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THE CARTESIAN PRODUCT OF TWO SUBCATEGORIES

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Abstract. We examine a categorial construction which permits to obtained a new reflective subcategorie with a special properties.

Key words: Reflective subcategories, pairs of conjugated subcategories, right product of the two subcategories.
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PRODUSUL CARTEZIAN A DOUĂ SUBCATEGORII

Rezumat. Se examinează o construcţie categorială care permite de a obţine noi subcategorii reflective cu anumite proprietăţi.

Cuvinte-cheie: Subcategorii reflective, perechi de subcategorii conjugate, produsul de dreapta a două subcategorii.

Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ a reflective subcategory of the category of locally convex topological vector Hausdorff spaces $\mathcal{C}_2\mathcal{V}$ with respective functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$.

Concerning the terminology and notation see [1]. Note by $\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid k(m) \in \mathcal{I}so\}$, $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so\}$. Further for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ we examine the following construction: let $k^X : kX \longrightarrow X$ is $\mathcal{K}$-coreplique, and $r^{kX} : kX \longrightarrow rkX$-replique of the respective objects. On the morphisms k^X and r^{kX} we construct the cocartesian square

$$\overline{v}^X \cdot k^X = u^X \cdot r^{kX}. \tag{1}$$

Definition 1. *1. The full subcategory of all isomorphic objects with the type of objects is called* $\overline{v}X$ *cartesian product of the subcategories* $\mathcal{K}$ *and* $\mathcal{R}$*, noted by* $\overline{v} = \mathcal{K} *_{dc} \mathcal{R}$.
2.The diagram of cartesian product is called the diagram of cartesian product of the pair of conjugate subcategories $(\mathcal{K}, \mathcal{R})$ *(Diagram (RCP)).*

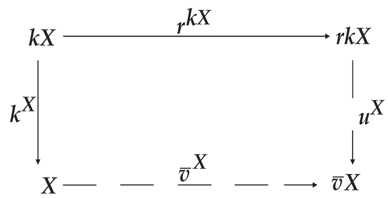


Diagram (RCP)

Reciprocally. Let $\mathcal{R}$ be a reflective subcategory, and $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. Let X be an object of the category $\mathcal{C}_2\mathcal{V}$, $r^X : X \longrightarrow rX$-$\mathcal{R}$-replique and $k^{rX} : krX \longrightarrow rX$ be $\mathcal{K}$-coreplique of the respective objects. On the morphisms r^X and k^{rX} we construct the cartesian product

$$r^X \cdot \overline{w}^X = k^{rX} \cdot t^X \tag{2}$$

Definition 2. *1. The full subcategory of all isomorphic objects with the objects of type* $\overline{v}X$ *is called cartesian product of the subcategories* $\mathcal{K}$ *and* $\mathcal{R}$*, noted* $\overline{\mathcal{W}} = \mathcal{K} *_{sc} \mathcal{R}$.

2. The diagram of the cartesian square (2) is called the diagram of the left cartesian product of the pair of conjugate subcategories $(\mathcal{K}, \mathcal{R})$ *(Diagram (LCP)).*

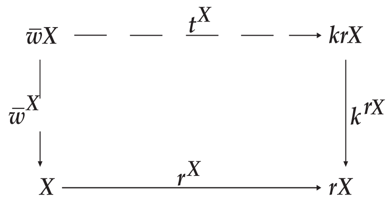
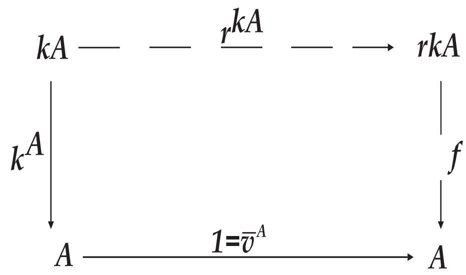


Diagram (LCP)

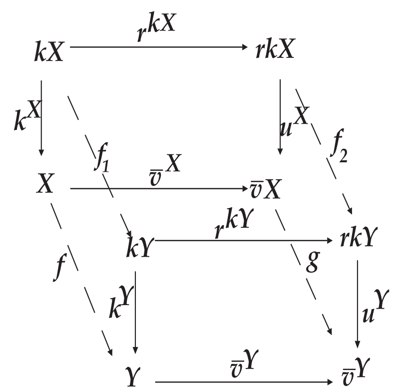
Lemma 1. $\mathcal{R} \subset \mathcal{K} *_{dc} \mathcal{R}$.
Proof. Let $\mid \mathcal{A} \mid \in \mid \mathcal{R} \mid$ and $k^A : kA \longrightarrow A$ be $\mathcal{K}$-coreplique, $r^{kA} : kA \longrightarrow rkA$, $\mathcal{R}$-replique of the respective objects. Then $k^A = f \cdot r^{kA}$ for an morphism f. It is obvious that $f \cdot r^{kA} = 1 \cdot k^A$ is cocartesian square construct on the morphisms k^A and r^{kA}. So $\overline{v}^A = 1$.



Theorem 1. *The application* $X \mapsto \overline{v}X$ *define a functor*

$$\overline{v} : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K} *_{dc} \mathcal{R}.$$

Proof. We define the functor $\overline{v}$ on the morphism. Let $f : X \longrightarrow Y \in \mathcal{C}_2\mathcal{V}$. We examine the diagram (RCP) constructed for the objects X and Y.



For the morphism $f \cdot k^X$ exists one single morphism $f_1 : kX \longrightarrow kY$ so that

$$f \cdot k^X = k^Y \cdot f_1. \tag{3}$$

The same for the morphism $r^{kY} \cdot f_1$ exists one single morphism $f_2 : rkX \longrightarrow rkY$. It follows that

$$r^{kY} \cdot f_1 = f_2 \cdot r^{kX}. \tag{4}$$

Then we have

$$\overline{v}^Y \cdot f \cdot k^X = (from3) = \overline{v}^Y \cdot k^Y \cdot f_1 = (from1) = u^Y \cdot r^{kY} \cdot f_1 = (from4) = u^Y \cdot f_2 \cdot r^{kX}$$

or

$$(\overline{v}^Y \cdot f) \cdot k^X = (u^Y \cdot f_2) \cdot r^{kX}. \tag{5}$$

From equality (5) concerning that (3) is cocartesian square, it results the existence of a single morphism g, such that

$$\overline{v}^Y \cdot f = g \cdot \overline{v}^X, \tag{6}$$

$$u^Y \cdot f_2 = g \cdot u^X. \tag{7}$$

Define $g = t(f)$. In equality (6) $\overline{v}^Y$ is an epimorphism. Thus, we deduce that the morphism g verifing equality (6), is unique. And here we come out with the result $\overline{v}(1) = 1$ and $\overline{v}(f \cdot h) = \overline{v}(f) \cdot \overline{v}(h)$.

Concerning the functor $\overline{v} : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K} *_{dc} \mathcal{R}$ appears the following problem: When $\overline{v}$ is a reflector functor?

We examine the following condition:

(RCP) For any object X of the category $\mathcal{C}_2\mathcal{V}$ in the diagram (RCP) the morphism u^X belongs to the class $\mu\mathcal{K}$.

Theorem 2. *Let it be a pairs of the subcategories* $(\mathcal{K}, \mathcal{R})$ *verify the condition (RCP). Then* $\overline{v}$ *it is a reflector functor.*

Proof. We examine the diagram (RCP) constructed for objects X and Y of the category $\mathcal{C}_2\mathcal{V}$. Let $f : X \longrightarrow \overline{v}Y$. Since $u^Y \in \mu\mathcal{K}$, it follows that

$$f \cdot k^X = u^Y \cdot g \tag{8}$$

for a morphism g. Further, r^{kX} is $\mathcal{R}$-replique of object kX. So

$$g = h \cdot r^{kX} \tag{9}$$

for a morphism h. We have

$$f \cdot k^X = (from8) = u^Y \cdot g = (from9) = u^Y \cdot h \cdot r^{kX}$$

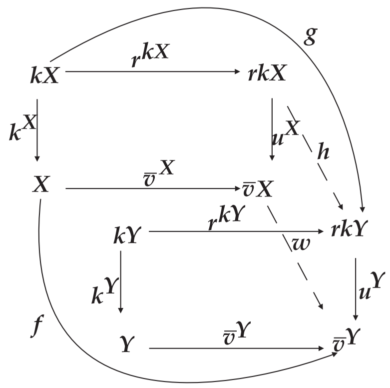
or

$$f \cdot k^X = (u^Y \cdot h) \cdot r^{kX}. \tag{10}$$

I mean that square (1) is cocartesian, we deduce that:

$$f = w \cdot \overline{v}^X, \tag{11}$$

$$u^Y \cdot h = w \cdot u^X. \tag{12}$$



So morphism f extends through morphism $\overline{v}^X$. The uniqueness of this extension results from the fact that $\overline{v}^X$ it is like r^{kX} an epimorphism.

Theorem 3. *Let $\mathcal{K}$ be a coreflective subcategory, but $\mathcal{R}$ is a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$, $\widetilde{\mathcal{M}}$ - the subcategory of the spaces with Mackey topology, $\mathcal{S}$ is the subcategory of the spaces with weak topology. If $\mathcal{K} \subset \widetilde{\mathcal{M}}$, but $\mathcal{S} \subset \mathcal{R}$, then the pair of subcategories $(\mathcal{K}, \mathcal{R})$ verify condition (RCP) the cartesian product is a reflective subcategory.*

Proof. Since $\mathcal{S} \subset \mathcal{R}$, it follows that $\varepsilon\mathcal{R} \subset \varepsilon\mathcal{S} = \mathcal{E}_u \cap \mathcal{M}_u = \mu\widetilde{\mathcal{M}} \subset \mu\mathcal{K}$. We examine the diagram (RCP) for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$. We have $r^{kX} \in \varepsilon\mathcal{R}$. So, and $\overline{v}^X \in \varepsilon\mathcal{R}$. Thus $\overline{v}^X$, $k^X \in \mu\mathcal{K}$. On the other hand $\overline{v}^X \cdot k^X \in \mu\mathcal{K}$. In equality

$$\overline{v}^X \cdot k^X = u^X \cdot r^{kX},$$

where r^{kX}, k^X, $\overline{v}^X$ are bijective application. In other words u^X is a bijective application. Thus $u^X \in \mu\mathcal{K}$.

Example. **1.** *For any coreflective subcategory $\mathcal{K}$ we have $\mathcal{K} *_{dc} \Pi = \Pi$, Π-reflective subcategory of the complete space with weak topology.*

2. *For any coreflective subcategory $\mathcal{K}$ we have $\mathcal{K} *_{dc} \mathcal{S} = \mathcal{S}$, $\mathcal{S}$-reflective subcategory of the space with weak topology.*

Proof. We construct the (RCP) diagram for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ in relation to the pair of subcategories $(\mathcal{K}, \Pi)$. We represent the reflector functor $\pi : \mathcal{C}_2\mathcal{V} \longrightarrow \Pi$ as a composition

$$\pi = g_0 \cdot s.$$

So either $s^{kX} : kX \longrightarrow skX$ $\mathcal{S}$-replique of the object kX, but $g_0^{skX} : skX \longrightarrow g_0skX$ is a Γ_0-replique of the object skX, where Γ_0 is subcategory of the complete space.

Thus $g_0^{skX} \cdot s^{kX}$is a replique of the object kX. We construct the cocartesian square on the morphism k^X and s^{kX}:

$$u_1^X \cdot s^{kX} = \overline{v}_1^X,$$

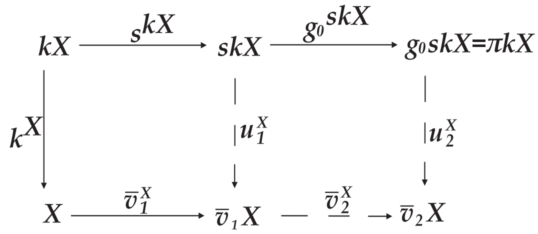
on the morphisms u_1^X and g_0^{skX}:

$$\overline{v}_2^X \cdot u_1^X = u_2^X \cdot g_0^{skX}.$$

Then

$$(\overline{v}_{1X} \cdot \overline{v}_2^X) \cdot k^X = u_2^X \cdot (g_0^{skX} \cdot s^{kX})$$

is a cocartesian square construct on morphisms k^X and $g_0^{skX} \cdot s^{kX}$ or morphisms k^X and π^{kX}.



Since k^X is an epimorphism it results as well u_1^X and u_2^X are epimorphisms. Therefore u_2^X is retractable, but $\overline{v}_2^X \in\mid \Pi \mid$. Further $g_0^{skX} \in \varepsilon\Gamma_0$. So $\overline{v}_2^X \in \varepsilon\Gamma_0$, but $\overline{v}_1^X \in\mid \mathcal{S} \mid$. Thus we have proved that $\mathcal{K} *_{dc} \Pi = \Pi$ and $\mathcal{K} *_{dc} \mathcal{S} = \mathcal{S}$.

Return to previous diagram. If for any object $X \in\mid \mathcal{C}_2\mathcal{V} \mid$ we have $u_2^X \in \mu\mathcal{K}$, then u_2^X is an isomorphism, and from equality

$$g_0^{skX} \cdot s^{kX} = (u_2^X)^{-1} \cdot \overline{v}_2^X \cdot \overline{v}_1^X \cdot k^X$$

it results that $k^X \in \mathcal{M}_u$, and $\widetilde{\mathcal{M}} \subset \mathcal{K}$.

Remark. 1. May it be $\widetilde{\mathcal{M}} \notin \mathcal{K}$. Then the pair $(\mathcal{K}, \Pi)$ do not check the condition (RCP), but $\mathcal{K} *_{dc} \Pi = \Pi$. So the condition (RCP) is sufficient, but not necessary that the respective product is a reflective subcategory.

2. Lemma 1 indicates inclusion $\mathcal{R} \subset \mathcal{K} *_{dc} \mathcal{R}$, and the preceding examples indicate the equality of these subcategories.

Definition 3 (see [1]). *Let $\mathcal{K}$ a coreflective subcategory and $\mathcal{L}$ a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ with those functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$. Pair $(\mathcal{K}, \mathcal{L})$ is called a pair of conjugate subcategories if*

$$\mu\mathcal{K} = \varepsilon\mathcal{L}.$$

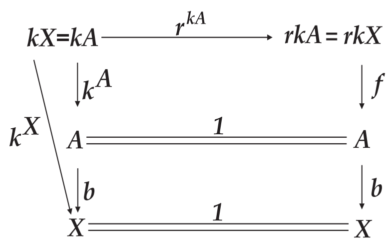
Theorem 4. *Let $(\mathcal{K}, \mathcal{L})$ a pair of conjugate subcategories, and $\mathcal{R}$ a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. Then:*

1. *$\mathcal{K} *_{dc} \mathcal{R} = \mathcal{Q}_{\varepsilon\mathcal{L}}(\mathcal{R})$, where $\mathcal{Q}_{\varepsilon\mathcal{L}}(\mathcal{R})$ is the full subcategory of all $\varepsilon\mathcal{L}$-factorobjects of objects of the subcategory $\mathcal{R}$.*
2. *$\mathcal{K} *_{dc} \mathcal{R}$ is a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$.*
3. *The subcategory $\mathcal{K} *_{dc} \mathcal{R}$ is closed in relation to $\varepsilon\mathcal{L}$-factorobjects.*
4. *$\overline{v} \cdot k = r \cdot k$.*
5. *If $r(\mathcal{K}) \subset \mathcal{K}$, then the coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and the reflector $\overline{v} : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K} *_{dc} \mathcal{R}$ commute: $k \cdot \overline{v} = \overline{v} \cdot k$.*

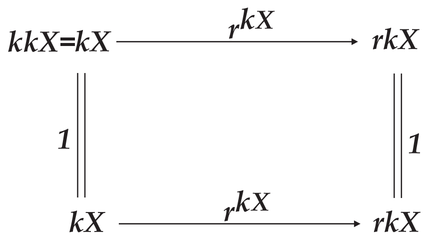
Proof. 1. In the (RCP) diagram $k^X \in \mu\mathcal{K} = \varepsilon\mathcal{L}$. So $k^X \in \varepsilon\mathcal{L} = \mu\mathcal{K}$. Thus $\mathcal{K} *_{dc} \mathcal{R} \subset \mathcal{Q}_{\varepsilon\mathcal{L}}(\mathcal{R})$. *Reciprocally*: Let $b : A \longrightarrow X \in \varepsilon\mathcal{L}$ and $A \in\mid \mathcal{R} \mid$. Then $b \cdot k^A : kA \longrightarrow X$ is $\mathcal{K}$-coreplique of the object X, and

$$k^A = f \cdot r^{kA} \tag{13}$$

is an cocartesian square construct on the morphisms k^X and r^{kX}. So $X \in\mid \mathcal{K} *_{dc} \mathcal{R} \mid$.



2. Result from 1. and the Theorem 2.
3. Result from 1.
4. For an object of form kX, diagram (RCP) is the next one



Thus $\overline{v}kX = rkX$.

5. Examine the diagram (RCP) construct for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$. Then k^{rX} it is also $\mathcal{V}$-replique of the object kX. Further, $u^X \in \mu\mathcal{K}$ and $rkX \in \mid \mathcal{K} \mid$, according to the hypothesis $r(\mathcal{K}) \subset \mathcal{K}$. So

$$k\overline{v}X = rkX = \overline{vkX}$$

or

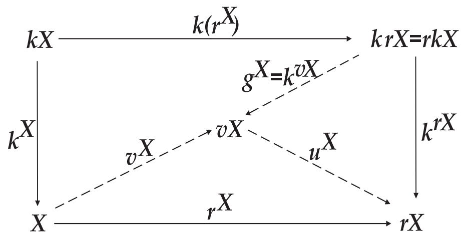
$$k \cdot \overline{v} = \overline{v} \cdot k.$$

In the paper [2] was introduced the right product of the product $\mathcal{K}*_d\mathcal{R}$ of the coreflective subcategory $\mathcal{K}$ and of the reflective subcategory $\mathcal{R}$, the properties of this product have been examined and examples have been construct.

Theorem 5. *Let $\mathcal{K}$ (respective $\mathcal{R}$) a coreflective subcategory (respective: reflective)of the category $\mathcal{C}_2\mathcal{V}$, those functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $k \cdot r = r \cdot k$. Then*

$$\mathcal{K} *_{dc} \mathcal{R} = \mathcal{K} *_d \mathcal{R}.$$

Proof. Let's examine the diagram of the right product constructed for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ in relation to the subcategories $\mathcal{K}$ and $\mathcal{L}$.



Because functors k and r commute, be sure to verify that $k(r^X) = r^{kX}$. Thus, the right product is obtained by constructing the cocartesian square on morphisms k^X and $k(r^X)$, and the right cocartesian product is obtained by constructing the cocartesian square on morphisms k^X and r^{kX}. So these products coincide.

References

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2. Botnaru D., Turcanu A. The factorization of the right product of two subcategories. ROMAI Journal, v.6, N.2, 2010, p. 41-53.