CENTER-AFFINE INVARIANT CONDITIONS OF STABILITY OF UNPERTURBED MOTION FOR DIFFERENTIAL SYSTEM $s(1,2,3)$ WITH QUADRATIC PART OF DARBOUX TYPE

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Abstract. The Lie algebra, the Lyapunov series and the center-affine invariant conditions of stability of unperturbed motion have been determined by critical Lyapunov system with quadratic part of Darboux type.

Keywords: Differential system, stability of unperturbed motion, center-affine comitant and invariant, Lie algebra, Sibirsky graded algebra, group.
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CONDIȚII CENTROAFIN-INVARIANTE DE STABILITATE A MIȘCĂRII NEPERTURBATE PENTRU SISTEMUL DIFERENȚIAL $s(1,2,3)$ CU PARTEA PĂTRATICĂ DE TIP DARBOUX

Rezumat. A fost determinată algebra Lie, seria Lyapunov și condițiile centroafin-invariante de stabilitate a mișcării neperturbate guvernate de sistemul critic de tip Lyapunov cu partea pătratică de tip Darboux.

Cuvinte-cheie: Sistem diferențial, stabilitatea mișcării neperturbate, comitanți și invatianți centro-afini, algebră Lie, algebră Sibirschi graduată, grup.

Introduction

A lot of papers were written in the field of stability of motion. The universal scientific literature, concerning the stability of motion contains thousands of papers, including hundreds of monographs and textbooks of many authors. This literature is rich in the development of this theory, as well as in its applications in practice.

Note that many problems on stability treated in these works are governed by two-dimensional (or multidimensional) autonomous polynomial differential systems. Methods of the theory of invariants for such systems were elaborated in the school of differential equations from Chișinău. Moreover, there was developed the theory of the Lie algebras and Sibirsky graded algebras [1-5] with applications in the qualitative theory of these equations.

With a special weight, in this domain, it is published the Lyapunov (1857-1918) PhD thesis concerning the stability of motion in 1882 [6]. This work contains many fruitful ideas and results of great importance. It is considered that all history related to the theory on stability of motion is divided into periods before and after Lyapunov.

First of all, A.M. Lyapunov gave a strict definition of the stability of motion, which was so successful that all scientists took it as fundamental one for their researches.

In this paper and [7], with these visions was studied the Lie algebra, was built the Lyapunov series and was determined the stability of the unperturbed motion for two-dimensional critical differential system $s(1,2,3)$ with quadratic part of Darboux type.

1. The Lie algebra allowed of Lyapunov canonical form of the differential system $s(1,2,3)$ with quadratic part of Darboux type

We will examine the differential system $s(1,2,3)$ with quadratic part of Darboux type of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \qquad (j, \alpha, \beta, \gamma = 1,2), \tag{1}$$

where $a^j_{\alpha\beta}$ and $a^j_{\alpha\beta\gamma}$ are a symmetric tensors in lower indices in which the total convolution is done. Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$.

Remark 1.1. *The characteristic equation of system* (1) *has one zero root and the other ones real and negative if and only if the following invariant conditions* [7] *hold*

$$I_1^2 - I_2 = 0, \qquad I_1 < 0, \tag{2}$$

where

$$I_1 = a^\alpha_\alpha, \qquad I_2 = a^\alpha_\beta a^\beta_\alpha. \tag{3}$$

When the characteristic equation of (1) has one zero root and the other one is negative, i.e. the conditions (2) and $R_2 \equiv 0$ from (18) are satisfied, then this system by a center-affine transformation can be brought to its critical form

$$\begin{aligned} \frac{dx}{dt} &= x(gx + 2hy) + px^3 + 3qx^2y + 3rxy^2 + sy^3 \equiv P, \\ \frac{dy}{dt} &= ex + fy + y(gx + 2hy) + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \equiv Q, \end{aligned} \tag{4}$$

where $a^1_1 = a^1_2 = a^1_{22} = a^2_{11} = 0$ and $a^2_1 = e, a^2_2 = f, a^1_{11} = 2a^2_{12} = g, a^1_{12} = \frac{1}{2}a^2_{22} = h,$ $a^1_{111} = p,\ a^1_{112} = q, a^1_{122} = r,\ a^1_{222} = s,\ a^2_{111} = t,\ a^2_{112} = u,\ a^2_{122} = v,\ a^2_{222} = w.$

We examine the determined equations [8] for system (4)

$$\begin{aligned} \xi^1_x P + \xi^1_y Q &= \xi^1 P_x + \xi^2 P_y + D(P), \\ \xi^2_x P + \xi^2_y Q &= \xi^1 Q_x + \xi^2 Q_y + D(Q), \end{aligned} \tag{5}$$

where

$$\begin{aligned} D = \eta^1 \frac{\partial}{\partial e} + \eta^2 \frac{\partial}{\partial f} + \eta^3 \frac{\partial}{\partial g} + \eta^4 \frac{\partial}{\partial h} + \eta^5 \frac{\partial}{\partial p} + \eta^6 \frac{\partial}{\partial q} + \eta^7 \frac{\partial}{\partial r} + \eta^8 \frac{\partial}{\partial s} + \\ + \eta^9 \frac{\partial}{\partial t} + \eta^{10} \frac{\partial}{\partial u} + \eta^{11} \frac{\partial}{\partial v} + \eta^{12} \frac{\partial}{\partial w}. \end{aligned} \tag{6}$$

The polynomials P, Q are given in (4) and η^j $(j = \overline{1,12})$ are functions of the parameters *e*, *f*, *g*, *h*, *p*, *q*, *r*, *s*, *t*, *u*, *v*, *w*.

Let us consider

$$\xi^j = A^i x + B^i y \quad (i = \overline{1,2}), \tag{7}$$

where A^i, B^i are unknown parameters.

We write the operator

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + D, \tag{8}$$

where ξ^1, ξ^2 are given in (7) and D is defined in (6).

Solving the system of equations (5) with respect to the operators (6), (8) with coordinates (7) we obtain 3 independent linear operators

$$\begin{gathered} X_1 = x\frac{\partial}{\partial x} - e\frac{\partial}{\partial e} - g\frac{\partial}{\partial g} - 2p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} + s\frac{\partial}{\partial s} - 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \\ X_2 = y\frac{\partial}{\partial y} + e\frac{\partial}{\partial e} - h\frac{\partial}{\partial h} - q\frac{\partial}{\partial q} - 2r\frac{\partial}{\partial r} - 3s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}, \\ X_3 = x\frac{\partial}{\partial y} - f\frac{\partial}{\partial e} - 2h\frac{\partial}{\partial g} - 3q\frac{\partial}{\partial p} - 2r\frac{\partial}{\partial q} - s\frac{\partial}{\partial r} + (p-3u)\frac{\partial}{\partial t} + \\ +(q-2v)\frac{\partial}{\partial u} + (r-w)\frac{\partial}{\partial v} + s\frac{\partial}{\partial w}. \end{gathered} \tag{9}$$

Remark 1.2. *The system* (4) *admits a solvable three-dimensional Lie algebra* L_3 *composed of operators* (9).

The following transformation of the phase plan

$$x = \bar{x}, \qquad y = -\alpha\bar{x} + \bar{y}$$

corresponds to the representation operator X_3 from (9) of the system (4) .

With this transformation, for $f \neq 0$, we can always get the equality *e = 0.*

Remark 1.3. *This property, for* $f \neq 0$*, is true for any Lyapunov canonical two-dimensional system.*

2. Invariant conditions of stability of unperturbed motion for critical system $s(1,2,3)$ of Lyapunov type (4) with quadratic part of Darboux type

According to Lyapunov's Theorem [6, §32], we examine the non-critical equation of the system (4)

$$ex + fy + gxy + 2hy^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 = 0. \tag{10}$$

Then from this relation we express *y* and obtain

$$y = -\frac{e}{f}x - -\frac{g}{f}xy - 2\frac{h}{f}y^2 - \frac{t}{f}x^3 - 3\frac{u}{f}x^2y - 3\frac{v}{f}xy^2 - \frac{w}{f}y^3. \tag{11}$$

We seek *y* as a holomorphic function of *x*. Then we can write

$$y = -\frac{e}{f}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \cdots \tag{12}$$

Substituting (12) into (11) and identifying the coefficients of the same powers of *x* in the obtained relation we have

$$B_2 = \frac{e}{f^2}\left(g - 2\frac{eh}{f}\right),$$

$$B_3 = -[\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_2 + \frac{1}{f}(t - 3\frac{eu}{f} + 3\frac{e^2v}{f^2} - \frac{e^3w}{f^3})],$$

$$B_4 = -[2\frac{h}{f}B_2^2 + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_3 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_2],$$

$$B_5 = -[4\frac{h}{f}B_2B_3 + 3(\frac{v}{f} + \frac{ew}{f^2})B_2^2 + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_4 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_3],$$

$$B_6 = -[2\frac{h}{f}(2B_2B_4 + B_3^2) + \frac{w}{f}B_2^3 + 6\left(\frac{v}{f} + \frac{ew}{f^2}\right)B_2B_3 + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_5 +$$

$$+\frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_4],$$

$$B_7 = -[4\frac{h}{f}(B_2B_5 + B_3B_4) + 3\frac{w}{f}B_2^2B_3 + 3\left(\frac{v}{f} + \frac{ew}{f^2}\right)(2B_2B_4 + B_3^2) +$$

$$+\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_6 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_5],$$

$$B_8 = -[2\frac{h}{f}(2B_2B_6 + 2B_3B_5 + B_4^2) + 3\frac{w}{f}(B_2^2B_4 + B_2B_3^2) +$$

$$+6\left(\frac{v}{f} + \frac{ew}{f^2}\right)(B_2B_5 + B_3B_4) + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_7 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_6],$$

$$B_9 = -[4\frac{h}{f}(B_2B_7 + B_3B_6 + B_4B_5) + \frac{w}{f}(3B_2^2B_5 + 6B_2B_3B_4 + B_3^3) +$$

$$+3\left(\frac{v}{f} + \frac{ew}{f^2}\right)(2B_2B_6 + 2B_3B_5 + B_4^2) + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_8 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_7],$$

$$B_{10} = -[2\frac{h}{f}(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2) + 3\frac{w}{f}(B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 +$$

$$+B_3^2B_4) + 6\left(\frac{v}{f} + \frac{ew}{f^2}\right)(B_2B_7 + B_3B_6 + B_4B_5) + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_9 +$$

$$+\frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_8],$$

$$B_{11} = -[4\frac{h}{f}(B_2B_9 + B_3B_8 + B_4B_7 + B_5B_6) + 3\frac{w}{f}(B_2^2B_7 + 2B_2B_3B_6 +$$

$$+2B_2B_4B_5 + B_3^2B_5 + B_3B_4^2) + 3\left(\frac{v}{f} + \frac{ew}{f^2}\right)(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2) +$$

$$+\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_{10} + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_9],$$

$$B_{12} = -[2\frac{h}{f}(2B_2B_{10} + B_3B_9 + B_4B_8 + B_5B_7 + B_6^2) + \frac{w}{f}(3B_2^2B_8 + 6B_2B_3B_7 +$$

$$+6B_2B_4B_6 + 3B_2B_5^2 + 3B_3^2B_6 + 6B_3B_4B_5 + B_4^3) + 6\left(\frac{v}{f} + \frac{ew}{f^2}\right)(B_2B_9 + B_3B_8 +$$

$$B_4B_7 + B_5B_6) + +\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_{11} + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_{10}], ... \quad (13)$$

Substituting (12) into the right-hand side of the critical differential equation (4) we obtain

$$gx^2 + 2hxy + px^3 + 3qx^2y + 3rxy^2 + sy^3 =$$
$$= A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots$$

From this, taking into account (12) and (13), we get

$$A_2 = g - 2\frac{eh}{f},$$

$$A_3 = 2hB_2 + (t - 3\frac{eq}{f} + 3\frac{e^2r}{f^2} - \frac{e^3s}{f^3})],$$

$$A_4 = 2hB_3 + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_2,$$

$$A_5 = 2hB_4 + 3(r - \frac{es}{f})B_2^2 + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_3,$$

$$A_6 = sB_2^3 + 2hB_5 + 6(r - \frac{es}{f})B_2B_3 + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_4,$$

$$A_7 = 3sB_2^2B_3 + 2hB_6 + 3\left(r - \frac{es}{f}\right)(2B_2B_4 + B_3^2) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_5,$$

$$A_8 = 3s(B_2^2B_4 + B_2B_3^2) + 2hB_7 + 6\left(r - \frac{es}{f}\right)(B_2B_5 + B_3B_4) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_6,$$

$$A_9 = s(3B_2^2B_5 + 6B_2B_3B_4 + B_3^3) + 2hB_8 + 3\left(r - \frac{es}{f}\right)(2B_2B_6 + 2B_3B_5 + B_4^2) +$$
$$+3\left(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\right)B_7, \quad (14)$$

$$A_{10} = 3s(B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 + B_3^2B_4) + 2hB_9 + 6\left(r - \frac{es}{f}\right)(B_2B_7 +$$
$$+B_3B_6 + B_4B_5) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_8,$$

$$A_{11} = 3s(B_2^2B_7 + 2B_2B_3B_6 + 2B_2B_4B_5 + B_3^2B_5 + B_3B_4^2) + 2hB_{10} +$$
$$+3\left(r - \frac{es}{f}\right)(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_9,$$

$$A_{12} = s3(B_2^2B_8 + 6B_2B_3B_7 + 6B_2B_4B_6 + 3B_2B_5^2 + 3B_3^2B_6 + 6B_3B_4B_5 + B_4^3) +$$
$$+2hB_{11} + 6\left(r - \frac{es}{f}\right)(B_2B_9 + B_3B_8 + B_4B_7 + B_5B_6) + 3\left(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\right)B_{10}, ...$$

We introduce the following notations:

$$N_1 = fg - 2eh; \quad N_2 = f^3p - 3ef^2q + 3\,e^2f\,r - e^3\,s;$$
$$N_3 = f^3t - 3ef^2u + 3\,e^2f\,v - e^3\,w; \quad N_4 = f^2q - 2\,e^2f\,r - e^3\,s; \quad (15)$$
$$N_5 = fr - es.$$

Then, from (13) and (14) with this notations we obtain

$$B_2 = \frac{e}{f^3}N_1, \qquad B_3 = -(\frac{1}{f^2}B_2N_1 + \frac{1}{f^4}N_3),$$

$$A_2 = \frac{1}{f}N_1, \qquad A_3 = 2hB_2 + \frac{1}{f^3}N_2, \qquad A_4 = 2hB_3 + \frac{3}{f^2}B_2N_4,$$

$$A_5 = 2hB_4 + \frac{3}{f}B_2^2N_5 + \frac{3}{f^2}B_3N_4,$$

$$A_6 = sB_2^3 + 2hB_5 + \frac{6}{f}B_2B_3N_5 + \frac{3}{f^2}B_4N_4,$$

$$A_7 = 3sB_2^2B_3 + 2hB_6 + \frac{3}{f}(2B_2B_4 + B_3^2)N_5 + \frac{3}{f^2}B_5N_4,$$

$$A_8 = 3s(B_2^2B_4 + B_2B_3^2) + 2hB_7 + \frac{6}{f}(B_2B_5 + B_3B_4)N_5 + \frac{3}{f^2}B_6N_4,$$

$$A_9 = s(3B_2^2B_5 + 6B_2B_3B_4 + B_3^3) + 2hB_8 + \frac{3}{f}(2B_2B_6 + 2B_3B_5 + B_4^2)N_5 + \frac{3}{f^2}B_7N_4,$$

$$A_{10} = 3s(B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 + B_3^2B_4) + 2hB_9 + \frac{6}{f}(B_2B_7 + B_3B_6 + B_4B_5)N_5 + \frac{3}{f^2}B_8N_4, \ldots \qquad (16)$$

Lemma 2.1. *The stability of unperturbed motion in the system of perturbed motion* (4) *is described by one of the following twelve possible cases, if for expressions* (15) $I_1 = f < 0$*) the following conditions are satisfied:*

I. $N_1 \neq 0$, *then the unperturbed motion is unstable;*

II. $N_1 = 0, N_2 > 0$, *then the unperturbed motion is stable;*

III. $N_1 = 0, N_2 < 0$, *then the unperturbed motion is unstable;*

IV. $N_1 = N_2 = 0,\ hN_3 \neq 0$, *then the unperturbed motion is unstable;*

V. $N_1 = N_2 = h = 0;\ N_3N_4 < 0$, *then the unperturbed motion is unstable;*

VI. $N_1 = N_2 = h = 0;\ N_3N_4 > 0$, *then the unperturbed motion is stable;*

VII. $N_1 = N_2 = N_4 = h = 0,\ N_3 \neq 0;\ N_5 > 0$, *then the unperturbed motion is stable;*

VIII. $N_1 = N_2 = N_4 = h = 0,\ N_3 \neq 0;\ N_5 < 0$, *then the unperturbed motion is unstable;*

IX. $N_1 = N_2 = N_4 = N_5 = h = 0;\ sN_3 < 0$, *then the unperturbed motion is unstable;*

X. $N_1 = N_2 = N_4 = N_5 = h = 0;\ sN_3 > 0$, *then the unperturbed motion is stable;*

XI. $N_1 = N_2 = N_3 = 0$, *then the unperturbed motion is stable;*

XII. $N_1 = N_2 = N_4 = N_5 = h = s = 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, this motion is also asymptotic stable in Cases II, VI, VII and X. The expressions N_i $(i = \overline{1,5})$ *are given in* (15).

Proof. According to Lyapunov Theorem [6, §32], the coefficients of the A_i series from (14) are analyzed.

If $A_2 \neq 0$, then from (16) we get $N_1 \neq 0$ (taking into account that $I_1 = f < 0$). According to Lyapunov Theorem [6, §32], we have proved the Case I.

If $A_2 = 0$, i.e. $N_1 = 0$ respectively $B_2 = 0$, then by (16) the stability or the instability of unperturbed motion is determined by the sign of the expression A_3 (the sign of the product N_2). Using the Lyapunov Theorem [6, §32] we obtain the Cases II and III.

If $N_1 = N_2 = 0$, then from (16) we get $A_4 = -2\frac{h}{f^4}N_3$. If $hN_3 \neq 0$. Then we obtain the Cases IV (see the Lyapunov Theorem [6, §32]).

Suppose $N_1 = N_2 = h = 0$. Then from (16) it results that $A_5 = -\frac{3}{f^6}N_3N_4$. So the stability or the instability of the unperturbed motion is determined by the sign of expression N_3N_4. Using the Lyapunov Theorem [6, §32] we get the Cases V and VI.

If $N_1 = N_2 = N_3 = 0$, then all $B_i = 0$ $(i \geq 3)$ and respectively $A_i = 0$ $(i \geq 5)$. By the Lyapunov Theorem [6, §32] we have the Case XI.

If $N_1 = N_2 = N_4 = h = 0$ and $N_3 \neq 0$, then $A_6 = 0$, but $A_7 = \frac{3}{f^9}N_3^2N_5$. So the stability or the instability of the unperturbed motion is determined by the sign of expression N_5. Using the Lyapunov Theorem [6, §32] we get the Cases VII and VIII.

If $N_1 = N_2 = N_4 = N_5 = h = 0$ and $N_3 \neq 0$, then $A_8 = 0$, but $A_9 = -\frac{s}{f^{12}}N_3^3$. So the stability or the instability of the unperturbed motion is determined by the sign of expression sN_3. Using the Lyapunov Theorem [6, §32] we get the Cases IX and X.

If $N_1 = N_2 = N_4 = N_5 = h = s = 0$ then all $A_i = 0$ $(\forall i)$ vanish. By the Lyapunov Theorem [6, §32] we get the Case XII. Lemma 2.1 is proved.

Let φ and ψ be homogeneous comitants of degree ρ_1 and ρ_2 respectively of the phase variables x and y of a two-dimensional polynomial differential system. Then the transvectant

$$(\varphi, \psi)^{(j)} = \frac{(\rho_1 - j)(\rho_2 - j)}{\rho_1!\,\rho_2!}\sum_{i=0}^{j}(-1)^j\binom{j}{i}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}} \tag{17}$$

is also a comitant for this system.

In the Iu. Calin's works, see for example [9], it is shown that by means of the transvectant (17) all generators of the Sibirsky algebras of comitants and invariant for any system of type (1) can be constructed.

According to [10] we write the following comitants of the system (1)

$$R_i = P_i(x,y)y - Q_i(x,y)x, \qquad S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right), (i = \overline{1,3}). \tag{18}$$

Later on, we will need the following comitants and invariants from [10] of system (1) built by operations (17) and (18):

$$I_1 = S_1,\ \ I_2 = (R_1, R_1)^{(2)},\ \ I_3 = ((R_3, R_1)^{(2)}, R_1)^{(2)},\ \ I_4 = (S_3, R_1)^{(2)},$$
$$K_2 = R_1,\ \ K_5 = S_2,\ \ K_8 = R_3,\ \ K_9 = (R_3, R_1)^{(1)},\ \ K_{10} = (R_3, R_1)^{(2)}, \quad (19)$$
$$K_{11} = ((R_3, R_1)^{(2)}, R_1)^{(1)},\ \ K_{14} = (S_2, R_1)^{(1)},\ \ K_{15} = S_3,\ \ K_{16} = (S_3, R_1)^{(1)}.$$

We consider for system (1) the following expressions composed of comitants and invariants from (19) that can be written in the form:

$$\mathcal{N}_1 = 2K_{14} - I_1 K_5,$$
$$\mathcal{N}_2 = 2I_1^2\, K_{10} - 4\, I_1\, K_{11} - 3\, I_1\, I_2 K_{15} - 3\, I_1^2 K_{16} +\ 4I_3 K_2 + 3I_1\, I_4 K_2,$$
$$\mathcal{N}_3 = -12 I_1\, K_{10}\, K_2 + 8\, K_{11}\, K_2 + 3\, I_1^2\, K_{15}\, K_2 - 6\, I_1\, K_{16}\, K_2 + 6\, I_4\, K_2^2 -$$
$$-4\, I_1^3\, K_8 +\ 8\, I_1^2\, K_9,\ \ \mathcal{N}_4 = 2\, I_3\ +\ I_1\, I_4,\ \ \mathcal{N}_5 = 2\, K_{10}\ +\ I_1\, K_{15} - K_{16}, \qquad (20)$$
$$S = 3\, K_{15}\, K_2\ -\ 2\, I_1\, K_8\ -\ 4\, K_9.$$

Theorem [11]. *Let for system of perturbed motion* (1) *the invariant conditions* (2)-(3) and $R_2 \equiv 0$ from (18) *are satisfied. Then the stability of unperturbed motion is described by one of the following twelve possible cases:*

I. $\mathcal{N}_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $\mathcal{N}_1 \equiv 0, \mathcal{N}_2 > 0$, *then the unperturbed motion is stable;*

III. $\mathcal{N}_1 \equiv 0, \mathcal{N}_2 < 0$, *then the unperturbed motion is unstable;*

IV. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv 0,\ K_5\mathcal{N}_3 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv\ K_5 \equiv 0;\ \mathcal{N}_3\mathcal{N}_4 < 0$, *then the unperturbed motion is unstable;*

VI. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv\ K_5 \equiv 0;\ \mathcal{N}_3\mathcal{N}_4 > 0$, *then the unperturbed motion is stable;*

VII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ K_5 \equiv 0, \mathcal{N}_3 \not\equiv 0;\ N_5 > 0$, *then the unperturbed motion is stable;*

VIII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ K_5 \equiv 0\ ,\ \mathcal{N}_3 \not\equiv 0\ ;\ N_5 < 0\ ,$ *then the unperturbed motion is unstable;*

IX. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ \mathcal{N}_5 \equiv K_5 = 0;\ S\mathcal{N}_3 < 0$, *then the unperturbed motion is unstable;*

X. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ \mathcal{N}_5 \equiv K_5 \equiv 0;\ S\mathcal{N}_3 > 0$, *then the unperturbed motion is stable;*

XI. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_3 \equiv\ 0$, *then the unperturbed motion is stable;*

XII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ \mathcal{N}_5 \equiv K_5 \equiv S \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, this motion is also asymptotic stable in Cases II, VI, VII and X. The expressions $S, K_5, \mathcal{N}_i\ (i = \overline{1,5})$ *are given in* (19)-(20).

Proof. Observe that the first three expressions from (20), for critical system (4), look as follows:

$$\mathcal{N}_1 = -3N_1 x,\ \ \mathcal{N}_2 = 4N_2 x^2,\ \ \mathcal{N}_3 = 8N_3 x^4 - 8N_2 x^3 y,\ \ \mathcal{N}_4 = 2N_4,$$
$$\mathcal{N}_5 = \frac{2}{f} N_5 (ex + fy)^2,\ \ K_5 = 3\frac{h}{f}(ex + fy),\ \ S = -4\frac{s}{f^3} N_5 (ex + fy)^4. \quad (21)$$

Using the expressions (21) and the last assertion together with Lemma 2.1, we obtain the Cases I-XII. We note that the comitants $\mathcal{N}_2, \mathcal{N}_3\mathcal{N}_4, \mathcal{N}_5, S\mathcal{N}_3$ from (20), used in the Cases II-X of Theorem, are even-degree comitants with respect to *x* and *y* and have the weights [1] equal to 0, 0, 0, -2, respectively. Moreover, each one of these comitants (in the case when it is applied) is a binary form with a well-defined sing. This ensures that any center-affine transformation cannot change their sign. Theorem is proved.

Conclusions

In this paper the Lie algebra allowed by differential system $s(1,2,3)$ of the Lyapunov canonical form with quadratic part of the Darboux type was determined, which is a solvable three-dimensional algebra. Based on the constructed Lyapunov series, all center-affine invariant conditions of stability of the unperturbed motion were obtained and they are included in twelve cases.

References

1. Sibirsky K. S. Introduction to the algebraic theory of invariants of differential equations. Nonlinear Science: Theory and Applications. Manchester: Manchester University Press, 1988.
2. Vulpe N. I. Polynomial bases of comitants of differential systems and their applications in qualitative theory. Kishinev: Știința, 1986 (in Russian).
3. Popa M.N. Algebraic methods for differential system. Editura the Flower Power, Universitatea din Pitești, Seria Matematică Aplicată și Industrială, 2004, 15 (in Romanian).
4. Diaconescu O. Lie algebras and invariant integrals for polynomial differential systems. Chișinău: PhD thesis, 2008 (in Russian).
5. Popa M. N., Pricop V. The center-focus problem: algebraic solutions and hypotheses. Chișinău: AȘM, IMI, 2018, 256 p. (in Russian).
6. Liapunov A. M. Obshchaia zadacha ob ustoichivosti dvijenia, Sobranie sochinenii, II – Moskva-Leningrad: Izd. Acad. Nauk SSSR, 1956 (in Russian).
7. Neagu N., Orlov V., Popa M.N. Invariant conditions of stability of unperturbed motion governed by some differential systems in the plane. In: Bull. Acad. Sci. of Moldova, Mathematics, 2017, vol. 85, no. 3, p. 88-106.
8. Popa M.N. Algebre Lie și sisteme diferențiale. Univ. de Stat din Tiraspol, Chișinău: Tipogr. AȘM, 2008, 164 p.
9. Calin Iu. On rational bases of $\mathrm{GL}(2,\mathbb{R})$ - comitants of planar polynomial systems of differential equations. Bul. AȘM. Mat., 2003, No. 2(42), p. 69-86.
10. Ciubotaru S. Rational bases of $\mathrm{GL}(2,\mathbb{R})$ - comitants and $\mathrm{GL}(2,\mathbb{R})$ - invariants for the planar systems of differential equations with nonlinearities of the fourth degree. Bul. Acad. Științe Repub. Moldova. Mat., 2015, No. 3(79), p. 14-34.
11. Neagu N., Orlov V. Invariant conditions of stability of unperturbed motion described by cubic differential systems with quadratic part of Darboux type. The 26th Conference on Applied and Industrial Mathematics (CAIM 2018), Chișinău. Book of Abstracts, p. 37-39.