INTEGRABILITY CONDITIONS FOR LOTKA-VOLTERRA DIFFERENTIAL SYSTEM WITH A BUNDLE OF TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT CUBIC

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Abstract. For Lotka-Volterra differential system, we find conditions for the existence of a bundle of two invariant straight lines and one irreducible invariant cubic. We apply the Darboux theory to study the integrability of the obtained systems having three algebraic solutions.
Keywods: Lotka-Volterra differential system, invariant algebraic curves, integrability.
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CONDIŢII DE INTEGRABILITATE PENTRU SISTEMUL DIFERENŢIAL LOTKA-VOLTERRA CU UN FASCICOL DIN DOUĂ DREPTE INVARIANTE ŞI O CUBICĂ INVARIANTĂ

Rezumat. Pentru sistemul diferențial Lotka-Volterra sunt determinate condițiile de existență a unui fascicol format din două drepte invariante și o cubică invariantă ireductibilă. Aplicând teoria Darboux de integrabilitate se studiază integrabilitatea sistemelor obținute cu trei soluții algebrice.
Cuvinte-cheie: sistemul diferențial Lotka-Volterra, curbe invariante algebrice, integrabilitate.

1. Introduction

A planar polynomial differential system is a differential system of the form

$$\dot{x} = P(x,y), \ \ \dot{y} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials, $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$ denotes the derivatives with respect to independent variable t. We say that the polynomial differential system (1) has degree n, if $n = \max\{degP(x,y), degQ(x,y)\}$. In particular, when $n = 2$, a differential system (1) will be called *a quadratic system*.

In this paper we consider the quadratic system of differential equations

$$\dot{x} = x(a_1x + b_1y + c_1) \equiv P(x,y), \ \ \dot{y} = y(a_2x + b_2y + c_2) \equiv Q(x,y), \tag{2}$$

in which all coefficients $a_1, b_1, c_1, a_2, b_2, c_2$ and variables $x = x(t), \ y = y(t)$ are assumed to be real. The system (2) introduced by Lotka and Volterra appears in chemistry and ecology where it models two species in competition. It has been widely used in applied mathematics and in a large variety of physical topics such as laser physics, plasma physics, neural networks, hydrodynamics, etc [1]. Many authors have examined the integrability of system (2).

The Darboux integrability of (2) by using invariant straight lines and conics was investigated in [2]. The integrability of (2) via polynomial first integrals and polynomial inverse integrating factors was studied in [1]. The complete classification of systems (2) in the plane having a global analytic first integral was provided in [3]. The family of systems (2) according to their geometric properties encoded in the configurations of invariant straight lines which these systems possess was classified in [4].

The integrability conditions for some classes of quadratic systems (2) having an irreducible invariant cubic curve were obtained in [5] and [6].

In this paper we study the integrability of system (2) using invariant algebraic curves, two invariant straight lines and one irreducible invariant cubic curve, passing through one singular point, i.e. forming a bundle of invariant algebraic curves.

The integrability conditions will be found modulo the symmetry

$$(x, y, a_1, b_1, c_1, a_2, b_2, c_2) \to (y, x, b_2, a_2, c_2, b_1, a_1, c_1). \tag{3}$$

2. Invariant cubic curves

In this section we find the conditions under which the Lotka-Volterra system (2) has a bundle of two invariant straight lines and one irreducible invariant cubic.

Definition 2.1. An algebraic curve $\Phi(x, y) = 0$ in $\mathbb{C}^2$ with $\Phi(x, y) \in \mathbb{C}[x, y]$ is an *invariant algebraic curve* of a differential system (2) if the following identity holds

$$\frac{\partial\Phi(x, y)}{\partial x} P(x, y) + \frac{\partial\Phi(x, y)}{\partial y} Q(x, y) \equiv \Phi(x, y)K(x, y) \tag{4}$$

for some polynomial $K(x, y) \in \mathbb{C}[x, y]$ called the *cofactor* of the curve $\Phi(x, y) = 0$.

By Definition 2.1, a straight line $C + Ax + By = 0$, $A, B, C \in \mathbb{C}$, $(A, B) \neq 0$ is an *invariant straight line* of system (2) if and only if there exists a polynomial $K(x, y) = \gamma + \alpha x + \beta y$ such that the following identity holds

$$A \cdot P(x, y) + B \cdot Q(x, y) \equiv (C + Ax + By)(\gamma + \alpha x + \beta y). \tag{5}$$

If the quadratic system (2) has complex invariant straight lines then obviously they occur in complex conjugated pairs $C + Ax + By = 0$ and $\bar{C} + \bar{A}x + \bar{B}y = 0$.

By using the identity (5), it is easy to verify that the quadratic system (2) has always two invariant straight lines $x = 0$ and $y = 0$ with cofactors $K_1 = a_1x + b_1y + c_1$ and $K_2 = a_2x + b_2y + c_2$, respectively.

By Definition 2.1, a cubic curve

$$\begin{aligned}\Phi(x, y) \equiv a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \\ + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y = 0,\end{aligned} \tag{6}$$

where $a_{ij} \in \mathbb{R}$, $i + j = 1,2,3$ and $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$ is said to be an *invariant cubic curve* of system (2) if the identity (4) holds for some polynomial $K(x, y) = \gamma + \alpha x + \beta y$, called the cofactor of the invariant cubic curve $\Phi(x, y) = 0$.

Identifying the coefficients of the monomials $x^i y^j$ in (4) for cubic curve (6), we reduce this identity to an algebraic system of fourteen equations

$U_{40} \equiv a_{30}(3a_1 - \beta) = 0,$

$U_{31} \equiv a_{21}(2a_1 + a_2 - \beta) + a_{30}(3b_1 - \gamma) = 0,$

$U_{22} \equiv a_{12}(a_1 + 2a_2 - \beta) + a_{21}(2b_1 + b_2 - \gamma) = 0,$

$$
\begin{aligned}
&U_{13} \equiv a_{12}(b_1 + 2b_2 - \gamma) + a_{03}(3a_2 - \beta) = 0,\\
&U_{04} \equiv a_{03}(3b_2 - \gamma) = 0,\\
&U_{30} \equiv a_{20}(2a_1 - \beta) + a_{30}(3c_1 - \alpha) = 0,\\
&U_{21} \equiv a_{11}(a_1 + a_2 - \beta) + a_{20}(2b_1 - \gamma) + a_{21}(c_2 + 2c_1 - \alpha) = 0,\\
&U_{12} \equiv a_{11}(b_1 + b_2 - \gamma) + a_{02}(2a_2 - \beta) + a_{12}(2c_2 + c_1 - \alpha) = 0,\\
&U_{03} \equiv a_{02}(2b_2 - \gamma) + a_{03}(3c_2 - \alpha) = 0,\\
&U_{20} \equiv a_{10}(a_1 - \beta) + a_{20}(2c_1 - \alpha) = 0,\\
&U_{11} \equiv a_{01}(a_2 - \beta) + a_{10}(b_1 - \gamma) + a_{11}(c_2 + c_1 - \alpha) = 0,\\
&U_{02} \equiv a_{01}(b_2 - \gamma) + a_{02}(2c_2 - \alpha) = 0,\\
&U_{10} \equiv a_{10}(c_1 - \alpha) = 0,\\
&U_{01} \equiv a_{01}(c_2 - \alpha) = 0,
\end{aligned} \tag{7}
$$

for the unknowns $a_{30}, a_{21}, a_{12}, a_{03}, a_{20}, a_{11}, a_{02}, a_{10}, a_{01}$ and α, β, γ.

To simplify derivation of the invariant cubic curves from (7) we use the following assertion proved in [7] .

Lemma 2.1. Suppose that a polynomial system (1) of degree n has the invariant algebraic curve $\Phi(x, y) = 0$ of degree m. Let P_n, Q_n and Φ_m be the homogeneous components of P, Q and Φ of degree n and m, respectively. Then the irreducible factors of Φ_m must be factors of $yP_n - xQ_n$.

According to Lemma 2.1, the irreducible factors of Φ_3 must be the factors of

$$yP_2 - xQ_2 = xy[(a_1 - a_2)x + (b_1 - b_2)y].$$

The symmetry (3) implies $\Phi(x, y) = 0$ to have one of the following forms

$$\Phi(x, y) \equiv x^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y = 0, \tag{8}$$

$$\Phi(x, y) \equiv x^2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y = 0, \tag{9}$$

$$\Phi(x, y) \equiv xy[(a_1 - a_2)x + (b_1 - b_2)y] + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \tag{10}$$

$$\Phi(x, y) \equiv x^2[(a_1 - a_2)x + (b_1 - b_2)y] + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \tag{11}$$

$$\Phi(x, y) \equiv x[(a_1 - a_2)x + (b_1 - b_2)y]^2 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \tag{12}$$

$$\Phi(x, y) \equiv [(a_1 - a_2)x + (b_1 - b_2)y]^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \tag{13}$$

where $a_{20}, a_{11}, a_{02}, a_{10}, a_{01}$ are unknown coefficients.

We study the consistency of system (7) for each cubic curve (8) - (13) and determine the conditions under which the Lotka-Volterra system (2) has an irreducible invariant cubic. We assume that

$$(a_1^2 + b_1^2 + a_2^2 + b_2^2)(a_1^2 + c_1^2)(b_2^2 + c_2^2) \neq 0 \tag{14}$$

and that

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} \tag{15}$$

do not hold simultaneously. These conditions ensure the system (2) to be not linear and the vector field defined by (2) to be not constant.

There are proved the following theorems:

Theorem 2.1. The quadratic differential system (2) has an irreducible invariant cubic of the form (8) if and only if one of the following sets of conditions holds:

(i) $a_2 = \frac{3a_1}{2}, b_1 = b_2 = 0, c_2 = c_1;$ (ii) $b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(iii) $a_2 = 3a_1, b_1 = b_2 = 0,\ c_2 = c_1;$ (iv) $a_2 = 3a_1, b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(v) $a_2 = \frac{5a_1}{3}, b_2 = \frac{3b_1}{2}, c_2 = c_1;$ (vi) $a_2 = \frac{3a_1}{2},\ b_1 = b_2 = 0, c_1 = 2c_2;$

(vii) $a_2 = \frac{5a_1}{2}, b_2 = \frac{3b_1}{2}, c_1 = 2c_2;$ (viii) $a_2 = 2a_1,\ b_1 = b_2 = 0,\ c_2 = 3c_1;$

(ix) $a_2 = 3a_1,\ b_1 = b_2 = 0,\ c_2 = 2c_1;$ (x) $a_2 = 2a_1, b_2 = \frac{3b_1}{2},\ c_2 = 3c_1;$

(xi) $a_2 = \frac{15a_1}{8}, b_2 = \frac{3b_1}{2}, c_2 = 2c_1.$

Proof. Let $\Phi(x,y) = 0$ be of the form (8). We study the consistency of system (7) with $a_{30} = 1$, $a_{21} = a_{12} = a_{03} = 0$. In this case the equations $U_{40} = 0$, $U_{31} = 0$ of (7) yield $\beta = 3a_1, \gamma = 3b_1$ and $U_{10} \equiv a_{10}(\alpha - c_1) = 0, U_{01} \equiv a_{01}(\alpha - c_2) = 0$.

1) Assume that $a_{10} = a_{01} = 0$. In this case, the equations $U_{02} = 0$ and $U_{03} = 0$ imply $\alpha = 2c_2$ and $b_2 = (3b_1)/2$.

Let $c_2 = c_1$, then $a_{20} = c_1/a_1$. If $b_1 = 0$, then $a_2 = (3a_1)/2$, $a_{11} = 0$ and we get the invariant cubic

$$(a_1x + c_1)x^2 + a_{02}a_1y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2c_1$, where $a_1c_1a_{02} \neq 0$. We obtain the set of conditions (i) of Theorem 2.1. If $b_1 \neq 0$ and $a_2 = 2a_1$, then $c_1 = 0$. In this case we obtain a set of conditions which is contained in (x).

Suppose that $b_1(2a_1 - a_2) \neq 0$. Then express a_{11} from $U_{12} = 0$ and a_{02} from $U_{21} = 0$. We get the invariant cubic

$$2(3a_1 - 2a_2)((a_1x + c_1)(2a_1 - a_2)x - b_1c_1y)x + b_1^2c_1y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_1y + 2c_1$, where $a_1c_1b_1(2a_1 - a_2)(3a_1 - 2a_2) \neq 0$. We obtain the set of conditions (ii) of Theorem 2.1.

Let $c_2 \neq c_1$. Then $a_{20} = a_{11} = 0$. In this case the system (7) has no solutions.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$ and $c_1a_1 \neq 0$. We express a_{20} from $U_{30} = 0$, a_{10} from $U_{20} = 0$, a_{02} from $U_{02} = 0$ and a_{11} from $U_{11} = 0$. Then

$$U_{03} \equiv (3b_1 - b_2)(3b_1 - 2b_2) = 0.$$

Let $b_2 = 3b_1$ and $a_2 = 3a_1$. In this case $b_1 = 0$ and the system (2) has an invariant cubic curve

$$(a_1x + c_1)^2x + a_{01}a_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + c_1$, where $a_1c_1a_{01} \neq 0$. We get the conditions (iii).

When $b_2 = 3b_1$ and $a_2 \neq 3a_1$, the system (7) is not consistent.

Let $b_2 = (3b_1)/2$, $b_1 \neq 0$ and express a_{01} from $U_{12} = 0$. In this case we have $U_{21} \equiv (5a_1 - 3a_2)(3a_1 - a_2) = 0$. If $a_2 = 3a_1$, then we get the set of conditions (iv). The invariant cubic is

$$9a_1x(a_1x + c_1)^2 + 18a_1b_1c_1xy + 3b_1^2c_1y^2 + 2b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

If $a_2 = (5a_1)/3$, then we obtain the set of conditions (v). The invariant cubic is

$$a_1x(a_1x + c_1)^2 - 6a_1b_1c_1xy - 9b_1^2c_1y^2 - 6b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

3) Assume that $a_{10} \neq 0$ and let $a_{01} = 0$. Then $a_{02} \neq 0$ and $\alpha = c_1$. In this case we express c_1, b_2, a_{20}, a_{10}, a_{11} from the equations $U_{02} = 0$, $U_{03} = 0$, $U_{30} = 0$, $U_{20} = 0$, $U_{11} = 0$, respectively. If $b_1 = 0$ and $a_2 = (3a_1)/2$, then we obtain the invariant cubic

$$(a_1x + 2c_2)^2x + a_{02}a_1^2y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2c_2$, where $a_1c_2a_{02} \neq 0$. We get the conditions (vi).

If $b_1 \neq 0$ and $a_2 = (5a_1)/2$, then we find the invariant cubic curve

$$a_1x(a_1x + 2c_2)^2 + 8a_1b_1c_2xy + 2b_1^2c_2y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_1y + 2c_2$, where $a_1b_1c_2 \neq 0$. We determine the set of conditions (vii).

4) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Suppose that $a_{02} = a_{20} = 0$. Then $c_2 = 3c_1$ and $b_2 = 3b_1$. If $a_{11} = 0$, then the system (7) is not consistent. If $a_{11} \neq 0$, then $b_1 = 0$ and $a_2 = 2a_1$. In this case we obtain the invariant cubic

$$a_1x^3 + a_{11}y(c_1 + a_1x) = 0$$

with cofactor $K_3(x,y) = 3(a_1x + c_1)$, where $a_1c_1a_{11} \neq 0$. We get the conditions (viii).

Let $a_{02} = 0$ and $a_{20} \neq 0$. In this case from the equations of (7) we find that $\alpha = c_2$, $c_2 = 2c_1$, $b_2 = 3b_1$, $a_{20} = c_1/a_1$, $a_{11} = a_{01}(3a_1 - a_2)/c_1$.

If $b_1 = 0$ and $a_2 = 3a_1$. Then (2) has an invariant cubic curve

$$a_1x^3 + c_1x^2 + a_{01}a_1y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2c_1$, where $a_1c_1a_{01} \neq 0$. We obtain the set of conditions (ix). If $b_1 = 0$ and $a_2 = 2a_1$, then the cubic curve (8) is reducible.

Let $a_{20} = 0$ and $a_{02} \neq 0$. In this case the equations of (7) yield $\alpha = c_2$, $c_2 = 3c_1$, $b_2 = (3b_1)/2$, $a_{01} = (2c_1a_{02})/b_1$, $a_{11} = 2a_{02}(3a_1 - a_2)/b_1$.

When $a_2 = 2a_1$, we get the set of conditions (x). The invariant cubic is

$$b_1x^3 + 2a_{02}a_1xy + a_{02}b_1y^2 + 2a_{02}c_1y = 0$$

with cofactor $K_3(x,y) = 3(a_1x + b_1y + c_1)$, where $a_1c_1b_1a_{02} \neq 0$.

Let $a_{20}a_{02} \neq 0$. In this case the equations of (7) yield $\alpha = c_2,\ c_2 = 2c_1,\ b_2 = (3b_1)/2,\ a_{20} = c_1/a_1,\ a_{02} = (3b_1a_{01})/(4c_1),\ a_{11} = a_{01}(3a_1 - a_2)/c_1$.

If $a_2 = (15a_1)/8$, then we obtain the set of conditions (xi). The invariant cubic is

$$9a_1^2x^2(a_1x + c_1) - 72a_1b_1c_1xy - 48b_1^2c_1y^2 - 64b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_1y + 2c_1$, where $a_1c_1b_1 \neq 0$. Theorem 2.1 is proved.

Theorem 2.2. The quadratic differential system (2) has an irreducible invariant cubic of the form (9) if and only if one of the following sets of conditions is realized:

(i) $a_2 = 0,\ b_2 = 2b_1,\ c_2 = c_1$; (ii) $a_2 = -a_1,\ b_1 = 0,\ c_2 = c_1$;

(iii) $a_2 = -a_1,\ b_2 = 2b_1,\ c_2 = c_1$; (iv) $a_2 = -a_1,\ b_2 = 2b_1,\ c_1 = 2c_2$;

(v) $a_2 = 0,\ b_2 = 2b_1,\ c_1 = 2c_2$; (vi) $a_2 = 0,\ b_1 = 0,\ c_2 = 2c_1$.

Proof. Let $\Phi(x,y) = 0$ be of the form (9). We study the consistency of system (7) with $a_{21} = 1,\ a_{30} = a_{12} = a_{03} = 0$. In this case the equations $U_{31} = 0,\ U_{22} = 0$ of (7) yield $\beta = 2a_1 + a_2, \gamma = 2b_1 + b_2$.

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{20}a_{02} \neq 0$ and the equations of (7) yield $\alpha = 2c_1,\ c_2 = c_1, a_2 = 0, b_2 = 2b_1, a_{11} = (-2a_1a_{02})/b_1, a_{20} = (2a_{02}a_1^2 + b_1c_1)/(2b_1^2)$. In this case obtain the set of conditions (i) of Theorem 2.2. The invariant cubic is

$$x^2(2a_{02}a_1^2 + 2b_1^2y + b_1c_1) - 4a_1b_1a_{02}xy + 2b_1^2a_{02}y^2 = 0$$

with cofactor $K_3(x,y) = 2(a_1x + 2b_1y + c_1)$, where $a_{02}b_1 \neq 0$.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$. Let $a_{20} = 0$, then the equations $U_{20} = 0, U_{21} = 0, U_{12} = 0$ yield $a_2 = -a_1, a_{11} = (2c_1)/a_1,\ a_{02} = (-2b_1c_1)/(3a_1^2)$.

If $b_1 = 0$, then we get the set of conditions (ii). The invariant cubic is

$$(2a_1^2x + 4a_1c_1)xy + 2a_1^2a_{10}x + (2c_1^2 - a_1a_{10}b_2)y = 0$$

with cofactor $K_3(x,y) = a_1x + b_2y + c_1)$, where $a_{10}a_1(2c_1^2 - a_1a_{10}b_2) \neq 0$.

If $b_1 \neq 0$, then $b_2 = 2b_1$ and we obtain the set of conditions (iii) of Theorem 2.2. The invariant cubic (9) looks

$$9a_1b_1xy(a_1x + 2c_1) - 6b_1^2c_1y^2 + 8a_1c_1^2x - 3b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

Suppose that $a_{20} \neq 0$ and let $a_2 = 0$. Then the equations $U_{20} = 0, U_{02} = 0, U_{11} = 0$ yield $a_{20} = (a_1a_{10})/c_1,\ a_{02} = (2b_1a_{01})/c_1,\ a_{11} = (2a_1a_{01} + b_1a_{10} + b_2a_{10})/c_1$.

When $b_1 = 0$, the cubic is reducible. If $b_1 \neq 0$, then we express a_{01} from $U_{12} = 0$ and a_{10} from $U_{21} = 0$. In this case the cubic (9) is also reducible.

3) Assume $a_{01} = 0$ and let $a_{10} \neq 0$. Then $a_{02} \neq 0$ and $\alpha = c_1$. The equations $U_{02} = 0, U_{03} = 0$ yield $c_1 = 2c_2,\ b_2 = 2b_1$. If $a_{20} = 0$, then $a_2 = -a_1, a_{11} = (3c_2)/a_1$, $a_{02} = (-b_1c_2)/(a_1^2),\ a_{10} = c_2^2/(a_1b_1)$. In this case we get the set of conditions (iv) of Theorem 2.2. The invariant cubic (9) is

$$2a_1b_1xy(2a_1x + 3c_1) - 2b_1^2c_1y^2 + a_1c_1^2x = 0$$

with cofactor $K_3(x,y) = a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

If $a_{20} \neq 0$, then $a_2 = 0$. In this case $a_{10} = (2c_2a_{20})/a_1$, $a_{11} = (6b_1a_{20})/a_1$, $a_{20} = (3c_2)/(8b_1)$ and we obtain the set of conditions (v). The invariant cubic is

$$a_1b_1xy(8a_1x + 18c_2) + 3a_1^2c_2x^2 - 9b_1^2c_2y^2 + 6a_1c_2^2x = 0$$

with cofactor $K_3(x,y) = 2(a_1x + 2b_1y + c_2)$, where $a_1c_2b_1 \neq 0$.

4) Assume $a_{10} = 0$ and let $a_{01} \neq 0$. Then $a_{20} \neq 0$, $\alpha = c_2, a_2 = 0$ and $c_2 = 2c_1$.

Let $a_{02} = 0$. Then $b_1 = 0$ and $a_{11} = (2a_1a_{01})/c_1$. In this case we have the invariant cubic

$$2a_1^2x^2(y + a_{20}) + 2a_1(2c_1 - b_2a_{20})xy + c_1(2c_1 - b_2a_{20})y = 0$$

with cofactor $K_3(x,y) = 2a_1x + b_2y + 2c_1$, where $a_1c_1a_{20}(2c_1 - b_2a_{20}) \neq 0$. We get the set of conditions (vi) of Theorem 2.2.

Let $a_{02} \neq 0$. Then $b_2 = 2b_1$, $a_{02} = (b_1a_{01})/c_1$, $a_{11} = (2a_1a_{01})/c_1$, $a_1 = 0$ and $a_{20} = c_1/b_1$. In this case the cubic curve (9) is reducible. Theorem 2.2 is proved.

Theorem 2.3. The quadratic differential system (2) has an irreducible invariant cubic of the form (10) if and only if the following set of conditions is satisfied

(i) $a_2 = -a_1$, $b_1 = 0$, $c_2 = c_1$.

Proof. Let $\Phi(x,y) = 0$ be of the form (10). We study the consistency of system (7) with $a_{21} = a_1 - a_2$, $a_{12} = b_1 - b_2$, $a_{30} = a_{03} = 0$. In this case the equations $U_{31} = 0$, $U_{13} = 0$ of (7) yield $\beta = 2a_1 + a_2, \gamma = b_1 + 2b_2$.

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{20}a_{02} \neq 0$ and the equations of (7) imply $\alpha = 2c_1$, $c_2 = c_1, a_2 = b_1 = 0$, $a_{11} = (-2a_1a_{02} - b_2c_1)/b_2$, $a_{20} = a_1(a_1a_{02} + b_2c_1)/b_2^2$. In this case the invariant cubic (10) is reducible.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$. When $a_{20} = a_{02} = 0$, the equations of (7) yield $a_2 = -a_1, b_2 = -b_1$, $a_{11} = c_1 = 0$, $a_{10} = (a_{01}a_1)/b_1$. In this case the cubic curve is reducible.

Suppose that $a_{02} \neq 0$ and let $a_{20} = 0$. Then from the equations of (7) we find that $a_2 = -a_1, b_1 = 0, a_{11} = 4c_1, a_{02} = (-2b_2c_1)/a_1$, $a_{01} = (-2c_1^2)/a_1$, $a_{10} = (4c_1^2)/b_2$. In this case the cubic curve (10) is reducible.

Suppose that $a_{20} \neq 0$ and let $a_{02} = 0$. Then from (7) we determine that $a_2 = 0$, $b_2 = -b_1, a_{11} = -4c_1, a_{10} = (-2c_1^2)/b_1, a_{20} = (-2a_1c_1)/b_1$, $a_{01} = (-4c_1^2)/a_1$. In this case the cubic curve (10) is also reducible.

When $a_{20}a_{02} \neq 0$, the equations of (7) yield $b_1 = a_2 = a_{11} = 0$, $a_{10} = c_1^2/b_2$, $a_{01} = (-c_1^2)/a_1$, $a_{20} = (a_1c_1)/b_2$ and $a_{02} = (-b_2c_1)/a_1$. The cubic (10) is reducible.

3) Assume $a_{10} \neq 0$ and let $a_{01} = 0$. Then $a_{02} \neq 0$, $b_1 = 0$ and $\alpha = c_1 = 2c_2$. We express a_{20}, a_{11}, a_{10} from the equations $U_{21} = 0, U_{12} = 0, U_{11} = 0$ of (7).

If $c_2 = 0$ or $a_2 = 0$, then the cubic curve (10) is reducible. Suppose that $a_2c_2 \neq 0$, then $a_2 = -a_1$ and $a_{02} = (-8b_2c_2)/(3a_1)$. In this case we get the set of conditions (i) of Theorem 2.3. The invariant cubic (10) looks

$$3a_1b_2xy(2a_1x - b_2y + 6c_2) - 8b_2^2c_2y^2 + 9a_1c_2^2x = 0$$

with cofactor $K_3(x,y) = a_1x + 2b_2y + 2c_2$, where $a_1c_2b_2 \neq 0$.

4) Assume $a_{01} \neq 0$ and let $a_{10} = 0$. This case is symmetric to the case 3) and we obtain the set of conditions symmetric to (i). Theorem 2.3 is proved.

Theorem 2.4. The quadratic differential system (2) has an irreducible invariant cubic of the form (11) if and only if one of the following sets of conditions holds:

(i) $a_2 = \dfrac{3a_1}{2}, b_2 = 2b_1,\ c_2 = c_1$; (ii) $a_2 = 3a_1,\ b_1 = 0,\ c_2 = c_1$;

(iii) $a_2 = \dfrac{5a_1}{3},\ b_2 = 2b_1,\ c_2 = c_1$; (iv) $b_1 = 0,\ c_2 = 3c_1$;

(v) $a_2 = \dfrac{15a_1}{7},\ b_2 = 2b_1,\ c_2 = 3c_1$; (vi) $a_2 = 3a_1,\ b_1 = 0,\ c_2 = 2c_1$;

(vii) $a_2 = \dfrac{5a_1}{2},\ b_2 = 2b_1,\ c_2 = \dfrac{c_1}{2}$; (viii) $a_2 = \dfrac{7a_1}{6},\ b_2 = 2b_1,\ c_2 = \dfrac{c_1}{2}$.

Proof. Let $\Phi(x,y) = 0$ be of the form (11). We study the consistency of system (7) with $a_{30} = a_1 - a_2$, $a_{21} = b_1 - b_2$, $a_{12} = a_{03} = 0$. In this case the equations $U_{40} = 0$, $U_{22} = 0$ of (7) yield $\beta = 3a_1, \gamma = 2b_1 + b_2$.

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{02} \neq 0$ and $\alpha = 2c_2$. Suppose that $c_2 = c_1$, then $b_2 = 2b_1$. We express a_{11} and a_{20} from the equations $U_{12} = 0$ and $U_{21} = 0$ of (7).

If $a_2 = (3a_1)/2$, then we get the set of conditions (i). The invariant cubic is

$$a_1x^3 + 2b_1x^2y + c_1^2x^2 - 2a_{02}y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 4b_1y + 2c_1$, where $a_1b_1a_{02} \neq 0$. If $a_2 \neq (3a_1)/2$, then $a_{02} = (b_1^2c_1)/[a_1(2a_1 - a_2)]$. In this case the invariant cubic (11) is reducible.

Suppose that $c_2 \neq c_1$. Then $a_{20} = a_{11} = 0$, $b_2 = 2b_1$ and $a_2 = (3a_1)/2$. In this case the algebraic system (7) is not consistent.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1$, $c_2 = c_1$ and $c_1a_1 \neq 0$. We express a_{20}, a_{02}, a_{11}, a_{10} from the equations $U_{20} = 0$, $U_{02} = 0$, $U_{11} = 0$, $U_{30} = 0$ of (7).

Let $b_1 = 0$. If $a_2 = 2a_1$, then the invariant cubic (11) is reducible.

If $a_2 = 3a_1$, then we obtain the set of conditions (ii). The invariant cubic is

$$a_1x^2(2a_1x + b_2y + 4c_1) + 2b_2c_1xy + 2c_1^2x - a_{01}a_1y = 0$$

with cofactor $K_3(x,y) = 3a_1x + b_2y + c_1$, where $a_1c_1b_2a_{01} \neq 0$.

If $(a_2 - 2a_1)(a_2 - 3a_1) \neq 0$ and $a_{01} = (-b_2c_1^2)/a_1^2$, then (11) is reducible. Suppose that $b_1 \neq 0$. Then $U_{03} = 0$ yields $b_2 = 2b_1$. If $a_2 = 3a_1$, then $a_{01} = (-b_1c_1^2)/a_1^2$ and the invariant cubic (11) is reducible.

Let $a_2 \neq 3a_1$. Then express a_{01} from $U_{21} = 0$. In this case $U_{12} \equiv (3a_1 - 2a_2)(5a_1 - 3a_2) = 0$. If $a_2 = (3a_1)/2$, then the invariant cubic (11) is reducible.

If $a_2 = (5a_1)/3$, then we get the set of conditions (iii). The invariant cubic is

$$a_1^2x^2(2a_1x + 3b_1y + 4c_1) - 6a_1b_1c_1xy - 18b_1^2c_1y^2 + 2a_1c_1^2x - 9b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

3) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Then $\alpha = c_2$. Suppose that $a_{20} = a_{02} = 0$, then $b_1 = 0$, $c_2 = 3c_1$ and $a_{11} = (2b_2c_1)/(a_2 - 2a_1)$. When $a_2 = 3a_1$, we get a set of conditions which is contained in (ii). When $a_2 \neq 3a_1$, we express a_{01} from $U_{11} = 0$. In this case we get the set of conditions (iv). The invariant cubic is

$$(3a_1 - a_2)(2a_1 - a_2)(2a_1x + b_2y)x^2 + 2b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + b_2y + 3c_1$, where $a_1c_1b_2(3a_1 - a_2)(2a_1 - a_2) \neq 0$.

Let $a_{20}a_{02} \neq 0$. Then $c_2 = 2c_1$ and $b_2 = 2b_1$. We express a_{11} from $U_{12} = 0$, a_{20} from $U_{21} = 0$, a_{01} from $U_{02} = 0$ and a_2 from $U_{11} = 0$. In this case the invariant cubic (11) is reducible.

Let $a_{20} = 0$ and $a_{02} \neq 0$. Then from the equations of (7) we find that $a_{02} = (2b_1a_{01})/(3c_1)$, $a_{11} = (6a_1a_{01})/(7c_1)$, $a_{01} = (49b_1c_1^2)/(3a_1^2)$ and $a_2 = (15a_1)/7$. In this case we get the set of conditions (v) of Theorem 2.4. The invariant cubic is

$$a_1^2(72a_1x + 63b_1y)x^2 - 882a_1b_1c_1xy - 686b_1^2c_1y^2 - 1029b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 4b_1y + 3c_1$, where $a_1c_1b_1 \neq 0$.

Let $a_{20} \neq 0$ and $a_{02} = 0$. Then from the equations of (7) we find that

$$b_1 = 0,\ c_2 = 2c_1,\ a_{20} = (a_1a_{11})/b_2,\ a_{11} = a_{01}(3a_1 - a_2).$$

If $a_2 = 3a_1$, then we obtain the set of conditions (vi). The invariant cubic is

$$(2a_1x + b_2y + 2c_1)x^2 - a_{01}y = 0$$

with cofactor $K_3(x, y) = 3a_1x + b_2y + 2c_1$, where $a_1b_2c_1a_{01} \neq 0$.

If $a_2 \neq 3a_1$, then express a_{01} from $U_{30} = 0$. The invariant cubic is reducible.

4) Assume that $a_{10} \neq 0$ and let $a_{01} = 0$. Then $\alpha = c_1$, $b_2 = 2b_1$ and $c_2 = c_1/2$. We express a_{20}, a_{11}, a_{02} from the equations of (7) and obtain that

$$U_{21} \equiv (7a_1 - 6a_2)(5a_1 - 2a_2) = 0.$$

If $a_2 = (5a_1)/2$, then we get the set of conditions (vii). The invariant cubic is

$$a_1^2x^2(3a_1x + 2b_1y + 6c_1) + 18a_1b_1c_1xy + 3a_1c_1^2x + 9b_1^2c_1y^2 = 0$$

with cofactor $K_3(x, y) = 3a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

If $a_2 = (7a_1)/6$, then we get the set of conditions (viii). The invariant cubic is

$$a_1^2x^2(a_1x + 6b_1y + 2c_1) + 6a_1b_1c_1xy + a_1c_1^2x - 9b_1^2c_1y^2 = 0$$

with cofactor $K_3(x, y) = 3a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$. Theorem 2.4 is proved.

Theorem 2.5. The quadratic differential system (2) has an irreducible invariant cubic of the form (12) if and only if one of the following sets of conditions is realized:

(i) $a_2 = \dfrac{3a_1}{2},\ b_1 = 0,\ c_2 = c_1;$

(ii) $a_2 = 3a_1, b_2 = -b_1,\ c_2 = c_1;$

(iii) $a_2 = \dfrac{5a_1}{3},\ b_2 = -b_1,\ c_2 = c_1;$

(iv) $a_2 = \dfrac{5a_1}{3}, b_2 = -b_1, c_2 = 3c_1;$

(v) $a_2 = \dfrac{5a_1}{2},\ b_1 = 0,\ c_2 = 3c_1;$

(vi) $a_2 = \dfrac{3a_1}{2},\ b_1 = 0,\ c_2 = 3c_1;$

(vii) $a_2 = 7a_1,\ b_2 = -b_1, c_2 = 2c_1;$

(viii) $a_2 = \dfrac{5a_1}{3},\ b_2 = -b_1,\ c_2 = 2c_1;$

(ix) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = 2c_1;$ (x) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = \frac{c_1}{2};$

(xi) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = \frac{c_1}{2}.$

Proof. Let $\Phi(x,y) = 0$ be of the form (12). We study the consistency of system (7) with $a_{30} = (a_1 - a_2)^2, a_{21} = 2(a_1 - a_2)(b_1 - b_2),\ a_{12} = (b_1 - b_2)^2, a_{03} = 0.$

In this case we have $U_{20} \equiv a_{20}(c_1 - c_2) = 0,\ U_{11} \equiv a_{11}(c_1 - c_2) = 0$ and the equations $U_{40} = 0,\ U_{31} = 0$ of (7) yield $\beta = 3a_1,\ \gamma = b_1 + 2b_2$. We divide the investigation into the following cases:

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{02} \neq 0$ and $\alpha = 2c_2, b_1 = 0$.

Let $c_2 = c_1$. Then express a_{11} from $U_{12} = 0$ and a_{20} from $U_{30} = 0$. If $a_2 = 2a_1$, then the cubic curve (12) is reducible. If $a_2 = (3a_1)/2$, then we get the set of conditions (i) of Theorem 2.5. The invariant cubic is

$$x(a_1x + 2b_2y)^2 + a_1c_1x^2 + 4b_2c_1xy + 4a_{02}y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + 2c_1$, where $a_{02}a_1b_2 \neq 0$.

If $(a_2 - 2a_1)(2a_2 - 3a_1) \neq 0$, then $U_{21} = 0$ implies $a_{02} = (b_2^2c_1)/a_1$ and the invariant cubic (12) is reducible.

Let $c_2 \neq c_1$, then $a_{20} = a_{02} = 0$. In this case the system (7) is not consistent.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$ and $c_1b_2a_1 \neq 0$. We express a_{20} from $U_{20} = 0$, a_{02} from $U_{02} = 0$ and a_{11} from $U_{11} = 0$, then $U_{02} \equiv b_1(b_1 + b_2) = 0$.

Suppose that $b_1 = 0$ and express a_{10} from $U_{12} = 0$. If $a_2 = 2a_1$, then the cubic curve (12) is reducible. Let $a_2 \neq 2a_1$and express a_{01} from $U_{30} = 0$. If $a_2 = 3a_1$ or $a_2 = (3a_1)/2$, then the cubic (12) is reducible.

Suppose that $b_2 = -b_1,\ b_1 \neq 0$ and express a_{10} from $U_{12} = 0$. If $a_2 = 3a_1$, then we obtain the set of conditions (ii) of Theorem 2.5. The invariant cubic is

$$4x(a_1x - b_1y)^2 + 8a_1c_1x^2 - 8b_1c_1xy + 4c_1^2x + a_{01}y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + c_1$, where $a_{01}a_1b_1c_1 \neq 0$.

If $a_2 \neq 3a_1$, then express a_{01} from $U_{21} = 0$. In this case $a_2 = (5a_1)/3$ and we get the set of conditions (iii) of Theorem 2.5. The invariant cubic is

$$a_1x(a_1x - 3b_1y)^2 + 2a_1^2c_1x^2 - 18a_1b_1c_1xy + a_1c_1^2x - 12b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + c_1$, where $a_1b_1c_1 \neq 0$.

3) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Then $\alpha = c_2$ and

$$U_{20} \equiv a_{20}(2c_1 - c_2) = 0.$$

Let $a_{20} = 0$. Then $U_{30} = 0$ yields $c_2 = 3c_1$. If $c_1 = 0$, then $b_2 = -b_1, a_2 = 3a_1$ and this case is contained in (ii). When $c_1 \neq 0$, we express a_{11} from $U_{11} = 0$, a_{02} from $U_{02} = 0$ and obtain that $U_{03} \equiv b_1(b_2 + b_1) = 0$.

If $b_2 = -b_1$, then express a_{01} from $U_{12} = 0$. In this case $a_2 = (5a_1)/3$ and we get the set of conditions (iv) of Theorem 2.5. The invariant cubic looks

$$a_1x(a_1x - 3b_1y)^2 - 36a_1b_1c_1xy - 27b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x - b_1y + 3c_1$, where $a_1b_1c_1 \neq 0$.

If $b_2 \neq -b_1$, then $b_1 = 0$. We express a_{01} from $U_{12} = 0$ and obtain that

$$U_{21} \equiv (5a_1 - 2a_2)(3a_1 - 2a_2) = 0.$$

Suppose that $a_2 = (5a_1)/2$. In this case we obtain the set of conditions (v). The invariant cubic is

$$a_1x(3a_1x + 2b_2y)^2 - 48a_1b_2c_1xy - 32b_2^2y^2 - 96b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2b_2y + 3c_1$, where $a_1b_2c_1 \neq 0$.

Suppose that $a_2 = (3a_1)/2$, then we obtain the set of conditions (vi) of Theorem 2.5. The invariant cubic is

$$9a_1x(a_1x + 2b_2y)^2 + 144a_1b_2c_1xy + 32b_2^2c_1y^2 + 96b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2b_2y + 3c_1$, where $a_1b_2c_1 \neq 0$.

Let $a_{20} \neq 0$. Then $U_{30} = 0$ yields $c_2 = 2c_1$. We express a_{11} from $U_{11} = 0$, a_{02} from $U_{02} = 0$ and obtain that $U_{03} \equiv b_1(b_2 + b_1) = 0$. If $b_2 = -b_1$, then express a_{01} from $U_{12} = 0$, a_{20} from $U_{21} = 0$ and we get $U_{30} \equiv (a_2 - 7a_1)(3a_2 - 5a_1) = 0$.

When $a_2 = 7a_1$, we get the set of conditions (vii). The invariant cubic is

$$4a_1x(3a_1x - b_1y)^2 + 36a_1^2c_1x^2 - 12a_1b_1c_1xy + 3b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x - b_1y + 2c_1$, where $a_1b_1c_1 \neq 0$.

When $a_2 = (5a_1)/3$, we obtain the set of conditions (viii). The invariant cubic is

$$4a_1x(a_1x - 3b_1y)^2 + 4a_1^2c_1x^2 - 108a_1b_1c_1xy - 81b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x - b_1y + 2c_1$, where $a_1b_1c_1 \neq 0$.

If $b_2 \neq -b_1$, then $U_{03} = 0$ yields $b_1 = 0$. We express a_{01} from $U_{12} = 0$, a_{20} from $U_{21} = 0$ and we find that $U_{30} \equiv (5a_1 - 2a_2)(3a_1 - 2a_2)(3a_1 - a_2) = 0$.

When $a_2 = 3a_1$ or $a_2 = (3a_1)/2$, the cubic curve (12) is reducible. When $a_2 = (5a_1)/2$, we obtain the set of conditions (ix). The invariant cubic is

$$a_1x(3a_1x + 2b_2y)^2 + 9a_1^2c_1x^2 - 12a_1b_2c_1xy - 12b_2^2c_1y^2 - 24b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2b_2y + 2c_1$, where $a_1b_2c_1 \neq 0$.

4) Assume that $a_{10} \neq 0$ and let $a_{01} = 0$. Then $a_{02} \neq 0$, $\alpha = c_1$, $b_1 = 0$ and $c_2 = c_1/2$. We express a_{20} from $U_{20} = 0$, a_{11} from $U_{11} = 0$, a_{10} from $U_{30} = 0$ and obtain that $U_{21} \equiv (5a_1 - 2a_2)(3a_1 - 2a_2) = 0$.

If $a_2 = (3a_1)/2$, then we get the set of conditions (x). The invariant cubic is

$$x(a_1x + 2b_2y)^2 + 2a_1c_1x^2 + 4b_2c_1xy + 4a_{02}y^2 + c_1^2x = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2b_2y + c_1$, where $a_1b_2c_1 \neq 0$.

If $a_2 = (5a_1)/2$, then $a_{02} = (4b_2^2c_1)/a_1$ and we obtain the set of conditions (xi). The invariant cubic is

$$a_1x(3a_1x + 2b_2y)^2 + 18a_1^2c_1x^2 + 36a_1b_2c_1xy + 16b_2^2c_1y^2 + 9a_1c_1^2x = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2b_2y + c_1$, where $a_1b_2c_1 \neq 0$. Theorem 2.5 is proved.

Theorem 2.6. The quadratic differential system (2) has an irreducible invariant cubic of the form (13) if and only if one of the following sets of conditions holds:

(i) $a_2 = 2a_1,\ b_1 = 2b_2,\ c_2 = c_1 = 0$; (ii) $a_1(b_1 - 4b_2) + a_2(3b_2 - 2b_1) = 0,$ $c_2 = c_1$;

(iii) $a_2 = \frac{5a_1}{3},\ b_1 = 3b_2,\ c_2 = c_1$; (iv) $b_1 = \frac{7b_2}{3},\ a_2 = 5a_1,\ c_2 = 3c_1$;

(v) $b_1 = \frac{7b_2}{3},\ a_2 = \frac{9a_1}{5}, c_2 = 3c_1$; (vi) $b_1 = \frac{4b_2}{3},\ a_2 = 0,\ c_2 = 3c_1$;

(vii) $b_1 = \frac{5b_2}{2},\ a_2 = 4a_1,\ c_2 = 2c_1$; (viii) $b_1 = \frac{5b_2}{2},\ a_2 = -5a_1,\ c_2 = 2c_1$;

(ix) $a_2 = \frac{7a_1}{4},\ b_1 = \frac{5b_2}{2},\ c_2 = 2c_1$; (x) $a_2 = \frac{3a_1}{4},\ b_1 = \frac{7b_2}{6},\ c_2 = 2c_1$.

Proof. Let $\Phi(x,y) = 0$ be of the form (13). We study the consistency of system (7) with $a_{30} = (a_1 - a_2)^3,\ \ a_{12} = 3(a_1 - a_2)(b_1 - b_2)^2,\ a_{21} = 3(a_1 - a_2)^2(b_1 - b_2),\ \ a_{03} = (b_1 - b_2)^3$. In this case the equations $U_{40} = 0$, $U_{31} = 0$ of (7) yield $\beta = 3a_1$, $\gamma = 3b_2$. We divide the investigation into the following cases:

1) Assume that $a_{10} = a_{01} = 0$. Let $a_{02} = 0$ and $a_{11} = 0$. Then $\alpha = 2c_2,\ c_2 = c_1 = 0$ and $a_1 = 0$. We obtain a contradiction with conditions (14).

If $a_{02} = 0$ and $a_{11} \neq 0$, then $c_1 = 2c_2,\ \alpha = c_1 + c_2$. In this case the system (7) is consistent only if $a_{20} = 0$. We get the set of conditions (i) of Theorem 2.6. The invariant cubic is

$$(a_1x - b_2y)^3 - a_{11}xy = 0$$

with cofactor $K_3(x,y) = 3(a_1x + b_2y)$, where $a_{11}a_1b_2 \neq 0$.

Let $a_{02} \neq 0$. Then $\alpha = 2c_2$ and $U_{20} \equiv (c_1 - c_2)a_{20} = 0,\ U_{11} \equiv (c_1 - c_2)a_{11} = 0$. Suppose that $c_2 = c_1$. Then $b_2a_1 \neq 0$. We express a_{20} from $U_{30} = 0$ and a_{02} from $U_{03} = 0$. If $a_2 = 2a_1$, then the cubic curve (13) is reducible. If $a_2 \neq 2a_1$, then express a_{11} from $U_{21} = 0$, and $U_{12} = 0$ becomes $U_{12} \equiv e_1e_2e_3 = 0$, where $e_1 = a_1b_1 - 2a_1b_2 + a_2b_2$, $e_2 = 2a_1b_1 - 3a_1b_2 - a_2b_1 + 2a_2b_2,\ e_3 = 3a_1b_1 - 4a_1b_2 - 2a_2b_1 + 3a_2b_2$.

If $e_1 = 0$ or $e_2 = 0$, then the cubic curve (13) is reducible. If $e_3 = 0$, then we obtain the set of conditions (ii) of Theorem 2.6. The invariant cubic looks

$$a_1b_2(2a_1 - a_2)[(a_1 - a_2)x - (b_1 - b_2)y]^3 + c_1b_2(a_1 - a_2)^3(2a_1 - a_2)x^2 +$$
$$+2a_1b_2c_1(b_1 - b_2)(a_1 - a_2)^2xy + a_1c_1(b_1 - b_2)^3(2a_1 - a_2)y^2 = 0,$$

where $K_3(x,y) = 3a_1x + 3b_2y + 2c_1$ and $c_1a_1b_2(2a_1 - a_2)(a_1 - a_2)(b_1 - b_2) \neq 0$.

Suppose $c_2 \neq c_1$, then $a_{20} = a_{11} = 0$ and the system (7) has no solutions.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$ and $c_1b_2a_1 \neq 0$. We express a_{20} from $U_{20} = 0$, a_{02} from $U_{02} = 0$, a_{11} from $U_{11} = 0$, a_{01} from $U_{03} = 0$ and a_{10} from $U_{30} = 0$. In this case the equations $U_{21} = 0$ and $U_{12} = 0$ have a common factor $h = a_1b_1 - 2a_1b_2 + a_2b_2$. If $h = 0$, then the cubic (13) is reducible.

Let $h \neq 0$ and suppose $b_1 = 3b_2$. Then we obtain an irreducible cubic curve of the form (13) only if $a_2 = (5a_1)/3$. We get the set of conditions (iii). The invariant cubic is

$$(a_1x - 3b_2y)^3 + 2a_1^2c_1x^2 - 36a_1b_2c_1xy - 54a_2^2c_1y^2 + a_1c_1^2x - 27b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_2y + c_1$, where $c_1a_1b_2 \neq 0$.

Suppose that $h(b_1 - 3b_2) \neq 0$ and let $a_2 = 3a_1$. Then (2) has an irreducible cubic curve only if $b_1 = (5b_2)/3$. In this case we obtain the set of conditions symmetric to (iii).

Let $h(b_1 - 3b_2)(a_2 - 3a_1) \neq 0$. Then the system of equations (7) ($U_{21} = 0$, $U_{12} = 0$) is not consistent.

3) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Then $\alpha = c_2$ and

$$U_{20} \equiv a_{20}(2c_1 - c_2) = 0.$$

Suppose that $a_{20} = 0$. Then $U_{30} = 0$ yields $c_2 = 3c_1$ and $c_1b_2 \neq 0$. We express a_{02} from $U_{03} = 0$, a_{01} from $U_{02} = 0$ and a_{11} from $U_{11} = 0$. In this case

$$U_{21} \equiv (3b_1 - 7b_2)[(3a_1 - a_2)b_1 - 2(2a_1 - a_2)b_2] = 0.$$

If $b_1 = (7b_2)/3$ and $a_2 = 5a_1$, then we obtain the set of conditions (iv) of Theorem 2.6. The invariant cubic is

$$(3a_1x - b_2y)^3 + 18a_1b_2c_1xy - 6b_2^2c_1y^2 - 9b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3(a_1x + b_2y + c_1)$, where $c_1a_1b_2 \neq 0$.

If $b_1 = (7b_2)/3$ and $a_2 = (9a_1)/5$, then we obtain the set of conditions (v) of Theorem 2.6. The invariant cubic is

$$(3a_1x - 5b_2y)^3 - 1350a_1b_2c_1xy - 750b_2^2c_1y^2 - 1125b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3(a_1x + b_2y + c_1)$, where $c_1a_1b_2 \neq 0$.

Suppose that $3b_1 - 7b_2 \neq 0$ and let $(3a_1 - a_2)b_1 - 2(2a_1 - a_2)b_2 = 0$. Then $U_{12} = 0$ imply $a_2 = 0$. We get the set of conditions (vi). The invariant cubic looks

$$(3a_1x + b_2y)^3 + 27a_1b_2c_1xy + 6b_2^2c_1y^2 + 9b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3(a_1x + b_2y + c_1)$, where $c_1a_1b_2 \neq 0$.

Suppose that $a_{20} \neq 0$, then $U_{20} = 0$ yields $c_2 = 2c_1$ and $a_1c_1b_2 \neq 0$. We express a_{02} from $U_{02} = 0$, a_{11} from $U_{11} = 0$, a_{01} from $U_{03} = 0$ and a_{20} from $U_{30} = 0$. In this case $U_{12} \equiv (2b_1 - 5b_2)[2b_1(3a_1 - a_2) - b_2(9a_1 - 5a_2)] = 0$.

If $b_1 = (5b_2)/2$ and $a_2 = 4a_1$, then we obtain the set of conditions (vii) of Theorem 2.6. The invariant cubic is

$$(2a_1x - b_2y)^3 + 8a_1^2c_1x^2 + 4a_1b_2c_1xy - 6b_2^2c_1y^2 - 4b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_2y + 2c_1$, where $c_1a_1b_2 \neq 0$.

If $b_1 = (5b_2)/2$ and $a_2 = -5a_1$, then we obtain the set of conditions (viii) of Theorem 2.6. The invariant cubic is

$$(4a_1x + b_2y)^3 + 64a_1^2c_1x^2 + 32a_1b_2c_1xy + 4b_2^2c_1y^2 + 4b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_2y + 2c_1$, where $c_1a_1b_2 \neq 0$.

If $b_1 = (5b_2)/2$ and $a_2 = (7a_1)/4$, then we obtain the set of conditions (ix) of Theorem 2.6. The invariant cubic is

$$(a_1x-2b_2y)^3+a_1^2c_1x^2-40a_1b_2c_1xy-32b_2^2c_1y^2-32b_2c_1^2y=0$$

with cofactor $K_3(x,y)=3a_1x+3b_2y+2c_1$, where $c_1a_1b_2\neq 0$.

Suppose that $2b_1-5b_2\neq 0$ and let $2b_1(3a_1-a_2)-b_2(9a_1-5a_2)=0$. Then $U_{21}\equiv a_2(3a_1-4a_2)=0$. If $a_2=0$, then the cubic curve (13) is reducible.

If $a_2=(3a_1)/4$, then we obtain the set of conditions (x). The invariant cubic is

$$(3a_1x+2b_2y)^3+27a_1^2c_1x^2+72a_1b_2c_1xy+32b_2^2c_1y^2+32b_2c_1^2y=0$$

with cofactor $K_3(x,y)=3a_1x+3b_2y+2c_1$, where $c_1a_1b_2\neq 0$.

4) Assume that $a_{10}\neq 0$ and let $a_{01}=0$. In this case we obtain the sets of conditions symmetric to (iv) - (x). Theorem 2.6 is proved.

3. Darboux theory of integrability

Let the polynomial differential system (1) have the invariant algebraic curves $\Phi_j(x,y)=0$, $j=1,\dots,q$ with cofactors $K_j(x,y)$. Then in most cases a first integral (an integrating factor) can be constructed in the Darboux form [8]

$$\Phi_1^{h_1}\Phi_2^{h_2}\cdots\Phi_q^{h_q}$$

and we say that the polynomial system (1) is *Darboux integrable*.

Theorem 3.1. The system (1) has a Draboux first integral

$$F(x,y)\equiv\Phi_1^{h_1}\Phi_2^{h_2}\cdots\Phi_q^{h_q}=\mathrm{C} \tag{16}$$

if and only if there exists constants $\alpha_j\in\mathbb{C}$, not all identically zero, such that

$$h_1K_1(x,y)+h_2K_2(x,y)+\cdots+h_qK_q(x,y)\equiv 0, \tag{17}$$

where $K_j(x,y)$ are the cofactors of $\Phi_j(x,y)=0$, $j=1,\dots,q$.

Following [8], the relation (16) is a first integral for system (1) if and only if

$$\frac{\partial F(x,y)}{\partial x}P(x,y)+\frac{\partial F(x,y)}{\partial y}Q(x,y)\equiv 0.$$

If a first integral cannot be found, Darboux proposed to search for an integrating factor μ of the same form.

Theorem 3.2. The system (1) has a Draboux integrating factor

$$\mu=\Phi_1^{h_1}\Phi_2^{h_2}\cdots\Phi_q^{h_q} \tag{18}$$

if and only if there exists constants $\alpha_j\in\mathbb{C}$, not all identically zero, such that

$$h_1K_1(x,y)+h_2K_2(x,y)+\cdots+h_qK_q(x,y)+\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\equiv 0, \tag{19}$$

where $K_j(x,y)$ are the cofactors of $\Phi_j(x,y)=0$, $j=1,\dots,q$.

Following [8], the relation (18) is an integrating for system (1) if and only if

$$P(x,y)\frac{\partial\mu}{\partial x}+Q(x,y)\frac{\partial\mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\equiv 0.$$

How many invariant algebraic curves $\Phi_j(x,y)=0$ must admit the system (1) to have a Daroux first integral or a Darboux integrating factor? Darboux proved

Theorem 3.3. Suppose system (1) has q distinct invariant algebraic curves $\Phi_{\mathrm{j}}(x,y)=0$, $\mathrm{j}=1,\dots,q$. If $q\geq n(n+1)/2$, then either we have a Darboux first integral or a Darboux integrating factor.

By Theorem 3.3, in the case of quadratic system (2), if $q\geq 3$, then either we have a Darboux first integral or a Darboux integrating factor.

The method of Darboux is very useful and elegant one to prove integrability for some classes of differential systems depending on parameters [8].

4. Darboux first integrals

In this section we determine the sets of conditions from Theorems 2.1 – 2.5, under which the quadratic system (2) has Darboux first integrals of the form

$$x^{h_1}y^{h_2}\Phi^{h_3}=C, \tag{20}$$

where $x=0, y=0$ are invariant straight lines, $\Phi=0$ is an irreducible invariant cubic of the form (6) and h_1, h_2, h_3 are real numbers.

To construct the first integrals (20) we take into account the cofactors $K_1(x,y), K_2(x,y)$ and $K_3(x,y)$ of these algebraic solutions, obtained in the proofs of Theorems 2.1 – 2.5. Then we apply the identity (17)

$$h_1K_1(x,y)+h_2K_2(x,y)+h_3K_3(x,y)\equiv 0 \tag{21}$$

to each set of conditions from Theorems 2.1 – 2.5. It was proved the following theorem.

Theorem 4.1. The Lotka-Volterra system (2) has a Darboux first integral of the form (20) if one of the following conditions is satisfied:

(i) $a_2=\frac{3a_1}{2}, b_1=b_2=0, c_2=c_1;$

(ii) $a_2=3a_1,\ b_1=b_2=0,\ c_2=c_1;$

(iii) $a_2=\frac{3a_1}{2},\ b_1=b_2=0,\ c_1=2c_2;$

(iv) $a_2=2a_1,\ b_1=b_2=0,\ c_2=3c_1;$

(v) $a_2=3a_1,\ b_1=b_2=0,\ c_2=2c_1;$

(vi) $a_2=2a_1, b_2=\frac{3b_1}{2},\ c_2=3c_1;$

(vii) $a_2=\frac{3a_1}{2}, b_2=2b_1,\ c_2=c_1;$

(viii) $a_2=3a_1,\ b_1=0,\ c_2=c_1;$

(ix) $a_2=\frac{5a_1}{3},\ b_2=2b_1,\ c_2=c_1;$

(x) $a_2=3a_1,\ b_1=0,\ c_2=2c_1;$

(xi) $a_2=\frac{3a_1}{2},\ b_1=0,\ c_2=c_1;$

(xii) $a_2=3a_1, b_2=-b_1,\ c_2=c_1;$

(xiii) $a_2=\frac{5a_1}{2},\ b_1=0,\ c_2=2c_1;$

(xiv) $a_2=\frac{3a_1}{2},\ b_1=0,\ c_2=\frac{c_1}{2};$

(xv) $a_2=2a_1,\ b_1=2b_2,\ c_2=c_1=0;$

(xvi) $b_1=\frac{5b_2}{2},\ a_2=4a_1,\ c_2=2c_1;$

(xvii) $a_1(b_1-4b_2)+a_2(3b_2-2b_1)=0,$
$c_2=c_1.$

Proof. We use the identity (21) for each set of conditions from Theorems 2.1 – 2.5. Substituting in this identity the expressions of the cofactors and identifying the coefficients of x^0, x, y, we obtain systems of algebraic equations for the unknowns h_1, h_2 and h_3. Solving the obtained systems we determine the exponents h_1, h_2 and h_3.

Applying the identity (21) to the sets of conditions from Theorem 2.1, we obtain:

In case (i), $\Phi \equiv (a_1 x + c_1)x^2 + a_{02}a_1 y^2 = 0$ and $h_1 = 2,\ h_2 = 2,\ h_3 = -1$.

In case (ii), $\Phi \equiv (a_1 x + c_1)^2 x + a_{01}a_1^2 y = 0$ and $h_1 = 2,\ h_2 = 1,\ h_3 = -1$.

In case (iii), $\Phi \equiv (a_1 x + 2c_2)^2 x + a_{02}a_1^2 y^2 = 0$ and $h_1 = 2,\ h_2 = 2,\ h_3 = -1$.

In case (iv), $\Phi \equiv a_1 x^3 + a_{11} y(c_1 + a_1 x) = 0$ and $h_1 = 3,\ h_2 = 0,\ h_3 = -1$.

In case (v), $\Phi \equiv a_1 x^3 + c_1 x^2 + a_{01}a_1 y = 0$ and $h_1 = 0,\ h_2 = 1,\ h_3 = -1$.

In case (vi), $\Phi \equiv b_1 x^3 + 2a_{02}a_1 xy + a_{02}b_1 y^2 + 2a_{02}c_1 y = 0$ and

$$h_1 = 3, \qquad h_2 = 0, \qquad h_3 = -1 .$$

Applying the identity (21) to the sets of conditions from Theorem 2.4, we have:

In case (vii), $\Phi \equiv a_1 x^3 + 2b_1 x^2 y + c_1^2 x^2 - 2a_{02} y^2 = 0$ and $h_1 = 0, h_2 = 2, h_3 = -1$.

In case (viii), $\Phi \equiv a_1 x^2(2a_1 x + b_2 y + 4c_1) + 2b_2 c_1 xy + 2c_1^2 x - a_{01}a_1 y = 0$

and $h_1 = 0,\ h_2 = 1,\ h_3 = -1$.

In case (ix), $\Phi \equiv a_1^2 x^2(2a_1 x + 3b_1 y + 4c_1) - 6a_1 b_1 c_1 xy - 18b_1^2 c_1 y^2 + 2a_1 c_1^2 x -$

$-9b_1 c_1^2 y = 0$ and $h_1 = -2,\ h_2 = 3,\ h_3 = -1$.

In case (x), $\Phi \equiv (2a_1 x + b_2 y + 2c_1)x^2 - a_{01} y = 0$ and $h_1 = 0,\ h_2 = 1,\ h_3 = -1$.

Applying the identity (21) to the sets of conditions from Theorem 2.5, we get:

In case (xi), $\Phi \equiv x(a_1 x + 2b_2 y)^2 + a_1 c_1 x^2 + 4b_2 c_1 xy + 4a_{02} y^2 = 0$ and

$$h_1 = 0, \qquad h_2 = 2, \qquad h_3 = -1 .$$

In case (xii), $\Phi \equiv 4x(a_1 x - b_1 y)^2 + 8a_1 c_1 x^2 - 8b_1 c_1 xy + 4c_1^2 x + a_{01} y = 0$ and

$$h_1 = 0, \qquad h_2 = 1, \qquad h_3 = -1 .$$

In case (xiii), $\Phi \equiv a_1 x(3a_1 x + 2b_2 y)^2 + 9a_1^2 c_1 x^2 - 12a_1 b_2 c_1 xy - 12b_2^2 c_1 y^2 -$

$-24b_2 c_1^2 y = 0$ and $h_1 = -2,\ h_2 = 2,\ h_3 = -1$.

In case (xiv), $\Phi \equiv x(a_1 x + 2b_2 y)^2 + 2a_1 c_1 x^2 + 4b_2 c_1 xy + 4a_{02} y^2 + c_1^2 x = 0$ and

$$h_1 = 0, \qquad h_2 = 2, \qquad h_3 = -1 .$$

Applying the identity (21) to the set of conditions from Theorem 2.6, we obtain:

In case (xv), $\Phi \equiv (a_1 x - b_2 y)^3 - a_{11} xy = 0$ and $h_1 = 1,\ h_2 = 1,\ h_3 = -1$.

In case (xvi), $\Phi \equiv 4a_1 x(3a_1 x - b_1 y)^2 + 36a_1^2 c_1 x^2 - 12a_1 b_1 c_1 xy + 3b_1 c_1^2 y = 0$ and

$$h_1 = -2, \qquad h_2 = 2, \qquad h_3 = -1 .$$

In case (xvii), $\Phi \equiv a_1 b_2(2a_1 - a_2)[(a_1 - a_2)x - (b_1 - b_2)y]^3 + c_1 b_2(a_1 - a_2)^3 \cdot$

$\cdot (2a_1 - a_2)x^2 + 2a_1 b_2 c_1(b_1 - b_2)(a_1 - a_2)^2 xy + a_1 c_1(b_1 - b_2)^3(2a_1 - a_2)y^2 = 0$

and $h_1 = 2a_2 - 3a_1,\ h_2 = a_1,\ h_3 = a_1 - a_2$.

Theorem 4.1 is proved.

5. Darboux integrating factors

In this section we determine the sets of conditions from Theorems 2.1 – 2.5, under which the quadratic system (2) has Darboux integrating factors of the form

$$\mu = x^{h_1} y^{h_2} \Phi^{h_3}, \tag{22}$$

where $x = 0, y = 0$ are invariant straight lines, $\Phi = 0$ is an irreducible invariant cubic of the form (6) and h_1, h_2, h_3 are real numbers.

To construct the integrating factors (22) we take into account the cofactors $K_1(x,y), K_2(x,y)$ and $K_3(x,y)$ of these algebraic solutions, obtained in the proofs of Theorems 2.1 – 2.5. Then we apply the identity (19)

$$h_1 K_1(x,y) + h_2 K_2(x,y) + h_3 K_3(x,y) + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0 \tag{23}$$

for each set of conditions from Theorems 2.1 – 2.5. It was proved the following theorem.

Theorem 5.1. The Lotka-Volterra system (2) has a Darboux integrating factor of the form (22) if one of the following conditions is satisfied:

(i) $b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(ii) $a_2 = 3a_1, b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(iii) $a_2 = \frac{5a_1}{3}, b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(iv) $a_2 = \frac{5a_1}{2}, b_2 = \frac{3b_1}{2}, c_1 = 2c_2;$

(v) $a_2 = \frac{15a_1}{8}, b_2 = \frac{3b_1}{2}, c_2 = 2c_1;$

(vi) $a_2 = 0,\ b_2 = 2b_1,\ c_2 = c_1;$

(vii) $a_2 = -a_1,\ b_1 = 0,\ c_2 = c_1;$

(viii) $a_2 = -a_1,\ b_2 = 2b_1,\ c_2 = c_1;$

(ix) $a_2 = -a_1,\ b_2 = 2b_1,\ c_1 = 2c_2;$

(x) $a_2 = 0,\ b_2 = 2b_1,\ c_1 = 2c_2;$

(xi) $a_2 = 0,\ b_1 = 0,\ c_2 = 2c_1;$

(xii) $a_2 = -a_1,\ b_1 = 0,\ c_2 = c_1;$

(xiii) $b_1 = 0,\ c_2 = 3c_1;$

(xiv) $a_2 = \frac{15a_1}{7},\ b_2 = 2b_1,\ c_2 = 3c_1;$

(xv) $a_2 = \frac{5a_1}{2},\ b_2 = 2b_1,\ c_2 = \frac{c_1}{2};$

(xvi) $a_2 = \frac{7a_1}{6},\ b_2 = 2b_1,\ c_2 = \frac{c_1}{2};$

(xvii) $a_2 = \frac{5a_1}{3},\ b_2 = -b_1,\ c_2 = c_1;$

(xviii) $a_2 = \frac{5a_1}{3}, b_2 = -b_1, c_2 = 3c_1;$

(xix) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = 3c_1;$

(xx) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = 3c_1;$

(xxi) $a_2 = 7a_1,\ b_2 = -b_1, c_2 = 2c_1;$

(xxii) $a_2 = \frac{5a_1}{3},\ b_2 = -b_1,\ c_2 = 2c_1;$

(xxiii) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = \frac{c_1}{2};$

(xiv) $a_2 = \frac{5a_1}{3},\ b_1 = 3b_2,\ c_2 = c_1;$

(xxv) $b_1 = \frac{7b_2}{3},\ a_2 = 5a_1,\ c_2 = 3c_1;$

(xxvi) $b_1 = \frac{7b_2}{3},\ a_2 = \frac{9a_1}{5}, c_2 = 3c_1;$

(xxvii) $b_1 = \frac{4b_2}{3},\ a_2 = 0,\ c_2 = 3c_1;$

(xxviii) $b_1 = \frac{5b_2}{2},\ a_2 = -5a_1, c_2 = 2c_1;$

(xxix) $a_2 = \frac{7a_1}{4},\ b_1 = \frac{5b_2}{2},\ c_2 = 2c_1;$

(xxx) $a_2 = \frac{3a_1}{4},\ b_1 = \frac{7b_2}{6},\ c_2 = 2c_1.$

Proof. We use the identity (23) for each set of conditions from Theorems 2.1 – 2.5. Substituting in this identity the expressions of the cofactors and identifying the coefficients of x^0, x, y, we obtain systems of algebraic equations for the unknowns h_1, h_2 and h_3. Solving the obtained systems we determine the exponents h_1, h_2 and h_3.

Applying the identity (23) to the sets of conditions from Theorem 2.1, we obtain:
In case (i), $\Phi \equiv 2(3a_1 - 2a_2)((a_1x + c_1)(2a_1 - a_2)x - b_1c_1y)x + b_1^2c_1y^2 = 0$ and

$$h_1 = 2, \qquad h_2 = \frac{2(a_2 - 2a_1)}{3a_1 - 2a_2}, \qquad h_3 = \frac{3a_2 - 4a_1}{3a_1 - 2a_2}.$$

In case (ii), $\Phi \equiv 9a_1x(a_1x + c_1)^2 + 18a_1b_1c_1xy + 3b_1^2c_1y^2 + 2b_1c_1^2y = 0$ and

$$h_1 = 2, \qquad h_1 = -1/2, \qquad h_3 = -5/6\,.$$

In case (iii), $\Phi \equiv a_1x(a_1x + c_1)^2 - 6a_1b_1c_1xy - 9b_1^2c_1y^2 - 6b_1c_1^2y = 0$ and

$$h_1 = -5/2, \qquad h_2 = 2, \qquad h_3 = -3/2\,.$$

In case (iv), $\Phi \equiv a_1x(a_1x + 2c_2)^2 + 8a_1b_1c_2xy + 2b_1^2c_2y^2 = 0$ and

$$h_1 = -1/4, \qquad h_2 = -1/2, \qquad h_3 = -1.$$

In case (v), $\Phi \equiv 9a_1^2x^2(a_1x + c_1) - 72a_1b_1c_1xy - 48b_1^2c_1y^2 - 64b_1c_1^2y = 0$ and

$$h_1 = -2, \qquad h_2 = 1/3, \qquad h_3 = -5/6\,.$$

Applying the identity (23) to each set of conditions from Theorem 2.2, we get:
In case (vi), $\Phi \equiv x^2(2a_{02}a_1^2 + 2b_1^2y + b_1c_1) - 4a_1b_1a_{02}xy + 2b_1^2a_{02}y^2 = 0$ and

$$h_1 = 1, \qquad h_2 = 0, \qquad h_3 = -3/2.$$

In case (vii), $\Phi \equiv (2a_1^2x + 4a_1c_1)xy + 2a_1^2a_{10}x + (2c_1^2 - a_1a_{10}b_2)y = 0$ and

$$h_1 = 0, \qquad h_2 = -1/2, \qquad h_3 = -3/2.$$

In case (viii), $\Phi \equiv 9a_1b_1xy(a_1x + 2c_1) - 6b_1^2c_1y^2 + 8a_1c_1^2x - 3b_1c_1^2y = 0$ and

$$h_1 = -2/3, \qquad h_2 = -1/2, \qquad h_3 = -5/6.$$

In case (ix), $\Phi \equiv 2a_1b_1xy(2a_1x + 3c_1) - 2b_1^2c_1y^2 + a_1c_1^2x = 0$ and

$$h_1 = -1/3, \qquad h_2 = -1/3, \qquad h_3 = -1.$$

In case (x), $\Phi \equiv a_1b_1xy(8a_1x + 18c_2) + 3a_1^2c_2x^2 - 9b_1^2c_2y^2 + 6a_1c_2^2x = 0$ and

$$h_1 = -1/3, \qquad h_2 = -2/3, \qquad h_3 = -5/6.$$

In case (xi), $\Phi \equiv 2a_1^2x^2(y + a_{20}) + 2a_1(2c_1 - b_2a_{20})xy + c_1(2c_1 - b_2a_{20})y = 0$ and

$$h_1 = 1, \qquad h_2 = -1/2, \qquad h_3 = -3/2.$$

Applying the identity (23) to the set of conditions from Theorem 2.3, we obtain the case (xii), with $\Phi \equiv 3a_1b_2xy(2a_1x - b_2y + 6c_2) - 8b_2^2c_2y^2 + 9a_1c_2^2x = 0$ and

$$h_1 = 0, \qquad h_2 = 0, \qquad h_3 = -1.$$

Applying the identity (23) to the sets of conditions from Theorem 2.4, we have:
In case (xiii), $\Phi \equiv (3a_1 - a_2)(2a_1 - a_2)(2a_1x + b_2y)x^2 + 2b_2c_1^2y = 0$ and

$$h_1 = 2, \qquad h_2 = \frac{a_2 - 2a_1}{3a_1 - a_2}, \qquad h_3 = \frac{a_2 - 4a_1}{3a_1 - a_2}.$$

In case (xiv), $\Phi \equiv a_1^2(72a_1x + 63b_1y)x^2 - 882a_1b_1c_1xy - 686b_1^2c_1y^2 -$
$-1029b_1c_1^2y = 0$ and $h_1 = -2,\ h_2 = 1/6,\ h_3 = -5/6.$

In case (xv), $\Phi \equiv a_1^2x^2(3a_1x + 2b_1y + 6c_1) + 18a_1b_1c_1xy + 3a_1c_1^2x + 9b_1^2c_1y^2 = 0$ and $h_1 = -1/3,\ h_2 = -2/3,\ h_3 = -5/6.$

In case (xvi), $\Phi \equiv a_1^2x^2(a_1x + 6b_1y + 2c_1) + 6a_1b_1c_1xy + a_1c_1^2x - 9b_1^2c_1y^2 = 0$ and $h_1 = -1/3,\ h_2 = -2,\ h_3 = -1/6.$

Applying the identity (23) to the sets of conditions from Theorem 2.5, we get:

In case (xvii), $\Phi \equiv a_1x(a_1x - 3b_1y)^2 + 2a_1^2c_1x^2 - 18a_1b_1c_1xy + a_1c_1^2x - {}$ $-12b_1c_1^2y = 0$ and $h_1 = -1/2,\ h_2 = -1,\ h_3 = -1/2.$

In case (xviii), $\Phi \equiv a_1x(a_1x - 3b_1y)^2 - 36a_1b_1c_1xy - 27b_1c_1^2y = 0$ and $h_1 = -1/4, \quad h_2 = -1/4, \quad h_3 = -1.$

In case (xix), $\Phi \equiv a_1x(3a_1x + 2b_2y)^2 - 48a_1b_2c_1xy - 32b_2^2y^2 - 96b_2c_1^2y = 0$ and $h_1 = -5/2, \quad h_2 = 1, \quad h_3 = -3/2.$

In case (xx), $\Phi \equiv 9a_1x(a_1x + 2b_2y)^2 + 144a_1b_2c_1xy + 32b_2^2c_1y^2 + 96b_2c_1^2y = 0$ and $h_1 = -5/6,\ h_2 = -1/3,\ h_3 = -1/2.$

In case (xxi), $\Phi \equiv 4a_1x(3a_1x - b_1y)^2 + 36a_1^2c_1x^2 - 12a_1b_1c_1xy + 3b_1c_1^2y = 0$ and $h_1 = -1/3, \quad h_2 = -7/6, \quad h_3 = -1/6.$

In case (xxii), $\Phi \equiv 4a_1x(a_1x - 3b_1y)^2 + 4a_1^2c_1x^2 - 108a_1b_1c_1xy - 81b_1c_1^2y = 0$ and $h_1 = -1/3,\ h_2 = -1/2,\ h_3 = -5/6.$

In case (xxiii), $\Phi \equiv a_1x(3a_1x + 2b_2y)^2 + 18a_1^2c_1x^2 + 36a_1b_2c_1xy + 16b_2^2c_1y^2 + {}$ $+9a_1c_1^2x = 0$ and $h_1 = -1/2,\ h_2 = -1,\ h_3 = -1/2.$

Applying the identity (23) to the sets of conditions from Theorem 2.6, we obtain:

In case (xxiv), $\Phi \equiv (a_1x - 3b_2y)^3 + 2a_1^2c_1x^2 - 36a_1b_2c_1xy - 54a_2^2c_1y^2 + a_1c_1^2x - {}$ $-27b_2c_1^2y = 0$ and $h_1 = -5/6,\ h_2 = -1/2,\ h_3 = -2/3.$

In case (xxv), $\Phi \equiv (3a_1x - b_2y)^3 + 18a_1b_2c_1xy - 6b_2^2c_1y^2 - 9b_2c_1^2y = 0$ and $h_1 = -5/2, \quad h_2 = -3/2, \quad h_3 = 1.$

In case (xxvi), $\Phi \equiv (3a_1x - 5b_2y)^3 - 1350a_1b_2c_1xy - 750b_2^2c_1y^2 - 1125b_2c_1^2y = 0$ and $h_1 = -1/2,\ h_2 = -1/6,\ h_3 = -1.$

In case (xxvii), $\Phi \equiv (3a_1x + b_2y)^3 + 27a_1b_2c_1xy + 6b_2^2c_1y^2 + 9b_2c_1^2y = 0$ and $h_1 = -2,\ h_2 = -2/3,\ h_3 = 0.$

In case (xxviii), $\Phi \equiv (4a_1x + b_2y)^3 + 64a_1^2c_1x^2 + 32a_1b_2c_1xy + 4b_2^2c_1y^2 + {}$ $+4b_2c_1^2y = 0$ and $h_1 = -5/3,\ h_2 = -5/6,\ h_3 = 1/6.$

In case (xxix), $\Phi \equiv (a_1x - 2b_2y)^3 + a_1^2c_1x^2 - 40a_1b_2c_1xy - 32b_2^2c_1y^2 - {}$ $-32b_2c_1^2y = 0$ and $h_1 = -2/3,\ h_2 = -1/3,\ h_3 = -5/6.$

In case (xxx), $\Phi \equiv (3a_1x + 2b_2y)^3 + 27a_1^2c_1x^2 + 72a_1b_2c_1xy + 32b_2^2c_1y^2 + {}$ $+32b_2c_1^2y = 0$ and $h_1 = -\frac{5}{3},\ h_2 = -5/9,\ h_3 = -2/9$.

Theorem 5.1 is proved.

Conclusion

For Lotka-Volterra system (2) with a bundle of two invariant straight lines and one irreducible invariant cubic, modulo the symmetry (3), there were obtained 47 sets of Darboux integrability conditions.

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