AROUND THE POINCARÉ CENTER-FOCUS PROBLEM

Mitrofan M. CIOBAN, Academician
Tiraspol State University

Summary . It is well known that many mathematical models use differential equation systems and apply the qualitative theory of differential equations, introduced by Poincaré and Liapunoff. One of the problems that persists in order to control the behavior of systems of this type, is to distinguish between a focus or a center (the Center-Focus Problem). The solving of this problem goes through the computation of the Poincaré–Liapunoff quantities. The problem of estimating the maximal number of algebraically independent essential constants is called the Generalized Center-Focus Problem. The present article contains: some moments related to the history of the Center-Focus Problem; the contribution of the Academician C. Sibirschi's school in the solving of the Center-Focus Problem; methodological aspects of the M. N. Popa and V. V. Pricop solution of the Generalized Center-Focus Problem.

Key words: Poincaré-Liapunoff quantities, center-focus problem, generalized center-focus problem, Krull dimension, sober spaces.
2010 Mathematics Subject Classification: 34C05, 34C07

REFLECŢII ASUPRA PROBLEMEI LUI POINCARÉ DESPRE CENTRU ŞI FOCAR

Rezumat. Multe modele matematice folosesc sisteme de ecuaţii diferenţiale şi aplică teoria calitativă a ecuaţiilor diferenţiale, elaborată de Poincaré şi Liapunoff. Una din probleme ce persistă în studiul acestor sisteme constă în determinarea condiţiilor care asigură că punctul singular este un centru (Problema Centrului şi Focarului). Problema Generalizată a Centrului şi Focarului constă în estimarea de sus a numarului de elemete algebric independente din careva sistem complet de condiţii esenţiale. Problema Generalizată a Centrului şi Focarului a fost rezolvată de M. N.Popa şi V. V. Pricop. În articolul prezent: se expun unele momente din istoria rezolvării Problemei Centrului si Focarului; se mentionează contribuţia şcolii acad. C. Sibirschi la rezolvarea Problemei Centrului şi Focarului; se analizează aspectele metodologice ale soluţiei propusă de M. N.Popa şi V. V. Pricop.

Cuvinte-cheie: constantele Poincaré-Liapunoff, Problema Centrului şi Focarului, Problema Generalizată a Centrului şi Focarului, dimensiunea Krull, spaţiu sobru.

1. Introduction

Mathematical research has helped to solve a number of problems that have sprouted the scientists' minds for almost 2500 years, starting with Plato, Aristotle, Euclid, Archimedes. The nineteenth century brought to human civilization several surprising discoveries. Much of them is the result of the logical analysis and, in general, of the mathematical analysis of phenomena: Gauss discovered through calculus the asteroids Ceres, Palass, Vesta, Iunona; Galle also, based on the calculations, identified the planet Neptune; Mendeleev, starting from the atomic table, systematized the chemical elements and anticipated the existence of many new ones; Schliemann, based on Homer's descriptions, determined the place of Troy's placement, etc. At the end of the nineteenth century, the genius French mathematician Jules Henri Poincaré (1854 – 1912) created new areas of research such as topology, qualitative theory of dynamic systems, etc. We mention that by the quantitative methods, the Romanian mathematician Spiru Haret (1851 – 1912) demonstrated in 1878 the instability of the Solar System. He made a fundamental contribution to the n-body problem in celestial mechanics. Haret's major scientific contribution was made in 1878, in his Ph.D. thesis "Sur linvariabilité des grandes axes des orbites planétaires". At the time it was known that planets disturb each others orbits, thus deviating from the elliptic motion described by Johannes Kepler's First

Law. Pierre Laplace (in 1773) and Joseph Louis Lagrange (in 1776) had already studied the problem, both of them showing that the major axes of the orbits are stable, by using a first degree approximation of the perturbing forces. In 1808 Siméon Denis Poisson had proved that the stability also holds when using second degree approximations. In his thesis, Haret proved by using third degree approximations that the axes are not stable as previously believed, but instead feature a time variability, which he called secular perturbations. This result implies that planetary motion is not absolutely stable. Henri Poincaré considered this result a great surprise and continued Haret's research, which eventually led him to the creation of chaos theory and qualitative theory of dynamic systems [10, 19].

Henri Poincaré formulated a series of important problems, the solution of which determines the further development of mathematical sciences. One of them is the the Poincaré conjecture about the characterization of the 3-sphere, which is the hypersphere that bounds the unit ball in four-dimensional space. In 2000, it was named one of the seven Millennium Prize Problems, for which the Clay Mathematics Institute offered one million dollars prize for the first correct solution. The enigmatic Russian mathematician Grigori Perelman presented a proof of the conjecture in three papers made available in 2002 and 2003 on arXiv. On 22 December 2006, the scientific journal Science recognized Perelman's proof of the Poincaré conjecture as the scientific "Breakthrough of the Year", the first such recognition in the area of mathematics.

One of the famous problems of the qualitative theory of differential equations is the Center-Focus Problem, formulated by Poincaré about 135 years ago, in period 1881-1885 [10]. The Center-Focus Problem consists in distinguishing when a monodromic singular point is either a center or a focus. The Center-Focus Problem arises many open questions and it has deep links with Hilbert's 16th Problem.

Hilbert's 16th problem was posed by David Hilbert (1862 – 1943) at the Paris International Congress of Mathematicians in 1900, as part of his list of 23 problems in mathematics (see [4, 1, 6]). The original problem was posed as the Problem of the topology of algebraic curves and surfaces. Actually the problem consists of two similar problems in different fields of mathematics:

1. An investigation of the relative positions of the branches of real algebraic curves of degree n.

2. The determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n and an investigation of their relative positions.

In 1976, Academician Constantin Sibirschi (Sibirsky) (1928 – 1990), Head of Laboratory at the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova, founder of the scientific school of differential equations in the Republic of Moldova, published the monograph "Algebraic Invariants of Differential Equations and Matrices" (see [16, 15]), which had a great resonance in the world of mathematicians. Over three years in 1979, Professor C.S. Coleman has published a review of this scientific paper, in which he stated that it is written in the spirit of the research of Norwegian mathematician Marius Sophus Lie (1842 – 1899). Marius Sophus Lie obtained his PhD at the University of

Christiania (present day Oslo) in 1871 with the thesis entitled "Over en Classe Geometriske Transformationer". He created the theory of continuous symmetry, introducing the concept of algebra, those bearing his name today, and applied it to the study of geometry and differential equations. It would be described by Darboux as "one of the most handsome discoveries of Modern Geometry".

The mathematician Mihail Popa, who was a student of the Professor C. Sibirschi, went his own way, starting from establishing the link between the Lie algebras and the graduated algebra of Sibirschi invariants – a tool for further researches. M. Popa took as a basis the Generalized Center-Focus Problem for Polynomial Differential Systems, avoiding calculating Poincaré–Lyapunoff quantities for each system. Poincaré–Lyapunoff's quantities was substituted by a sequence of Lie algebras and a sequence of linear subspaces of the graduate algebra of Sibirsky's invariants (see [17, 13, 14]). When estimating the maximum number of algebraically independent focal constants, he applied these algebras. As a result, a finite numerical estimation was obtained for independent algebraic focal quantities, participating in the solving of the generalized Center–Focus Problem for any polynomial differential system (see Theorem 1). Currently, Professor Mihail Popa, along with his disciples, continues his research in the theory of polynomial differential systems, successfully using Lie algebras. An analysis of the activity of Professor Mihail Popa is contained in article [2].

2. The Center-Focus Problem

Consider the differential system

$$dx/dt = P(x, y),\ dy/dt = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials that contain the linear part and satisfy the conditions $P(0, 0) = Q(0, 0) = 0$. The coefficients of polynomials $P(x, y)$, $Q(x, y)$ and variables from the system (1) takes values from the field of the real numbers $\mathbb{R}$. It is known [7, 10] that the conditions which distinguish center from focus for the system (1) consist in study of an infinite sequence of polynomials (focal quantities, Lyapunoff constants, Poincaré–Lyapunoff quantities (constants))

$$L_1,\ L_2,\ ...,\ L_k,\ ... \tag{2}$$

in the coefficients of the polynomials from the right side of the system (1).

It was shown that if the focal quantities (2) are equal to zero then the origin of coordinates for the system (1) is a center, i.e. the trajectories near this point are closed. On the contrary the origin of coordinates is a focus and the trajectories are spirals.

We can assume that $P(x, y) = \Sigma\{P_{m_i} : i \in \{0, 1, 2, ...l\}\}$ and $Q(x, y) = \Sigma\{Q_{m_i} : i \in \{0, 1, 2, ...l\}\}$, where P_{m_i} and Q_{m_i} are homogeneous polynomials of degree $m_i \geq 1$ in x and y, $m_0 = 1$. In this case we denote the system (1) by $s(1, m_1, m_2, ..., m_l)$

It is known that if the roots of characteristic equation of the singular point $O(0, 0)$ of the system (1) are imaginary, then the singular point O is a center or a focus. In this case the origin of coordinates is a singular point of the second type.

The Center-Focus Problem can be formulated as follows: *Let for the system $s(1, m_1, m_2, ..., m_l)$ the origin of coordinates be a singular point of the second type (center or focus). Find the conditions which distinguish center from focus.* This problem was posed by H. Poincaré [10]. The basic results were obtained by A. M. Lyapunoff (1857 – 1918) [7].

It is well known that, from the Hilbert's theorem on the finiteness of basis of polynomial ideals, for any concrete system $s(1, m_1, m_2, ..., m_l)$ the set

$$PL(1, m_1, m_2, ..., m_l) = \{i \in \mathbb{N} = \{1, 2, ...\} : L_i \neq 0\} \quad (3)$$

is finite. Assume that

$$PL(1, m_1, m_2, ..., m_l) = \{n_1, n_2, ..., n_\beta\} \quad (4)$$

and

$$n_\alpha = n_1 < n_2 < ... < n_\beta. \quad (5)$$

The Poincaré's Center-Focus Problem determines the following problems:

P1. *The problem of finding the number n_α or obtaining for it an argued numerical upper bound.*

P2. *The problem of finding the number n_β or obtaining for it an argued numerical upper bound.*

P3. *The problem of finding the number β or obtaining for it an argued numerical upper bound.*

Problems P1 and P2 are open. Solution of the Problem P1 contains a solution of the Center-Focus Problem. Positive solution of Problem P2 contains the solution of Problem P1. Hence Problem P2 is the strong Center-Focus Problem. Problem P3 is the weakly Center-Focus Problem.

Denote by $\mathcal{D}$ the set of all systems (1). Since the Center-Focus Problem is very complicated, it presents interest the following problem: *Finding the subsets $\mathcal{H}$ of the set $\mathcal{D}$ for which Problems P1–P3 (or some of them) are positive solutions.* Monographs [16, 18, 11, 14, 12, 3] contain some results of that kind. The Center-Focus Problem is solved for the class QS of all quadratic systems (see [16, 18, 17, 15]). Using global geometric concepts, was completely studied the class $QW3$ of quadratic systems with a third order weak focus (see [15]). The class $QW2$ of all quadratic differential systems with a weak focus of second order is important for Hilbert's 16th problem (see [15, 1, 6]). Are important (see [3, 15]) the classes:

- the class of dynamical systems with special invariant algebraic curves;

- the class of dynamical systems with a Darboux first integral or a Darboux integrating factor.

3. Sibirschi graded algebras

C. S. Sibirschi (see [13, 11, 14]), for any system $s(1, m_1, m_2, ..., m_l)$, were introduced the graded algebra $SI = SI(1, m_1, m_2, ..., m_l)$ of unimodular invariants and the graded algebra $S = S(1, m_1, m_2, ..., m_l)$ of comitants of the system $s(1, m_1, m_2, ..., m_l)$. Obviously $SI(1, m_1, m_2, ..., m_l) \subset S(1, m_1, m_2, ..., m_l)$.

The maximal number of algebraically independent elements of the Sibirsky graded algebra S is denoted by $\rho(S)$.

Let R be a finitely generated algebra over a field K. By the virtue of Krull's theorem the maximum number of elements of R that are algebraically independent over K is the same as the Krull dimension of R. Hence $\rho(S)$ is the Krull dimension of the Sibirschi algebra S.

A natural question is of course: *Which properties of $s(1, m_1, m_2, ..., m_l)$ are described in $SI(1, m_1, m_2, ..., m_l)$ and $S(1, m_1, m_2, ..., m_l)$?* In particular, the following problem may be considered as the generalized Poincaré Center-Focus Problem (see [13, 2, 14]):

P4. *The problem of finding the number* $\rho(S(1, m_1, m_2, ..., m_l))$.

In [13, 14] was proved the following unexpected assertion.

Theorem 1. $\rho(S(1, m_1, m_2, ..., m_l)) = 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$ *for any system* $s(1, m_1, m_2, ..., m_l)$.

In this context, in [13] was formulated the following

Conjecture. $\beta \leq 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$ *for any system* $s(1, m_1, m_2, ..., m_l)$.

Present interest the following open question

P5. *For which n there exist two polynomials* $P(x, y) = \Sigma\{P_{m_i} : i \in \{0, 1, 2, ...l\}\}$ *and* $Q(x, y) = \Sigma\{Q_{m_i} : i \in \{0, 1, 2, ...l\}\}$ *for which:*

- $P(x, y)$ *is a polynomial of degree* n_1, $Q(x, y)$ *is a polynomial of degree* n_2 *and* $n = maximum\{n_1, n_2\}$;

- $\beta = 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$.

4. Krull's dimension of spaces

Any space X is considered to be a Kolmogorov space, i.e. for any two distinct points $x, y \in X$ there exists an open subset U of X for which the intersection $U \cap \{x, y\}$ is a singleton set.

A subset F of a space X is called an irreducible subset if for any two closed subsets F_1, F_2 of X for which $F \subset F_1 \cup F_2$ we have $F \subset F_i$ for some $i \in \{1, 2\}$. The closure $cl_X\{x\}$ of the singleton set $\{x\}$ is irreducible. A sober space is a topological space X such that every non-empty irreducible closed subset of X is the closure of one point of X. If $F = cl_X\{x\}$, then x is a generic point of the set F. A non-empty irreducible subset has a unique generic point.

Denote by $|L|$ the cardinality of a set L.

The following assertion is obvious.

Proposition 1. *A subset* L *of a space* X *is irreducible if and only if the its closure* $cl_X L$ *is irreducible.*

Example 1. Let $X = \{1, 2, 3\}$ with the topology $\{\varnothing, X, \{2\}, \{1, 2\}, \{2, 3\}\}$. Then X is a sober irreducible space and the closed subspace $Y = \{1, 3\}$ is discrete and not irreducible.

A closed subspace of a sober space is a sober space.

Example 2. Let $\omega = \{0, 1, 2, ..., n, ...\}$ and $X = \{0, 1, 2, ..., n, ..., \omega\}$ with the topology $\{\varnothing, X\} \cup \{X \setminus F : F$ *is a finite subset of* $\omega\}$. Then X is a sobre irreducible space and the subspace $Y = \{0, 1, 2, ..., n, ...\}$ is irreducible and not sober.

Define the Krull dimension $dk(X)$ of a space X to be the maximum n such that there exists a chain of pairwise distinct non-empty irreducible closed sets F_0, F_1, F_2 ..., F_n such that $F_0 \subset F_1 \subset F_2 \subset ... \subset F_n$. If Y is an irreducible closed subset of X the Krull co-dimension *co-*$dk_X(Y)$ of Y in X is the supremum over all n such that there is a chain of pairwise distinct non-empty irreducible closed sets F_0, F_1, F_2 ..., F_n such that $Y \subset F_0 \subset F_1 \subset F_2 \subset ... \subset F_n$. We observe that $dk(X) =$ *co-*$dk_X(\varnothing)$. We can assume that $dk(X) = -1$ for $X = \varnothing$.

From Proposition 1 it follows that $dk(Y) \leq dk(X)$ for any subspace Y of a space X.

If $\{X_i : i \in \mathbb{N}_n = \{1, 2, ..., n\}\}$ is a finite family of closed subspaces of a space X, $n \geq 2$ and $X = \cup\{X_i : i \in \mathbb{N}_n\}$, then $dk(X) = supremum\{dk(X_i) : i \in \mathbb{N}_n\}$. This fact follows from Claim 1 in the proof of the following proposition.

Proposition 2. *Let $\{X_i : i \in \mathbb{N}_n\}$ be a finite family of subspaces of a space X, $n \geq 2$ and $X = \cup\{X_i : i \in \mathbb{N}_n\}$. Then:*

1. If F is a closed irreducible subset of X, then there exists $i \in \mathbb{N}_n$ such that $F_i = F \cap X_i$ is an irreducible subset of the spaces X_i and X, and $F = cl_X F_i$.

2. If F is a closed irreducible subset of X, $i \in \mathbb{N}_n$, $F_i = F \cap X_i$ and $F = cl_X F_i$, then F_i is an irreducible subset of the spaces X_i and X.

3. $dk(X) = \Sigma\{dk(X_i) : i \in \mathbb{N}_n\}$.

Proof. In the first we prove the following assertion.

Clam 1. *Let F be an irreducible subset of the space X, γ is a finite family of closed subsets of X and $F \subset \cup\gamma$. Then $F \subset Y$ for some $Y \in \gamma$.*

The assertion follows from the definition for $|\gamma| \leq 2$. Assume that $k > 2$ and the assertion is true provided $|\gamma| < k$. Fix a collection γ of closed subsets of X for which $|\gamma| = k$ and $F \subset \cup\gamma$. Now fix $Y \in \gamma$ and put $\gamma_1 = \gamma \setminus \{Y\}$. We have two possible cases.

Case 1. $F \subset \cup\gamma_1$.

Since $\gamma_1| = k - 1 < k$, there exists $Z \in \gamma_1$ such that $F \subset Z$.

Case 2. $F \not\subset \cup\gamma_1$.

We put $Z = \cup\gamma_1$. Then $F \subset Z \cup Y$ and $F \not\subset Z$. Hence $F \subset Y \in \gamma$. The proof of Claim 1 is complete.

Clam 2. *If F is a closed irreducible subset of X, then there exists $i \in \mathbb{N}_n$ such that $F = cl_X(F \cap X_i)$.*

We put $F_i = F \cap X_i$ and $\Phi_i = cl_X F_i$. Then $\gamma = \{\Phi_i : i \in \mathbb{N}_n\}$ is a finite family of closed subsets of X and $F \subset \cup\gamma$. Thus $F \subset \Phi_i$ for some $i \in \mathbb{N}_n$. Claim is proved.

Assertion 2 follows from Proposition 1.

Fix a chain of pairwise distinct non-empty irreducible closed sets F_0, F_1, F_2 ..., F_m of the space X such that $F_0 \subset F_1 \subset F_2 \subset ... \subset F_n$. We put $F_{ij} = F_j \cap X_i$. Let $A_i = \{j : 0 \leq j \leq m, F = cl_X F_{ij}\}$ and $m_i = |A_i|$. Then $m_i \leq dk(X_i)$ and, by virtue of assertions 1 and 2, we have $m \leq \Sigma\{m_i : i \in \mathbb{N}_n\} \leq \Sigma\{dk(X_i) : i \in \mathbb{N}_n\}$. Assertion 3 is proved. The proof is complete.

Proposition 3. *Let $\{X_i : i \in \mathbb{N}_n\}$ be a finite family of sober subspaces of a space X, $n \geq 2$ and $X = \cup\{X_i : i \in \mathbb{N}_n\}$. Then:*

1. If F is a closed irreducible subset of X, $i \in \mathbb{N}_n$, $F_i = F \cap X_i$ and $F = cl_X F_i$, then F_i is an irreducible subset of X_i and the generic point $x \in X_i$ of F_i in X_i is a generic point of F in X.

2. X is a sober space.

Proof. Assertion 1 follows from Proposition 1. Assertion 2 follows from assertions 1 and Proposition 2.

5. Spectrum of a ring

Let R be a commutative ring [9, 5]. A subset I of R is called an ideal of R if:

1. $(I, +)$ is a subgroup of the group $(R, +)$.

2. $R \cdot I \subset I$.

3. If R is an algebra over field K, then $K \cdot I \subset I$ for any ideal I of R.

An ideal I of R is said to be prime ideal if $x, y \in R$ and $x \cdot y \in I$ implies $I \cap \{x, y\} \neq \varnothing$.

The set of all prime ideal of R, is denoted by $Spec(R)$, is called spectrum of the ring R. Let A be an ideal of R and let $V(A)$ be the collection of all prime ideal contains A. The collection of all $V(A)$ satisfies the axioms of closed subsets of a topology for $Spec(R)$, called the Zariski topology for $Spec(R)$. The space $Spec(R)$ is a compact Kolmogorov space.

For any commutative ring R and $m \in \mathbb{N}$ the following assertions are equivalent:

1. $dk(Spec(R) = m$.

2. If $I_0 \subset I_1 \subset ... \subset I_n$ is a chain of distinct prime ideals of R, then $n \leq m$.

From Theorem 1 it follows that $dk(S(1, m_1, m_2, ..., m_l)) = 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$ for any system $s(1, m_1, m_2, ..., m_l)$.

6. Representation of a class of problems

The problem of determining the finite numbers n_α, n_β and β (see (5) in Section 2), or obtaining for them some numerical boundaries from the top, is important for the complete solution of the Center-Focus Problem. Obviously the Center-Focus Problem is a difficult one. So far, no general methods have been found for studying the Poincaré-Liapunoff quantities (2). In particular, there is no a general strategy to solve. Another impediment is the enormous calculations that can not be overcome by the modern supercomputers, even for the system $s(1, 2, 3)$, not to mention more complicated systems. From a psychological point of view, there are also impediments to the human conservatism to explore the problems traditionally, classically. History confirms that new, unusual methods with great difficulty are approved and valued at their fair value. However, according to Kurt Gödel's incompleteness theorem, as a rule, the resources created up to now are not sufficient for further studies. Therefore, it is undeniable that the successes of the future depend to a large extent on the newly created tools.

The study of a new problem or an unsolved problem, applying the methods of solving the known problem is done by various methods: the method of substitution of the variables; the method of crossing on limit, etc. Some of them have been well-known since ancient times and have generated new methods, appropriate to the mathematical concepts of the respective period. For example, with the method of crossing on limit, Hopf has solved the quasi-linear equations. In [8] the method of substitution of algebraic operations was successfully used in the solving of some problems of the theory of differential equations.

The principle of contrast revealed in "matter and anti-matter", "parallel spaces", "world and anti-world" penetrates into the essence of the universe, thus constituting amazing "symmetries" in the world of known phenomena. From a mathematical point of view such "symmetries" are built based on the duality principle. To build a duality means to determine a correspondence between certain types of objects, where each property of the original object corresponds to a particular property of that object in that correspondence. In any duality, their "objects" and "properties" have dual "objects" and "properties". Any concrete duality is a valuable event for these theories. The dualities in the projective geometry, the duality of Pontryagin in the theory of the local compact Abelian groups, the

Kolmogorov–Gelfand duality of compact spaces and functional Banach algebras, the dualities of Serre and Alexander in the topology, the duality of Radu Miron of the Cartan spaces and the Finsler spaces, the duality of De Morgan in the theory of sets, the Stone duality between zero-dimensional compact spaces and Boolean rings, wave-particle duality in quantum mechanics, Kramers–Wannier dualism in statistical physics, etc. This method, which is also an "anti-analogy reasoning", determines from the point of view of formal logic that many objects different in form and content are built in a similar way.

From this point of view, represent interest some correspondences of concrete class of objects of one theory into other theory. Let $\mathcal{A}$ and $\mathcal{B}$ be two theories, $\mathcal{P}$ be a class of problems of the theory $\mathcal{A}$ and $\mathcal{S}(P)$ be a set of solution of the problem $P \in \mathcal{P}$. A correspondence $\Psi : \mathcal{P} \longrightarrow \mathcal{B}$ is a representation of the class of problems $\mathcal{P}$ in the theory $\mathcal{B}$ if:

- $\Psi(P)$ is a problem of the theory $\mathcal{B}$ for any problem $P \in \mathcal{P}$;

- if $P \in \mathcal{P}$ and $\Omega \in \mathcal{S}(P)$ is a given solution of the problem P, then $\Psi(\Omega)$ is a solution of the problem $\Psi(P)$.

In this case the problem $\Psi(P)$ is a generalized form of the initial problem $P \in \mathcal{P}$. Solving generalized forms is important if for a long time there is no solution for the initial problem. Moreover, the solutions of the generalized problem propose strategies and hypotheses to solve the initial problem. Some estimates in the generalized problem solution can serve as working hypotheses for the initial problem. Furthermore, the solution to the generalized problem reflects possible ways of examining some particular cases.

Denote by $\mathcal{E}$ the theory of polynomial differential systems (1), by $\mathcal{R}$ the theory of commutative algebras and by $\mathcal{T}$ the theory of topological spaces. For any problem $s(1, m_1, m_2, ..., m_l)$ is determined the number $\{\beta\}$ as the set of solutions $\mathcal{S}(s(1, m_1, m_2, ..., m_l))$.

The correspondence $\Psi_A : \mathcal{D} \longrightarrow \mathcal{R}$, where $\Psi_A(s(1, m_1, m_2, ..., m_l)) = S(s(1, m_1, m_2, ..., m_l)$ and $\mathcal{D}$ is the set of all equations (1), is a representation of the class of problems $\mathcal{D}$ in the theory $\mathcal{R}$. We have $\Psi_A(\beta) = dk(S(1, m_1, m_2, ..., m_l))$ for any problem $s(1, m_1, m_2, ..., m_l)$ (Theorem 1).

The correspondence $\Psi_T : \mathcal{D} \longrightarrow \mathcal{T}$, where

$$\Psi_T(s(1, m_1, m_2, ..., m_l)) = Spec(S(1, m_1, m_2, ..., m_l))$$

is a representation of the class of problems $\mathcal{D}$ in the theory $\mathcal{T}$. We have

$$\Psi_a(\beta) = dk(Spec(S(1, m_1, m_2, ..., m_l)))$$

for any problem $s(1, m_1, m_2, ..., m_l)$ (Theorem 1).

Therefore the number $dk(S(1, m_1, m_2, ..., m_l)) = dk(Spec(S(1, m_1, m_2, ..., m_l)))$ is a generalized solution of the Center-Focus Problem of the system $s(1, m_1, m_2, ..., m_l)$.

References

1. Arnold V. I. et all. Some unsolved problems in the theory of differential equations and mathematical physics. Uspekhi Mat. Nauk 44 (1989), no. 4, p. 191–202 (English translation in Russian Math. Surveys 44 (1989), no. 4, p. 157–171).
2. Ciobanu M., Rotaru T. 130 years of the effort in the solving of the Poincaré's center-focus problem (130 de ani de zbucium pentru soluţionarea problemei lui Poincaré despre centru şi focar). Akademos, (2013), no. 3, p. 13–21. (In Romanian)

3. Cozma D. Integrability of Cubic Systems with Invariant Straight Lines and Invariant Conics. Chişinău: Ştiinţa, 2013.
4. Hilbert D. Mathematical Problems. Lecture delivered before the International Congress of Mathematicians at Paris in 1900, Translated for Bulletin of the AMS 8 (1902), p. 437–479.
5. Hochster M. Prime ideal structure in commutative rings. Trans. Amer. Math. Soc. 142 (1969), p. 43–60.
6. Ilyashenko Yu. Centennial History of Hilbert's 16th problem. Bulletin of the AMS 39 (2002), no. 3, p. 301–354.
7. Liapunoff A. Probléme général de la stabilité du mouvement. Annales de la Faculté des Sciences de Touluose (Sér 2) 9 (1907) p. 204–477. Reproduction in Annals of Mathematics Studies 17, Princeton: Princeton University Press, 1947, reprinted 1965, Kraus Reprint Corporation, New York.
8. Maslov V. P. On a new principle of superposition for optimization problems. Uspekhi Mat. Nauk 42 (1987), no. 3, p. 39–48 (English translation in Russian Mathematical Surveys 42 (1987), no.3, p. 43–54).
9. Matsumura H. Commutative Algebra, New York: Benjamin, 1970.
10. Poincaré H. Mémoire sur les courbes définies par une equation différentielle. J. Math. Pures et Appl.: (Sér 3) 7, (1881) p. 375–422; (Sér 3) 8 (1882) p. 251–296; (Sér 4) 1 (1885) p. 167–244; (Sér 4) 2 (1886) p. 151–217.
11. Popa M. N. Applications of algebras to differential systems. Institute of Mathematics and Computer Sciences, ASM, Chişinau, 2001, 224 p. (in Russian).
12. Popa M. N. Algebraic methods for differential systems. Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, (in Romanian).
13. Popa M. N., Pricop V. V. Applications of algebraic methods in solving the center-focus problem. Bulletin of ASM. Mathematics, 71 (2013), no.1, p. 45–71.
14. Popa M. N., Pricop V. V. The center-focus problem: algebraic solutions and hypotheses. Chişinău, 2018 (in Russian).
15. Schlomiuk D. New developments based on the mathematical legacy of C. S. Sibirschi. Buletinul ASM, Matematica (2013), no. 1, p. 3–10.
16. Sibirsky C. S. Algebraic invariants of differential equations and matrices. Kishinev, Ştiinţa, 1976 (In Russian).
17. Sibirsky C. S., Lunchevich V. A. Quadratic null systems of differential equations. Differentsialnye Uravnenya 25 (1989), no. 6, p. 1056–1058 (in Russian).
18. Vulpe N. I. Polynomial bases of comitants of differential systems and their application in qualitative theory. Kishinev, Ştiinţa, 1986 (in Russian).
19. Wikipedia. Spiru Haret. The free encyclopedia, 15.05.2018.