiv 2+(c\pm\sqrt{c^2-4m})x=0,\ \Phi(x,y)\equiv a_{30}x^3+a_{20}x^2+a_{10}x+a_{00}-y^2=0,$$

where $a_{00}=[4(2b+3c)(b+c-g)+6m+12s]/(f_1f_2)$, $a_{10}=-2ba_{00}$,

$$a_{20}=((4b^2+2bc+3m)a_{00}-2)/2,\ a_{30}=-2(2bma_{00}+(b+c)a_{20}+g)/3.$$

The cofactors of the invariant algebraic curves are

$$K_{1,2}(x,y)=[y(c+2mx\pm\sqrt{c^2-4m})]/2,\ K_\Phi=-y(2b-3mx).$$

In Case (II) the system (3.4) looks

$$\begin{cases}F_{40}\equiv 9aa_{30}+2aba_{20}=0,\\ F_{31}\equiv(2b+3c)a_{30}-ma_{20}-m=0,\\ F_{22}\equiv(a+d)(9a_{30}+2ba_{20})+2bd=0,\\ F_{30}\equiv 6aa_{20}-8ab^2a_{00}=0,\\ F_{21}\equiv 3a_{30}+2(b+c)a_{20}+4bma_{00}+2(b+c)=0,\\ F_{12}\equiv(a+d)(3a_{20}-4b^2a_{00})+3d=0,\\ F_{11}\equiv 2a_{20}-(4b^2+2bc+3m)a_{00}+2=0.\end{cases}\qquad(3.6)$$

If $a=d=0$, then (3.6) yields $a_{30}=-a_{00}(4b^3+6b^2c+2bc^2+7bm+3cm)/3$, $a_{20}=((4b^2+2bc+3m)a_{00}-2)/2$ and $(4b^2+6bc+2c^2+m)(4b^2+6bc+9m)=0$. This subcase is contained in the case (iii) of Theorem 3.1.

If $a=0, d\neq 0$, then the equations of (3.6) imply

$$a_{20}=(4b^2a_{00}-3)/3,\ a_{30}=(-8a_{00}b^3)/27,\ m=-2b(2b+3c)/9.$$

In this subcase we obtain the set of conditions (iv) of Theorem 3.1. The invariant algebraic curves are

$$l_{1,2}\equiv x^2+y^2=0,\ \Phi(x,y)\equiv(2bx-3)^3a_{00}+27(x^2+y^2)=0,$$

and have the cofactors

$$K_{1,2}(x,y)=2by(-3-2bx-3cx-3dy)/3,\ K_\Phi=-2by(2bx+3cx+3dy+3)/3.$$

If $a\neq 0$, then from the equations of (3.6) we find

$$a_{30}=(8b)/81,\ a_{20}=(-4)/9,\ a_{00}=(-1)/(3b^2),\ d=0,\ m=(-8b^2)/9,\ c=(-7b)/3.$$

In this subcase we get the set of conditions (v). The invariant algebraic curves are

$$l_{1,2}\equiv x^2+y^2=0,\ \Phi(x,y)\equiv(2bx-3)^3-9b(2b-3c)y^2=0$$

and have the cofactors $K_{1,2}(x,y)=-2(2abx^2+3aby^2+4b^2xy-3ax+3by)/3$,

$$K_\Phi=-2b(3ax^2+3ay^2+4bxy+3y)/3.$$

In Case (III) the equations $F_{40}=0$, $F_{30}=0$ of (3.4) implies $a_{20}=(4b^2a_{20})/3$, $a_{30}=(-2ba_{20})/9$, where $b\neq 0$. From the relations (3.3) we obtain $p=(2b)/3$,

$g=(-2b)/3,\ f=-1,\ n=b(3c-b)/3$. In this case $F_{31}\neq 0$ and the system of algebraic equations (3.4) is not compatible.

In Case (IV) the relations (3.3) yield $f=-1,\ g=(3c-3ac-b-ab)/3$,

$$k=(-2ab)/3,\ s=[(1-a)(b^2-3bc-3d+9a-12)]/3$$

and the equations $F_{40}=0, F_{30}=0$ of (3.4) looks

$$F_{40}\equiv a(2ba_{20}+9a_{30})=0,\ F_{30}\equiv a(3a_{20}-4b^2a_{00})=0.$$

Assume $a=0$, then from equations $F_{12}=0$ and $F_{22}=0$ we express a_{20} and a_{30}, respectively. Then $F_{11}\equiv a_{00}(3-b^2)+d+1=0.$

If $b^2=3$, then $d=-1$ and we obtain the set of conditions (vi) of Theorem 3.1. The invariant algebraic curves looks $l_1\equiv 1+A_1x-y=0$, $l_2\equiv 1+A_2x-y=0$,

$$\Phi(x,y)\equiv(8bx^3+18bx-36x^2-9)a_{00}-8bx^3+9(x^2+y^2)=0,$$

where A_1, A_2 are distinct solutions of the equation $3A^2+(b-3c)A-3bc-6=0$. The cofactors of these invariant algebraic curves are $K_{\Phi}=2y(by-bcx-2x-b)$,

$$K_{1,2}=[(3c-b-3A)x^2+3A(c-A)xy+3x+3by^2+3Ay]/3.$$

If $a=0$ and $b^2\neq 3$, then from the equations $F_{11}=0, F_{21}=0$ of (3.4) we find that

$$a_{00}=(d+1)/(b^2-3), d=b(b^2-3bc-24)/(21b+9c).$$

In this subcase we get the set of conditions (vii) of Theorem 3.1. The invariant algebraic curves are $l_1\equiv 1+A_1x-y=0$, $l_2\equiv 1+A_2x-y=0$,

$$\Phi(x,y)\equiv 2(25b^4-66b^3c-27b^2c^2-315b^2+54bc+81c^2)x^3+54b(b-3c)x+$$
$$+9b(72-7b^2+21bc)x^2+27(3c-b)+81(7b+3c)y^2=0,$$

where A_1, A_2 are distinct solutions of the equation

$$3(7b+3c)A^2+(7b^2-9c^2-18bc)A+3(2b^3-5b^2c-3bc^2-20b-12c)=0.$$

The cofactors of these invariant algebraic curves are

$$K_{1,2}=((3c-b-3A)x^2+(bA-3bc+b^2-9)xy+3x+3by^2+3Ay)/3,$$

$$K_{\Phi}=2y(b^2x-3bcx+3by-9x-3b)/3.$$

If $a\neq 0$, then the equations $F_{30}=0, F_{40}=0$ of (3.4) yield $a_{20}=(4b^2a_{00})/3$, $a_{30}=(-8b^3a_{00})/27$. We express $a_{00}=1/(3a+b^2-1)$ from $F_{11}=0$. Then the equations of (3.4) imply $d=0,\ c=b(3-a)/(3a-3),\ a=(9-2b^2)/9,\ b^2=3/2$.

In this subcase we obtain the set of conditions (viii). The invariant algebraic curves are $l_1\equiv 1+A_1x-y=0$, $l_2\equiv 1+A_2x-y=0$, $\Phi(x,y)\equiv 8bx^3-36x^2+36bx-18+9y^2=0$, where A_1, A_2 are distinct solutions of the equation $3A^2+8bA+6=0$. The cofactors of these invariant algebraic curves are $K_{\Phi}=2(3by^2-2bx^2-3by+9xy)/3$ and

$$K_{1,2} = (-(A+4b)x^2 + (bA+9)xy + 3x + 3by^2 + 3Ay)/3.$$

The proof of Theorem 3.1 is complete.

4. Integrability conditions for cubic system (1.1) with three algebraic curves

Let the cubic system (1.1) have at least two invariant straight lines and one invariant elliptic cubic curve, i.e. one of the sets of conditions (i) - (viii) of Theorem 3.1 is satisfied. In this section, we pay attention to the problem of the center for system (1.1) and prove that the origin $O(0,0)$ is a weak focus of order at most two.

Lemma 4.1. The following three sets of conditions are sufficient conditions for the origin to be a center:

(1) $a = d = f = k = l = p = q = r = 0, n = 3(2b+c)(b+c),$
$m = -2(2b+c)(b+c), s = (4b^2 + 4bc + 2bg + 3cg)/3;$

(2) $a = d = f = k = l = p = q = r = 0, n = b(2b+3c)/3,$
$m = -2b(2b+3c)/9, s = (3g-3c-2b)(2b+3c)/9;$

(3) $a = k = r = 0, f = d, g = b+c, m = -2b(2b+3c)/9, q = bd/3,$
$l = bd, n = b(2b+3c)/3, p = (-2bd)/3, s = b(2b+3c)/9.$

Proof. In Cases (1), (2) and (3), the cubic system (1.1) has two invariant straight lines and one invariant elliptic curve. The system has a Darboux first integral of the form

$$l_1^{\alpha_1} l_2^{\alpha_2} \Phi^{\alpha_3} = C. \qquad (4.1)$$

The first integral (4.1) can be easily constructed by using the identity (2.3) and the cofactors K_1, K_2, K_Φ of the invariant algebraic curves $l_1 = 0$, $l_2 = 0$, $\Phi = 0$.

In Case (1): $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = -1$, $l_1 \equiv 2(b+c)x + 1 = 0, l_2 \equiv (2b+c)x - 1 = 0$ and

$$\Phi \equiv 2(g-c-b)x^3 + 3(x^2+y^2) - 3(2bx+2cx+1)(2bx+cx-1)^2 a_{00} = 0.$$

In Case (2): $\alpha_1 = 0, \alpha_2 = 3, \alpha_3 = -1$, $l_1 \equiv (2b+3c)x + 3 = 0, l_2 \equiv 2bx - 3 = 0$ and

$$\Phi \equiv 18(g-c-b)x^3 + 27(x^2+y^2) + (2bx-3)^3 a_{00} = 0.$$

In Case (3): $\alpha_1 = \alpha_2 = 1, \alpha_3 = -1$, $l_{1,2} \equiv x \pm iy = 0$ and

$$\Phi(x,y) \equiv (2bx-3)^3 a_{00} + 27(x^2+y^2) = 0.$$

Lemma 4.2. The following set of conditions is sufficient for the origin to be a center

(4) $a = d = f = k = l = p = q = r = 0, n = -3m/2.$

Proof. When the set of conditions (4) is satisfied, the cubic system (1.1) has two invariant straight lines $l_{1,2} \equiv 2 + (c \pm \sqrt{c^2 - 4m})x = 0$ and one invariant elliptic cubic

$$\Phi(x,y) \equiv a_{30}x^3 + a_{20}x^2 + a_{10}x + a_{00} - y^2 = 0,$$

where $a_{10} = -2ba_{00}$,

$$a_{00} = [4(2b+3c)(b+c-g) + 6m + 12s]/[(4b^2 + 6bc + 2c^2 + m)(4b^2 + 6bc + 9m)],$$

$$a_{20} = [(4b^2 + 2bc + 3m)a_{00} - 2]/2,\ a_{30} = -2[2bma_{00} + (b+c)a_{20} + g]/3.$$

The system (1.1) is Darboux integrable and has an integrating factor of the form

$$\mu = l_1^{-1} l_2^{-1} \Phi. \quad (4.2)$$

The existence of an integrating factor (4.2) can be easily verified by using the identity (2.3) with cofactors

$$K_{1,2}(x,y) = [y(c + 2mx \pm \sqrt{c^2 - 4m})]/2,\ K_\Phi = -y(2b - 3mx).$$

Lemma 4.3. The following set of conditions is sufficient for the origin to be a center:

(5) $a = k = r = 0, d = f = -1,\ g = (3c - b)/3, m = -2(bc+2)/3,$

$$l = -b, n = bc + 2, p = (2b)/3, q = b, s = -bc - 2, b^2 = 3.$$

Proof. When the set of conditions (5) holds, the cubic system (1.1) has three invariant straight lines

$$l_1 \equiv 1 + A_1 x - y = 0,\ l_2 \equiv 1 + A_2 x - y = 0,\ l_3 \equiv 1 - bx + y = 0,$$

where A_1, A_2 are distinct solutions of the equation $3A^2 + (b - 3c)A - 3bc - 6 = 0$ and one invariant elliptic cubic $\Phi(x,y) \equiv (8bx^3 + 18bx - 36x^2 - 9)a_{00} - 8bx^3 + 9(x^2 + y^2) = 0.$

The system (1.1) has a Darboux first integral

$$l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} \Phi^{\alpha_4} = C, \quad (4.3)$$

where $\alpha_1 = b\sqrt{\Delta} + b^2 - 3bc - 18,\ \alpha_2 = b\sqrt{\Delta} - (b^2 - 3bc - 18),\ \alpha_3 = 2b\sqrt{\Delta}, \alpha_4 = -2b\sqrt{\Delta}$ and $\Delta = b^2 + 30bc + 9c^2 + 72.$

The first integral (4.3) was constructed by using the identity (2.3) with cofactors

$$K_{1,2} = [(3c - b - 3A)x^2 + 3A(c - A)xy + 3x + 3by^2 + 3Ay]/3,$$

$$K_3 = [3by^2 - (2b + 3c)x^2 - (3bc + 3)xy - 3by - 3x]/3,$$

$$K_\Phi = 2y(by - bcx - 2x - b).$$

Theorem 4.1. *The origin $O(0,0)$ is a center for cubic system (1.1) with two invariant straight lines and one invariant elliptic cubic curve (3.1) if and only if the first two Lyapunov quantities vanish.*

Proof. By using the algorithm described in [4], we compute the first two Lyapunov quantities L_1, L_2 for each set of conditions (i)–(viii) of Theorem 3.1. In the expressions for L_j we will neglect the denominators and non-zero factors.

In Cases (i) and (ii) the first two Lyapunov quantities vanish. Then we obtain the center conditions (1) and (2) of Lemma 4.1.

In Case (iii) the first two Lyapunov quantities vanish. Then Lemma 4.2.

In Case (iv) the first two Lyapunov quantities vanish. Then Lemma 4.1, (3).

In Case (v) the vanishing of L_1 gives $c = (-5b)/3$ and the second Lyapunov quantity looks $L_2 = ab^3 \neq 0$. In this case a singular point $O(0,0)$ is a focus.

In Case (vi) the first two Lyapunov quantities vanish. Then Lemma 4.3.

In Case (vii) the vanishing of the first Lyapunov quantity gives $c = (-4b)/3$. The second one looks $L_2 = 5b^2 + 9 \neq 0$ and therefore a singular point $O(0,0)$ is a focus.

In Case (viii) the first Lyapunov quantity looks $L_1 = b \neq 0$. In this case a singular point $O(0,0)$ is a focus. Theorem 4.1 is proved.

References

1. Romanovski V.G and Shafer D.S. The center and cyclicity problems: a computational algebra approach. Boston, Basel, Berlin: Birkhäuser, 2009.
2. Sadovskii A.P. Polynomial ideals and varieties. Minsk: BGU, 2008. (in Russian).
3. Amel'kin V.V., Lukashevich N.A., Sadovskii A.P. Non-linear oscillations in the systems of second order. Minsk: Belarusian University Press, 1982. (in Russian).
4. Cozma D. Integrability of cubic systems with invariant straight lines and invariant conics. Chișinău: Știința, 2013. 240 p.
5. Schlomiuk D. Algebraic particular integrals, integrability and the problem of centre. In: Trans. of the Amer. Math. Soc., 1993, vol. 338, no. 2. pp. 799-841.
6. Cozma D. Darboux integrability and rational reversibility in cubic systems with two invariant straight lines. In: E. J. of Diff. Equa., 2013, vol. 2013, no. 23. pp. 1-19.
7. Cozma D. The problem of the center for cubic systems with two parallel invariant straight lines and one invariant cubic. In: Romai Journal, 2015, vol. 11, no. 2. pp. 63-75.
8. Sadovskii A.P., Shcheglova T.V. Solution of the center-focus problem for a nine-parameter cubic system. In: Differential Equations, 2011, vol. 47, no. 2. pp. 208-223.