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SELECTIONS OF SET-VALUED MAPPINGS AND LOCALLY FINITE PROPERTIES OF MAPPINGS

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Summary. Using some methods from the works of E. Michael [10, 11, 12], T. Dobrowol J. van Mill [5] and of one of the authors [1, 2, 3], two theorems of the existence of the selections with conditions of continuity are proved.

Keywords: set-valued mapping, selection, linear space.
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SELECŢII ALE FUNCŢIILOR POLIFORME ŞI PROPRIETĂŢILE LOCAL FINITE ALE FUNCŢIILOR

Rezumat. Folosind unele metode din lucrările lui E. Michael [10, 11, 12], T. Dobrowolski şi J. van Mill [5] şi ale unui din autori [1, 2, 3], se demonstrează două teoreme de existenţă a selecţiilor cu condiţii de continuitate.

Cuvinte-cheie: funcţie poliformă, selecţie, spaţiu liniar.

1. Introduction

A single-valued mapping $f : X \longrightarrow Y$ of a space X into a space Y is said to be a selection of a given set-valued mapping $F : X \longrightarrow Y$ if $f(x) \in F(x)$ for each $x \in X$. Note that by the Axiom of Choice selections always exist. In the category of topological spaces and continuous single-valued mappings the situation is more complex.

The following problem is important: Under what conditions there exist continuous selections? There exist many theorems on continuous selections. One of them is the following classical Michael selection theorem for convex-valued mappings.

Theorem M. (E. Michael, [10]). *A multivalued mapping $F : X \longrightarrow B$ admits a continuous single-valued selection, provided that the following conditions are satisfied:*

(1) X is a paracompact space;

(2) B is a Banach space or a locally convex complete metrizable linear space;

(3) F is a lower semicontinuous mapping;

(4) for every point $x \in X$, $F(x)$ is a nonempty convex subset of B;

(5) for every point $x \in X$, $F(x)$ is a closed subset of B.

A natural question arises concerning the essentiality of each of conditions (1)-(5). There are lower semicontinuous convex-valued mappings $F : X \longrightarrow Y$ without any continuous single-valued selections, even for $X = [0; 1]$ (see Example 6.2 from [10]. An important example is published in [7]. It was proved that every convex-valued lower semicontinuous mapping mapping of a metrizable domain into a separable Banach space admits a selection, provided that all values are finite-dimensional ([10], special case of Theorem 3.1). Distinct results of this kind were proved in [4, 5, 6, 8, 13, 14, 15].

2. Main results

Any space is considered to be a Hausdorff space.

Let X and Y be topological spaces. We say that $F : X \longrightarrow Y$ is a set-valued mapping if $F(x)$ is a non-empty subset of Y for any point $x \in X$.

The set-valued mapping $F : X \longrightarrow Y$ is called:

- lower semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \varnothing\}$ is an open subset of the space X for any open subset H of the space Y;

- upper semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \varnothing\}$ is a closed subset of the space X for any closed subset H of the space Y;

- locally closed-valued if for any point $a \in X$ and any point $b \in F(x)$ there exist an open subset U of X and an open subset V of Y such that $F(x) \cap \cap cl_Y V$ is a closed subset of Y for each point $x \in U$;

- locally linear finite dimensional, where Y is a linear space, if for any point $a \in X$ and any point $b \in F(x)$ there exist an open subset U of X and an open subset V of Y such that $F(x) \cap V$ is a subset of some finite dimensional linear subspace of Y for each point $x \in U$.

Theorem 1. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a normal metacompact or a hereditary metacompact space X into a complete metrizable space Y. If the mapping F is locally closed-valued, then there exists a lower semicontinuous compact-valued mapping $\phi : X \longrightarrow Y$ such that $\phi(x) \subset F(x)$ for each point $x \in X$.*

Proof. Let d be a complete metric on a space Y. For any point $a \in X$ we fix an open subset Ua of the space X and an open subset Va of Y such that $a \in Ua \subset F^{-1}(V_a)$ and $F(x) \cap cl_Y Va$ is a closed subset of Y for any point $x \in Ua$. Since X is a metacompact space, there exist a subset A of X and an open point-finite cover $\{W_a : a \in A\}$ of the space X such that $Wa \subset Ua$. If X is a normal space, then we can assume that Wa is an F_σ-subset of X for each $a \in A$. Hence Wa is a metacompact subspace of X for each $a \in A$. Since Va is an open subset of the complete space (Y, d), on Va there exists a complete metric d_a. For any $a \in A$ consider the lower semicontinuous closed-valued mapping $F_a : Wa \longrightarrow Va$, where $F - a(x)$ $= F9x) \cap Va$ for any $x \in Wa$, of a metacompact space Wa into a complete metrizable space (Va, d_a). Fix $a \in A$. As was proved in [1, 2], there exists a lower semicontinuous compact-valued mapping $\phi_a : Wa \longrightarrow Va$ such that $\phi_a(x) \subset F_a(x)$ for each point $x \in Wa$. Then $\phi(x) = \cup\{\phi_a(x) : a \in A, x \in Wa\}$ is the desired mapping. The proof is complete.

From the E.Michael result from [12] and Theorem 1 it follows

Corollary 1. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a paracompact space X into a complete metrizable space Y. If the mapping F is locally closed-valued, then there exist a lower semicontinuous compact-valued mapping $\varphi : X \longrightarrow Y$ and a upper semicontinuous compact-valued mapping $\psi : X \longrightarrow Y$ such that $\varphi(x) \subset \psi(x) \subset F(x)$ for each point $x \in X$.*

Theorem 2. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a normal metacompact or a hereditary metacompact space X into a linear metrizable locally convex space Y. If the mapping F is locally closed-valued and locally linear finite dimensional, then there exists a lower semicontinuous compact-valued mapping $\phi : X \longrightarrow Y$ such that $\phi(x) \subset F(x)$ for each point $x \in X$. Moreover, if the mapping F is convex-valued, then the mapping ϕ is convex-valued too.*

Proof. Let d be an invariant metric on a space Y. For any point $a \in X$ we fix an open subset Ua of the space X and an open subset Va of Y such that $a \in Ua \subset F^{-1}(V_a)$ and $F(x) \cap clYVa$ is a closed subset of some finite dimensional linear subspace $L(a, x)$ for any point $x \in Ua$. By virtue of the V.L. Klee theorem [9], the metric d is complete on any finite dimensional linear subspace L of Y. Hence the existence of the mapping ϕ follows from Theorem 1. Assume now that the sets $F(x)$ are convex. Then the $conv(\phi) : X \longrightarrow Y$ is lower semicontinuous too [10]. Fix $x \in X$. Then $\phi(x)$ is a compact subset of the finite dimensional subspace $L(a)$ which contains the linear subspaces $\{L(a, x) : x \in Wa\}$. Hence $conv(\phi)(x)$ is a compact convex subset of Y. The proof is complete.

From the E.Michael result [10] and Theorem 2 it follows

Corollary 2. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a paracompact space X into a linear metrizable locally convex space Y. If the mapping F is locally closed-valued and locally linear finite dimensional, then there exists a single-valued continuous mapping $f : X \longrightarrow Y$ such that $f(x) \subset F(x)$ for each point $x \in X$.*

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