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INVARIANT CONDITIONS OF STABILITY OF MOTION FOR SOME FOUR-DIMENSIONAL DIFFERENTIAL SYSTEMS

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Abstract. Center-affine invariant conditions of the stability of unperturbed motion were determined for four-dimensional quadratic differential system of Darboux type in non-degenerate invariant condition.
Keywords: differential system, unperturbed motion, invariant, comitant, Lie algebra, stability.
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CONDIȚIILE INVARIANTE DE STABILITATE ALE MIȘCĂRII PENTRU UNELE SISTEME DIFERENȚIALE PATRUDIMENSIONALE

Rezumat. Au fost obținute condițiile centroafin invariante de stabilitate a mișcării neperturbate pentru sistemul diferențial patru dimensional pătratic de tip Darboux în condiția invariantă nedegenerată.
Cuvinte-cheie: sistemul diferențial, mișcarea neperturbată, invariant, comitant, algebra Lie, stabilitatea.

1. Introduction

In mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions.

The differential systems with polynomial nonlinearities are important in various applied problems. For example: the Van der Pol oscillator; the Fitzhugh–Nagumo model for action potentials of neurons; in seismology to model the two plates in a geological fault; in studies of phonation to model the right and left vocal fold oscillators as well as many other applications.

The stability of unperturbed motions using the theory of algebras, of invariants and of Lie algebras was studied for the first time in [1].

In [2] the center-affine invariant conditions of stability of unperturbed motion, described by critical two-dimensional differential systems with quadratic nonlinearities s(1; 2), cubic nonlinearities s(1; 3) and fourth-order nonlinearities s(1; 4), were obtained.

In this paper, the similar investigations are done for some four-dimensional differential systems with quadratic nonlinearities.

2. Center-affine invariants and mixt comitants for four-dimensional differential system with quadratic nonlinearities

We consider the system of differential equations

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \equiv P^j(x,a) \quad \left(j,\alpha,\beta=\overline{1,4}\right), \tag{1}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices in which the total convolution is done, and the group of center-affine transformations $GL(4,\mathbb{R})$ given by formulas

$$\overline{x}^r = q^r_j x^j, \quad \det\left(q^r_j\right) \neq 0; \quad \left(r, j = \overline{1,4}\right). \tag{2}$$

Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$. The phase variables vector $x = \left(x^1, x^2, x^3, x^4\right)$ of system (1), which changes by formulas (2), is usually called *contravariant* [3]. Any other vector $y = \left(y^1, y^2, y^3, y^4\right)$ which changes by formulas (2), is called *cogradient* with vector x. The vector $u = \left(u_1, u_2, u_3, u_4\right)$, which changes by formulas

$$\overline{u}_r = p^j_r u_j, \quad \left(r, j = \overline{1,4}\right), \tag{3}$$

where $p^r_j q^j_s = \delta^r_s$ is the Kronecker's symbol, is called *covariant*. The vector u is also called *contragradient* with vector x.

Applying the transformation (2), the system (1) will be brought to the system

$$\frac{d\overline{x}^j}{dt} = \overline{a}^j_\alpha \overline{x}^\alpha + \overline{a}^j_{\alpha\beta} \overline{x}^\alpha \overline{x}^\beta \quad \left(j, \alpha, \beta = \overline{1,4}\right), \tag{4}$$

in which the coefficients are linear functions of the coefficients of system (1) and are rational functions of parameters of transformation (2). We will denote the set of coefficients of system (1) by a, the set of coefficients of transformed system (4) by $\overline{a}$, and the set of parameters of transformation (2) by q.

According to [3], we say that the polynomial $k(x,u,a)$ of the coefficients of system (1) and of the coordinates of vectors x and u is call *mixt comitant* of the system (1) with respect to $GL(4,\mathbb{R})$ group, if the following identity holds

$$k(\overline{x}, \overline{u}, \overline{a}) = \Delta^{-g} \cdot k(x,u,a), \tag{5}$$

for all q from $GL(4,\mathbb{R})$ and every coordinates of vectors x and u, as well as all the coefficients a of system (1), where g is an integer number called the *weight of comitant*. If the mixt comitant k does not depend on the coordinates of the vector u, then we call it simply *comitant*, but if k does not depend on the coordinates of the vector x we call it *contravariant*. If k does not depend on x and u, then we will call it *invariant* of system (1) with respect to $GL(4,\mathbb{R})$ group.

The following center-affine invariant polynomials of the system (1) are known from [4]:

$$I_{1,4} = a^\alpha_\alpha, \quad I_{2,4} = a^\alpha_\beta a^\beta_\alpha, \quad I_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta, \quad I_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_\gamma,$$

$$P_{1,4} = a^\alpha_{\alpha\beta} x^\beta, \quad P_{2,4} = a^\alpha_\beta a^\beta_{\alpha\gamma} x^\gamma, \quad P_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_{\beta\delta} x^\delta, \quad P_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_{\gamma\mu} x^\mu,$$

$$K_{6,4} = a^\alpha_\theta a^\beta_\gamma a^\gamma_\varphi a^\delta_\mu a^\mu_\nu a^\nu_\psi x^\theta x^\varphi x^\psi x^\tau \varepsilon_{\alpha\beta\delta\tau}, \quad S_{0,4} = u_\alpha x^\alpha, \quad S_{1,4} = a^\alpha_\beta x^\beta u_\alpha,$$

$$S_{2,4}=a_{\gamma}^{\alpha}a_{\alpha}^{\beta}x^{\gamma}u_{\beta}\,,\;\; S_{3,4}=a_{\delta}^{\alpha}a_{\alpha}^{\beta}a_{\beta}^{\gamma}x^{\delta}u_{\gamma}\,,\;\; \bar{R}_{6,4}=a_{p}^{\alpha}\,a_{q}^{\beta}\,a_{\beta}^{\gamma}\,a_{r}^{\delta}a_{\delta}^{\mu}\,a_{\mu}^{\nu}u_{s}u_{\alpha}u_{\gamma}u_{\nu}\varepsilon^{pqrs}\,,$$

$$\bar{R}_{6,4}=\det\left(\frac{\partial S_{i-1,4}}{\partial x^{j}}\right)_{i,j=\overline{1,4}}\,,\quad \tilde{K}_{1,4}=a_{\beta\gamma}^{\alpha}\,x^{\beta}x^{\gamma}y^{\delta}z^{\mu}\varepsilon_{\alpha\gamma\delta\mu}, \tag{6}$$

$I_{i,4}$ $\left(i=\overline{1,4}\right)$ are invariants, $P_{i,4}$ $\left(i=\overline{1,4}\right)$ and $K_{6,4}$ are comitants, $S_{j,4}$ $\left(j=\overline{0,3}\right)$ are mixed comitants, $\bar{R}_{6,4}$ is contravariant, and $\tilde{K}_{1,4}$ is comitant of cogradient vectors x, y, z [3]. The vectors $\varepsilon_{\alpha\beta\delta\tau}$ and ε^{pqrs} are four-dimensional unit vector with coordinates 1 when an even permutation of the indices holds, -1 when an odd permutation of the indices holds and 0 in other cases.

Remark 1. The characteristic equation of the system (1) has the form

$$\rho^{4}+L_{1,4}\,\rho^{3}+L_{2,4}\,\rho^{2}+L_{3,4}\,\rho+L_{4,4}=0, \tag{7}$$

where the coefficients of equation (7) are invariants of system (1) and have the following form:

$$L_{1,4}=-I_{1,4},\quad L_{2,4}=\frac{1}{2}\left(I_{1,4}^{2}-I_{2,4}\right),\quad L_{3,4}=\frac{1}{6}\left(3I_{1,4}I_{2,4}-2I_{3,4}-I_{1,4}^{3}\right),$$

$$L_{4,4}=\frac{1}{24}\left(8I_{1,4}I_{3,4}-6I_{4,4}-6I_{1,4}^{2}I_{2,4}+3I_{2,4}^{2}+I_{1,4}^{4}\right), \tag{8}$$

where $I_{1,4}$ $\left(i=\overline{1,4}\right)$ from (6).

3. Invariant conditions of stability of unperturbed motion for system (1) in case when the roots of the characteristic equation have nonzero real parts

Definition 1. If for any small positive value ε, however small, one can find a positive number δ such that for all perturbations $x^{j}(t_0)$ satisfying the condition

$$\sum_{j=1}^{2}\left(x^{j}(t_0)\right)^{2}\le\delta, \tag{9}$$

the inequality $\sum_{j=1}^{2}\left(x^{j}(t)\right)^{2}<\varepsilon,$ is valid for any $t\ge t_0$, then the unperturbed motion $x^{j}=0$ $\left(j=\overline{1,4}\right)$ is called *stable*, otherwise it is called unstable. If the unperturbed motion is stable and the number δ can be found however small such that for any perturbed motions satisfying (9) the condition $\lim_{t\to\infty}\sum_{j=1}^{2}\left(x^{j}(t)\right)^{2}=0,$ is valid, then the unperturbed motion is called *asymptotically stable.*

By means of the Lyapunov theorems on stability of unperturbed motion by the signs of the roots of the characteristic equation (7) of system (1) and the Hurwitz theorem on the signs of the roots of an algebraic equation (see, for example, [5]) we have

Theorem 1. Assume that the center-affine invariants (8) of system (1) satisfy inequalities

$$L_{i,4} > 0 \ \left(i = \overline{1,4}\right), \quad L_{1,4}L_{2,4}L_{3,4} - L_{3,4}^2 - L_{1,4}^2 L_{4,4} > 0 .$$

Then the unperturbed motion $x^j = 0 \ \left(j = \overline{1,4}\right)$ of this system is asymptotically stable.

Theorem 2. If at least one of the center-affine invariant expressions (8) of system (1) is negative, then the unperturbed motion $x^j = 0 \ \left(j = \overline{1,4}\right)$ of this system is unstable.

4. Invariant conditions of stability of unperturbed motion for system (1) in case when the characteristic equation has one zero root in conditions $\bar{R}_{6,4} \neq 0, \ \tilde{K}_{1,4} \equiv 0$

Lemma 1. [4] If in (6) we have $\tilde{K}_{1,4} \equiv 0$ then the system (1) takes the form

$$\frac{dx^j}{dt} = a_\alpha^j x^\alpha + 2x^j \left(a_{1\alpha}^1 x^\alpha\right) \ \left(j, \alpha = \overline{1,4}\right). \qquad (10)$$

The system (10) is called *four-dimensional differential system of Darboux type*.

Remark 2. The expression $K_{6,4} = 0$ from (6) is the invariant partucular $GL(4, \mathbb{R})-$ integral of system (10).

Remark 3. For any center-affine transformation of the system (6), its quadratic part retains its form changing only the variables and coefficients. This follows from the fact that the identity $\tilde{K}_{1,4} \equiv 0$ is preserved under any center-affine transformation.

From [4] with considering Remark 3 it follows

Lemma 2. If in system (10) we have $\bar{R}_{6,4} \neq 0$, then by the center-affine transformation

$$\bar{x}^1 = S_{0,4}, \ \bar{x}^2 = S_{1,4}, \ \bar{x}^3 = S_{2,4}, \ \bar{x}^4 = S_{3,4},$$

the system (10) can be brought to the following form :

$$\dot{x}^1 = x^2 + 2x^1\left(a_{1\alpha}^1 x^\alpha\right), \ \dot{x}^2 = x^3 + 2x^2\left(a_{1\alpha}^1 x^\alpha\right), \ \dot{x}^3 = x^4 + 2x^3\left(a_{1\alpha}^1 x^\alpha\right),$$
$$\dot{x}^4 = -L_{4,4}x^1 - L_{3,4}x^2 - L_{2,4}x^3 - L_{1,4}x^4 + 2x^4\left(a_{1\alpha}^1 x^\alpha\right), \qquad (11)$$

where $S_{i,4} \ \left(i = \overline{0,3}\right)$ are from (6) and $L_{j,4} \ \left(j = \overline{1,4}\right)$ are from (8).

Definition 2. The differential system (1) will be called *a critical system of Lyapunov type* if the characteristic equation of the system has one zero root and all other roots have negative real parts.

Notice that for system (11) the characteristic equation coincides with equation (7).

Lemma 3. The system (1) or (11) is critical of Lyapunov type if and only if the following invariant conditions hold:

$$L_{4,4}=0,\quad L_{i,4}>0\quad (i=1,2,3),\quad L_{1,4}L_{2,4}-L_{3,4}>0, \tag{12}$$

where $L_{j,4}\ \left(j=\overline{1,4}\right)$ are from (8).

The proof of Lemma 3 follows from the Hurwitz theorem on the signs of the roots of an algebraic equation and from equation (7) (see, for example [5]).

Notice that the system (11) in invariant conditions (12) by the center-affine transformation

$$\bar{x}^1=L_{3,4}x^1+L_{2,4}x^2+L_{1,4}x^3+x^4,\quad \bar{x}^2=x^2,\quad \bar{x}^3=x^3,\quad \bar{x}^4=x^1,$$

can be brought to the canonical form

$$\dot{x}=2x\left(ax+by+cz+du\right),\ \ \dot{y}=z+2y\left(ax+by+cz+du\right),$$

$$\dot{z}=x-L_{2,4}y-L_{1,4}z-L_{3,4}u+2z\left(ax+by+cz+du\right),\ \dot{u}=y+2u\left(ax+by+cz+du\right). \tag{13}$$

According to Lyapunov's theorem [6], we will build the power series by which we can determine the stability of unperturbed motion of system (13). The first equation in system (13) is called the *critical equation*, and the other three are called *non-critical equations.* Using the algorithm from Lyapunov's theorem [6] we examine the equations generated by right-hand sides of latest three equations of system (13). We have non-critical equations $z+2y\left(ax+by+cz+du\right)=0$, $x-L_{2,4}y-L_{1,4}z-L_{3,4}u+2z\left(ax+by+cz+du\right)=0$, $y+2u\left(ax+by+cz+du\right)=0$.

We express $x,\ y$ and z from non-critical equations in the following way:

$$y=-2u\left(ax+by+cz+du\right),\ \ z=-2y\left(ax+by+cz+du\right),$$

$$u=\frac{x}{L_{3,4}}-\frac{L_{2,4}}{L_{3,4}}y-\frac{L_{1,4}}{L_{3,4}}z+\frac{2z}{L_{3,4}}\left(ax+by+cz+du\right) \tag{14}$$

We will seek $x,\ y$ and z as a holomorphic function on x. Then we can write

$$y(x)=A_1x+A_2x^2+A_3x^3+...,\ z(x)=B_1x+B_2x^2+B_3x^3+...,\ u(x)=C_1x+C_2x^2+C_3x^3+... \tag{15}$$

Substituting (15) into (14) we get

$$A_1x+A_2x^2+A_3x^3+...=-2(C_1x+C_2x^2+C_3x^3+...)[ax+b(A_1x+A_2x^2+A_3x^3+...)+$$
$$+c(B_1x+B_2x^2+B_3x^3+...)+d(C_1x+C_2x^2+C_3x^3+...)],$$
$$B_1x+B_2x^2+B_3x^3+...=-2(A_1x+A_2x^2+A_3x^3+...)[ax+b(A_1x+A_2x^2+A_3x^3+...)+$$
$$+c(B_1x+B_2x^2+B_3x^3+...)+d(C_1x+C_2x^2+C_3x^3+...)],$$
$$C_1x+C_2x^2+C_3x^3+...=\frac{x}{L_{3,4}}-\frac{L_{2,4}}{L_{3,4}}(A_1x+A_2x^2+A_3x^3+...)-\frac{L_{1,4}}{L_{3,4}}(B_1x+B_2x^2+B_3x^3+...)+$$
$$+2(B_1x+B_2x^2+B_3x^3+...)[ax+b(A_1x+A_2x^2+A_3x^3+...)+c(B_1x+B_2x^2+B_3x^3+...)+$$
$$+d(C_1x+C_2x^2+C_3x^3+...)].$$

This implies that $A_1=0,\ B_1=0,\ \ C_1=\dfrac{1}{L_{3,4}},\ A_2=-2C_1(a+dC_1),\ B_2=0,$

$$C_2 = \frac{2C_1L_{2,4}(a+dC_1)}{L_{3,4}},\ \ A_3 = -2[bA_2C_1 + C_2(a+2dC_1)],\ \ B_3 = -2A_2(a+dC_1),$$

$$C_3 = \frac{2}{L_{3,4}}[A_2L_{1,4}(a+dC_1) + bA_2C_1L_{2,4} + C_2L_{2,4}(a+2dC_1)],$$

$$A_4 = -2[C_1(bA_3+cB_3) + C_2(bA_2+dC_2) + C_3(a+2dC_1)],\quad B_4 = -2[A_3(a+dC_1) + A_2(bA_2+dC_2)],$$

$$C_4 = \frac{2}{L_{3,4}}[(B_3 + A_3L_{1,4})(a+dC_1) + (A_2L_{1,4} + C_2L_{2,4})(bA_2+dC_2) + C_1L_{2,4}(bA_3+cB_3) +$$

$$+C_3L_{2,4}(a+2dC_1)],... \qquad (16)$$

Substituting (15) into right-hand side of the critical equation (13) we get

$$2x\left(ax+by+cz+du\right) = D_1x + D_2x^2 + D_3x^3 + ...,$$

or in expanded form we get

$$2x[ax + b(A_1x + A_2x^2 + A_3x^3 + ...) + c(B_1x + B_2x^2 + B_3x^3 + ...) + d(C_1x + C_2x^2 + C_3x^3 + ...)] =$$

$$= D_1x + D_2x^2 + D_3x^3 + ...,$$

This implies that

$$D_1 = 0,\ D_2 = 2(a+dC_1),\ D_3 = 2(bA_2+dC_2),\ D_4 = 2(bA_3+cB_3+dC_3),$$

$$D_5 = 2(bA_4+cB_4+dC_4),\ D_6 = 2(bA_5+cB_5+dC_5),\ D_7 = 2(bA_6+cB_6+dC_6),... \qquad (17)$$

Using the Lyapunov's theorem, in [7] was obtained

Lemma 4. The stability of the unperturbed motion corresponding to system (13) is described by one of the following two possible cases:

1) $L_{3,4}a + d \neq 0,$ then the unperturbed motion is unstable ;

2) $L_{3,4}a + d = 0,$ then the unperturbed motion is stable.

In the latter case the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, if perturbations are small enough then perturbed motion will tend Asymptotically to one of stabilized motions.

Proof. According to Lyapunov's theorem on stability of unperturbed motion in critical case [6], we examine the coefficients D_i from (17) taking into account (16). If $D_2 \neq 0$, then we have first case from Lemma 4. If $D_2 = 0$, then we obtain $A_i = B_i = C_i = 0\ (i \geq 2)$ from (16), therefore $D_i = 0,\ i = 1,2,3,...$. According to Lyapunov's theorem we have the second case of this lemma. Lemma 4 is proved.

Theorem 3. Let for differential system of the perturbed motion (1) the invariant conditions $\bar{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$ be satisfied. Then in conditions (12) the stability of unperturbed motion corresponding to this system is described by one of the following two possible cases:

1) $4(I_{1,4}^3P_{1,4} - 3I_{1,4}I_{2,4}P_{1,4} + 2I_{3,4}P_{1,4}) - 15(I_{1,4}^2P_{2,4} - I_{2,4}P_{2,4} - 2I_{1,4}P_{3,4} + 2P_{4,4}) \neq 0,$ then the unperturbed motion is unstable;

1) $4(I_{1,4}^3 P_{1,4} - 3I_{1,4}I_{2,4}P_{1,4} + 2I_{3,4}P_{1,4}) - 15(I_{1,4}^2 P_{2,4} - I_{2,4}P_{2,4} - 2I_{1,4}P_{3,4} + 2P_{4,4}) = 0,$ then the unperturbed motion is stable.

In the latter case the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, if perturbations are small enough then perturbed motion will tend Asymptotically to one of stabilized motions. The invariant polynomials $I_{i,4}$ $(i = \overline{1,4})$ and $P_{j,4}$ $(j = \overline{1,4})$ are given in (6).

Proof. Using the system (13), obtained as a result of center-affine transformation in conditions $\bar{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$ and (12) with the help of the invariant polynomials $I_{i,4}$ $(i = \overline{1,4})$ and $P_{j,4}$ $(j = \overline{1,4})$ from (6), we obtain

$$4(I_{1,4}^3 P_{1,4} - 3I_{1,4}I_{2,4}P_{1,4} + 2I_{3,4}P_{1,4}) - 15(I_{1,4}^2 P_{2,4} - I_{2,4}P_{2,4} - 2I_{1,4}P_{3,4} + 2P_{4,4}) = 30(L_{3,4}a + d)x.$$

Consequently taking into account Lemma 4 we obtain truth of this theorem. Theorem 3 is proved.

5. Invariant conditions of stability of unperturbed motion for system (1) in case when the characteristic equation (7) has two pure imaginary roots in conditions $\bar{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$

Lemma 5. The characteristic equation (7) has two pure imaginary roots $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$ and the other two real and negative if and only if the following invariant conditions

$$L_{1,4} > 0,\ \ L_{3,4} > 0,\ \ L_{1,4}L_{2,4} - L_{3,4} > 0,\ \ L_{1,4}^2 L_{4,4} + L_{3,4}^2 - L_{1,4}L_{2,4}L_{3,4} = 0 \qquad (18)$$

hold, where $L_{i,4}$ $(i = \overline{1,4})$ are from (8).

Proof. Denote by ρ_i $(i = \overline{1,4})$ the roots of characteristic equation (7). According to Vieta's theorem we have

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 = -L_{1,4},\quad \rho_1\rho_2 + \rho_1\rho_3 + \rho_1\rho_4 + \rho_2\rho_3 + \rho_2\rho_4 + \rho_3\rho_4 = L_{2,4},$$

$$\rho_1\rho_2\rho_3 + \rho_1\rho_2\rho_4 + \rho_1\rho_3\rho_4 + \rho_2\rho_3\rho_4 = -L_{3,4},\quad \rho_1\rho_2\rho_3\rho_4 = L_{4,4}. \qquad (19)$$

Let us suppose that $\rho_1 = \lambda i$ and $\rho_2 = -\lambda i$ $(i^2 = -1)$, where $\lambda \neq 0$ is real number. From (19) we obtain

$$\rho_3 + \rho_4 = -L_{1,4},\quad \lambda^2 + \rho_3\rho_4 = L_{2,4},\ \lambda^2(\rho_3 + \rho_4) = -L_{3,4},\ \lambda^2\rho_3\rho_4 = L_{4,4}. \qquad (20)$$

From the first and third equalities (20) we get

$$\lambda = \pm\sqrt{\frac{L_{3,4}}{L_{1,4}}}\quad (L_{1,4}L_{3,4} > 0). \qquad (21)$$

Taking into account the first and second equalities from (20) we obtain

$$\rho_j^2 + L_{1,4}\rho_j + L_{2,4} - \frac{L_{3,4}}{L_{1,4}} = 0 \quad (j = 3,4). \tag{22}$$

Using the Hurwitz theorem on the signs of the roots of an algebraic equation [5] and the inequality (21) we get first three conditions from (18). Substituting $\rho_3\rho_4$ from second equality (20) into last equality (20) we obtain equality from (18). Lemma 5 is proved.

Lemma 6. The characteristic equation (7) has two pure imaginary roots $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$ of multiplicity 2 if and only if the following invariant conditions

$$L_{2,4} > 0, \quad L_{1,4} = L_{3,4} = L_{2,4}^2 - 4L_{4,4} = 0, \tag{23}$$

hold, where $L_{i,4}$ $(i = \overline{1,4})$ are from (8).

Proof. Let us suppose that

$$\rho_1 = \rho_2 = \lambda i, \quad \rho_3 = \rho_4 = -\lambda i, \tag{24}$$

where $\lambda \neq 0$ is real number. From (19) we obtain

$$L_{1,4} = L_{3,4} = 0, \quad 2\lambda^2 = L_{2,4}, \quad \lambda^4 = L_{4,4}. \tag{25}$$

Because $\lambda \neq 0$ is real number, from (25) we get

$$\lambda = \pm\sqrt{\frac{1}{2}L_{2,4}} \quad (L_{2,4} > 0), \tag{26}$$

and

$$L_{2,4}^2 - 4L_{4,4} = 0. \tag{27}$$

The conditions (25)-(27) coincide with (23). Lemma 6 is proved.

Theorem 4. Let for differential system of the perturbed motion (1) the invariant conditions $\bar{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$ be satisfied. Then this system by center-affine transformation can be reduced to the form $(x = x^1,\ y = x^2,\ z = x^3,\ u = x^4)$

a) in conditions (18):

$$\dot{x} = -\lambda y + 2x\cdot\psi \equiv P, \quad \dot{y} = \lambda x + 2y\cdot\psi \equiv Q, \quad \dot{z} = u + 2z\cdot\psi \equiv R, \tag{28}$$
$$\dot{u} = y + (\lambda^2 - L_{2,4})z - L_{1,4}u + 2u\cdot\psi \equiv S,$$

where λ is from (21), $L_{i,4}$ is from (8) and $\psi = Ax + By + Cz + Du$ with A, B, C, D real constants.

b) in conditions (23):

$$\dot{x} = -\lambda y + 2x\cdot\psi, \quad \dot{y} = \lambda x + 2y\cdot\psi, \quad \dot{z} = u + 2z\cdot\psi, \quad \dot{u} = y - \lambda^2 z + 2u\cdot\psi, \tag{29}$$

where λ is from (26), $L_{i,4}$ is from (8) and $\psi = Ax + By + Cz + Du$ with A, B, C, D real constants.

Proof. a) As shown in the Lemmas 1 and 2 in conditions $\bar{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$ the system (1) by the center affine transformation is reduced to the form (11). In the case (18) the system (11) has the form $(x = x^1,\ y = x^2,\ z = x^3,\ u = x^4,\ a^1_{11} = \alpha,\ a^1_{12} = \beta,\ \ a^1_{13} = \gamma,\ a^1_{14} = \delta)$

$$\dot{x} = y + 2x \cdot \Phi, \quad \dot{y} = z + 2y \cdot \Phi, \quad \dot{z} = u + 2z \cdot \Phi, \ \ \dot{u} = \frac{b^2 + bcd}{d^2} x + by + cz + du + 2u \cdot \Phi, \quad (30)$$

where

$$b = -L_{3,4},\ \ c = -L_{2,4},\ \ d = -L_{1,4}, \quad \Phi = \alpha x + \beta y + \gamma z + \delta u \quad (\alpha, \beta, \gamma, \delta \in \mathbb{R}). \qquad (31)$$

Let’s consider the transformation

$$X = -(c + \lambda^2) y - dz + u,\ \ Y = -\lambda (c + \lambda^2) x - d\lambda y + \lambda z,\ \ Z = \lambda x,\ \ U = \lambda y, \qquad (32)$$

where according to (21) and (31) we have $\lambda^2 = \dfrac{b}{d}$ and determinant $\Delta = -\lambda^3 \neq 0.$

Making the transformation (32) in the system (30)-(31) we obtain for it the form (28).

b) In the case (23) the system (11) has the form

$$\dot{x} = y + 2x \cdot \Phi, \quad \dot{y} = z + 2y \cdot \Phi, \quad \dot{z} = u + 2z \cdot \Phi, \ \ \dot{u} = -\lambda^4 x - 2\lambda^2 z + 2u \cdot \Phi, \qquad (33)$$

where

$$\Phi = ax + by + cz + du,\ \ \lambda = \pm\sqrt{\frac{L_{2,4}}{2}},\ \ L_{1,4} = L_{3,4} = 0,\ \ L^2_{2,4} = 4L_{4,4}. \qquad (34)$$

Let’s consider the transformation

$$X = \lambda^2 y + u,\ \ Y = \lambda^3 x + \lambda z,\ \ Z = \lambda x,\ \ U = \lambda y. \qquad (35)$$

According to (16) the determinant of transformation (35) is $\Delta = -\lambda^3 \neq 0.$

Making the transformation (35) in the system (33)-(34) we obtain for it the form (29). Theorem 4 is proved.

6. The theorem on the integrating factor for a four-dimensional differential system

Let's suppose that the system (1) admits the $(n-1)$ - dimensional commutative Lie algebra with operators

$$X_\alpha = \xi^j_\alpha (x) \frac{\partial}{\partial x^j} \quad \left(j = \overline{1,4};\ \alpha = \overline{1,3} \right), \qquad (36)$$

and

$$\Lambda = P^j (x, a) \frac{\partial}{\partial x^j} \quad \left(j = \overline{1,4} \right). \qquad (37)$$

Let’s consider the determinant constructed on coordinates of operators (36)-(37)

$$\Delta = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 & \xi^4_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 & \xi^4_2 \\ \xi^1_3 & \xi^2_3 & \xi^3_3 & \xi^4_3 \\ P^1 & P^2 & P^3 & P^4 \end{vmatrix} \qquad (38)$$

Theorem 5. [4] If the four-dimensional differential system (1) admits three-dimensional commutative Lie algebra of operators (36), then the function $\mu=\frac{1}{\Delta}$ where $\Delta\neq 0$ from (38) is the integrating factor for Pfaff equations

$$\begin{vmatrix}\xi_2^2 & \xi_2^3 & \xi_2^4\\ \xi_3^2 & \xi_3^3 & \xi_3^4\\ P^2 & P^3 & P^4\end{vmatrix}dx^1-\begin{vmatrix}\xi_2^1 & \xi_2^3 & \xi_2^4\\ \xi_3^1 & \xi_3^3 & \xi_3^4\\ P^1 & P^3 & P^4\end{vmatrix}dx^2+\begin{vmatrix}\xi_2^1 & \xi_2^2 & \xi_2^4\\ \xi_3^1 & \xi_3^2 & \xi_3^4\\ P^1 & P^2 & P^4\end{vmatrix}dx^3-\begin{vmatrix}\xi_2^1 & \xi_2^2 & \xi_2^3\\ \xi_3^1 & \xi_3^2 & \xi_3^3\\ P^1 & P^2 & P^3\end{vmatrix}dx^4=0,$$

$$\begin{vmatrix}\xi_1^2 & \xi_1^3 & \xi_1^4\\ \xi_3^2 & \xi_3^3 & \xi_3^4\\ P^2 & P^3 & P^4\end{vmatrix}dx^1-\begin{vmatrix}\xi_1^1 & \xi_1^3 & \xi_1^4\\ \xi_3^1 & \xi_3^3 & \xi_3^4\\ P^1 & P^3 & P^4\end{vmatrix}dx^2+\begin{vmatrix}\xi_1^1 & \xi_1^2 & \xi_1^4\\ \xi_3^1 & \xi_3^2 & \xi_3^4\\ P^1 & P^2 & P^4\end{vmatrix}dx^3-\begin{vmatrix}\xi_1^1 & \xi_1^2 & \xi_1^3\\ \xi_3^1 & \xi_3^2 & \xi_3^3\\ P^1 & P^2 & P^3\end{vmatrix}dx^4=0,$$

$$\begin{vmatrix}\xi_1^2 & \xi_1^3 & \xi_1^4\\ \xi_2^2 & \xi_2^3 & \xi_2^4\\ P^2 & P^3 & P^4\end{vmatrix}dx^1-\begin{vmatrix}\xi_1^1 & \xi_1^3 & \xi_1^4\\ \xi_2^1 & \xi_2^3 & \xi_2^4\\ P^1 & P^3 & P^4\end{vmatrix}dx^2+\begin{vmatrix}\xi_1^1 & \xi_1^2 & \xi_1^4\\ \xi_2^1 & \xi_2^2 & \xi_2^4\\ P^1 & P^2 & P^4\end{vmatrix}dx^3-\begin{vmatrix}\xi_1^1 & \xi_1^2 & \xi_1^3\\ \xi_2^1 & \xi_2^2 & \xi_2^3\\ P^1 & P^2 & P^3\end{vmatrix}dx^4=0, \quad (39)$$

that determine the general integral of system (1).

7. The Lie algebra of operators admitted by the system (28). Some particular integrals and one first integral of Darboux type

Lemma 7. The Lie algebra of operators admitted by the system (28) has the form

$$X_1=[(Bd-D)\lambda(c+\lambda^2)x+Ad\lambda(c+\lambda^2)y-2\varphi_1x^2+2\varphi_2\varphi_3xz+2C\varphi_2xu]\frac{\partial}{\partial x}+$$

$$+[-Ad\lambda(c+\lambda^2)x+(Bd-D)\lambda(c+\lambda^2)y-2\varphi_1xy+2\varphi_2\varphi_3yz+2C\varphi_2yu]\frac{\partial}{\partial y}+$$

$$+[A(c+\lambda^2)y+(c+\lambda^2)\varphi_2z-2\varphi_1xz+2\varphi_2\varphi_3z^2+2C\varphi_2zu]\frac{\partial}{\partial z}+$$

$$+[A\lambda(c+\lambda^2)x+(c+\lambda^2)\varphi_2u-2\varphi_1xu+2\varphi_2\varphi_3zu+2C\varphi_2u^2]\frac{\partial}{\partial u},$$

$$X_2=[\lambda(c+\lambda^2)\varphi_5x+A\lambda(c+\lambda^2)(c+2\lambda^2)y-2\varphi_4x^2+2\lambda\varphi_3\varphi_6xz+2C\lambda\varphi_6xu]\frac{\partial}{\partial x}+$$

$$+[-A\lambda(c+\lambda^2)(c+2\lambda^2)x+\lambda(c+\lambda^2)\varphi_5y-2\varphi_4xy+2\lambda\varphi_3\varphi_6yz+2C\lambda\varphi_6yu]\frac{\partial}{\partial y}+$$

$$+[A\lambda(c+\lambda^2)x+\lambda(c+\lambda^2)\varphi_6z-2\varphi_4xz+2\lambda\varphi_3\varphi_6z^2+2C\lambda\varphi_6zu]\frac{\partial}{\partial z}+$$

$$+[-A\lambda^2(c+\lambda^2)y+\lambda(c+\lambda^2)\varphi_6u-2\varphi_4xu+2\lambda\varphi_3\varphi_6zu+2C\lambda\varphi_6u^2]\frac{\partial}{\partial u},$$

$$X_3=[-B\lambda(c+\lambda^2)x-A\lambda(c+\lambda^2)y+2\varphi_7x^2+2\varphi_8xz+2\varphi_9xu]\frac{\partial}{\partial x}+$$

$$+[A\lambda(c+\lambda^2)x-B\lambda(c+\lambda^2)y+2\varphi_7xy+2\varphi_8yz+2\varphi_9yu]\frac{\partial}{\partial y}+$$

$$+[-B\lambda(c+\lambda^2)z+A(c+\lambda^2)u+2\varphi_7 xz+2\varphi_8 z^2+2\varphi_9 zu]\frac{\partial}{\partial z}+$$

$$+[A(c+\lambda^2)y+A(c+\lambda^2)^2 z+(Ad-B\lambda)(c+\lambda^2)u+2\varphi_7 xu+2\varphi_8 zu+2\varphi_9 u^2]\frac{\partial}{\partial u},$$

$$X_4=[\lambda(c+\lambda^2)x-2(\varphi_5-B\lambda^2)x^2+2A(c+\lambda^2)xy-2\lambda\varphi_{10}xz+2C\lambda xu]\frac{\partial}{\partial x}+$$

$$+[\lambda(c+\lambda^2)y-2(\varphi_5-B\lambda^2)xy+2A(c+\lambda^2)y^2-2\lambda\varphi_{10}yz+2C\lambda yu]\frac{\partial}{\partial y}+$$

$$+[\lambda(c+\lambda^2)z-2(\varphi_5-B\lambda^2)xz+2A(c+\lambda^2)yz-2\lambda\varphi_{10}z^2+2C\lambda zu]\frac{\partial}{\partial z}+$$

$$+[\lambda(c+\lambda^2)u-2(\varphi_5-B\lambda^2)xu+2A(c+\lambda^2)yu-2\lambda\varphi_{10}zu+2C\lambda u^2]\frac{\partial}{\partial u}, \qquad (40)$$

where

$$\varphi_1=(A^2+B^2)cd-BCd-BDc+CD-AC\lambda+(A^2d+B^2d-BD)\lambda^2,$$

$$\varphi_2=Ac+(Bd-D)\lambda+2A\lambda^2,\ \varphi_3=-Cd+(c+\lambda^2)D,$$

$$\varphi_4=-2BCc+C^2+A(Cd-Dc)\lambda+3(A^2c+B^2c-BC)\lambda^2-AD\lambda^3+(A^2+B^2)(c^2+2\lambda^4),$$

$$\varphi_5=B(c+2\lambda^2)-C,\ \varphi_6=B(c+2\lambda^2)-C-Ad\lambda,\ \varphi_7=(A^2+B^2)(c+\lambda^2)-BC,$$

$$\varphi_8=AC(c+\lambda^2)+B(Cd-Dc)\lambda-BD\lambda^3,\ \varphi_9=AD(c+\lambda^2)-BC\lambda,\ \varphi_{10}=Cd-cD-D\lambda^2. \qquad (41)$$

Proof. Writing the operators (36) in a general form $X=\xi^j(x)\frac{\partial}{\partial x^j}$ and solving the determining equations

$$\xi^j_{x^1}P^1+\xi^j_{x^2}P^2+\xi^j_{x^3}P^3+\xi^j_{x^4}P^4=\xi^1P^j_{x^1}+\xi^2P^j_{x^2}+\xi^3P^j_{x^3}+\xi^4P^j_{x^4},\ (j=\overline{1,4})$$

we obtain that the system (28) admits the operators (40)-(41).

The operators $X_i\ (i=1,2,3,4)$ are linearly independent, since the determinant of fourth order constructed on coordinates of these operators is different from zero. Notice that commutators $[X_i,X_j]=0,\ (i,j=\overline{1,4})$. Therefore operators $X_i\ (i=\overline{1,4})$ form a four-dimensional Lie algebra. Further, using the theorem 5 on integrating factor we calculate determinant μ which is constructed on the coordinates of three operators $X_i\ (i=1,2,3,4)$ and on the right-hand sides of the system (28), we obtain

$$\mu_{134}=\mu_{234}=0,\ \ \mu_{123}=A^2B\lambda(c+\lambda^2)^2\varsigma_1\varsigma_2\varsigma_3,\ \mu_{124}=-A^2\lambda(c+\lambda^2)^2\varsigma_1\varsigma_2\varsigma_3,$$

where

$$\varsigma_1=x^2+y^2,\ \ \varsigma_2=\lambda^3+c\lambda-2(Bc-C+B\lambda^2)x+2A(c+\lambda^2)y+2\lambda(-Cd+cD+D\lambda^2)z+2C\lambda u,$$

$$\varsigma_3=\lambda^2x^2+d\lambda xy+cd\lambda xz+\lambda(2c+d^2+4\lambda^2)xu-(c+\lambda^2)y^2-[2c^2+(6c+d^2)\lambda^2+4\lambda^2]yz-$$

$$-cdyu-[c^3+c(5c+d^2)\lambda^2+(8c+d^2)\lambda^4+4\lambda^6]z^2-[c^2+(4c+d^2)\lambda^2+4\lambda^4](dzu-u^2). \qquad (42)$$

We denote the operator of system (28) by $\Lambda=P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}+S\frac{\partial}{\partial u}$. Then we obtain

$$\Lambda(\varsigma_1)=4\varsigma_1\psi,\quad \Lambda(\varsigma_2)=2\varsigma_2\psi,\quad \Lambda(\varsigma_3)=\varsigma_3(d+4\psi),\quad \Lambda(\varsigma_1^{\alpha}\varsigma_2^{\beta})=2(2\alpha+\beta)\varsigma_1^{\alpha}\varsigma_2^{\beta}\psi,$$

where $\psi=Ax+By+Cz+Du.$

From the last equalities we get

Theorem 6. The functions $\varsigma_1,\varsigma_2,\varsigma_3$ from (42) are particular integrals of the system (28) and the function $F=\varsigma_1\varsigma_2^{-2}$ is a first integral of Darboux type for this system.

Remark 4. The comitant $K_{6,4}$ from (6) for the system (28) has the form $K_{6,4}=\lambda\varsigma_1\varsigma_3$, where λ from (21) and ς_1, ς_3 are from (42).

8. The Lie algebra of operators admitted by the system (29). Some particular integrals and one first integral of Darboux type

Lemma 8. The Lie algebra of operators admitted by the system (28) has the form

$$Y_1=[\lambda^3x-2(C+B\lambda^2)x^2+2A\lambda^2xy+2D\lambda^3xz-2C\lambda xu]\frac{\partial}{\partial x}+$$

$$+[\lambda^3y-2(C+B\lambda^2)xy+2A\lambda^2y^2+2D\lambda^3yz-2C\lambda yu]\frac{\partial}{\partial y}+$$

$$+[\lambda^3z-2(C+B\lambda^2)xz+2A\lambda^2yz+2D\lambda^3z^2-2C\lambda zu]\frac{\partial}{\partial z}+$$

$$+[\lambda^3u-2(C+B\lambda^2)xu+2A\lambda^2yu+2D\lambda^3zu-2C\lambda u^2]\frac{\partial}{\partial u},$$

$$Y_2=[-D\lambda^3x+2(CD-AC\lambda+BD\lambda^2)x^2-2D^2\lambda^3xz+2CD\lambda xu]\frac{\partial}{\partial x}+$$

$$+[-D\lambda^3y+2(CD-AC\lambda+BD\lambda^2)xy-2D^2\lambda^3yz+2CD\lambda yu]\frac{\partial}{\partial y}+$$

$$+[A\lambda^2y-D\lambda^3z+2(CD-AC\lambda+BD\lambda^2)xz-2D^2\lambda^3z^2+2CD\lambda zu]\frac{\partial}{\partial z}+$$

$$+[A\lambda^3x-D\lambda^3u+2(CD-AC\lambda+BD\lambda^2)xu-2D^2\lambda^3zu+2CD\lambda u^2]\frac{\partial}{\partial u},$$

$$Y_3=[-B\lambda^3x-A\lambda^3y+2Ex^2+2\lambda^2(AC-BD\lambda)xz+2\lambda(BC+AD\lambda)xu]\frac{\partial}{\partial x}+$$

$$+[A\lambda^3x-B\lambda^3y+2Exy+2\lambda^2(AC-BD\lambda)yz+2\lambda(BC+AD\lambda)yu]\frac{\partial}{\partial y}+$$

$$+[-B\lambda^3z+A\lambda^2u+2Exz+2\lambda^2(AC-BD\lambda)z^2+2\lambda(BC+AD\lambda)zu]\frac{\partial}{\partial z}+$$

$$+A\lambda^2y-A\lambda^4z-B\lambda^3u+2Exu+2\lambda^2(AC-BD\lambda)zu+2\lambda(BC+AD\lambda)u^2]\frac{\partial}{\partial u},$$

$$Y_4=[-\lambda^3(C+B\lambda^2)x-A\lambda^5y+2Hx^2-2\lambda^3Fxz+2\lambda Gxu]\frac{\partial}{\partial x}+$$

$$+[A\lambda^5x-\lambda^3(C+B\lambda^2)y+2Hxy-2\lambda^3Fyz+2\lambda Gyu]\frac{\partial}{\partial y}+$$

$$+[A\lambda^3 x-\lambda^3(C+B\lambda^2)z+A\lambda^4 u+2Hxz-2\lambda^3 Fz^2+2\lambda Gzu]\frac{\partial}{\partial z}+$$

$$+-A\lambda^6 z-\lambda^3(C+B\lambda^2)u+2Hxu-2\lambda^3 Fzu+2\lambda Gu^2]\frac{\partial}{\partial u}, \tag{43}$$

where $E=BC+(A^2+B^2)\lambda^2$, $F=CD-AC\lambda+BD\lambda^2$, $G=C^2+BC\lambda^2+AD\lambda^3$, $H=C^2+BC\lambda^2+AD\lambda^3+(A^2+B^2)\lambda^4$.

The proof of Lemma 8 is similarly with the proof of Lemma 7.

The operators Y_i $(i=1,2,3,4)$ are linearly independent, since the determinant of fourth order constructed on coordinates of these operators is different from zero. Notice that commutators $[Y_i,Y_j]=0,\ (i,j=\overline{1,4})$. Therefore operators Y_i $(i=\overline{1,4})$ form a four-dimensional Lie algebra. Further, using the theorem 5 on integrating factor we calculate determinant μ which is constructed on the coordinates of three operators Y_i $(i=1,2,3,4)$ and on the right-hand sides of the system (29), we obtain

$$\mu_{123}=\mu_{134}=0,\ \ \mu_{124}=-A^2\lambda^7\varphi^2\phi,\ \ \mu_{234}=-A^2B\lambda^7\varphi^2\phi,$$

where

$$\varphi=x^2+y^2,\quad \phi=\lambda^3-2(C+B\lambda^2)x+2A\lambda^2 y+2D\lambda^3 z-2C\lambda u, \tag{44}$$

Direct calculation of the operator Λ for the system (29) gives

$$\Lambda(\varphi)=4\varphi\psi,\quad \Lambda(\phi)=2\phi\psi,\quad \Lambda(\varphi^\alpha\phi^\beta)=2(2\alpha+\beta)\varphi^\alpha\phi^\beta\psi,$$

where $\psi=Ax+By+Cz+Du.$

From the last equalities we get

Theorem 7. The functions φ and ϕ from (44) are particular integrals of the system (29) and the function $F=\varphi\phi^{-2}$ is a first integral of Darboux type for this system.

Remark 5. The comitant $K_{6,4}$ from (6) for the system (29) has the form $K_{6,4}=\lambda^3\varphi^2$, where λ from (26) and φ are from (44).

Remark 6. The first integral $F=\varsigma_1\varsigma_2^{-2}$ of the system (28) is the holomorphic integral of Lyapunov type, i.e. this integral can be written in the form $F=x^2+y^2+\tilde{F}(x,y,z,u)$, where $\tilde{F}(x,y,z,u)$ is the polynomial of the order more than two.

From [4] it is known the comitant of system (1) in the form

$$\Phi_{4,4}=L_{4,4}-2\left(\frac{4}{5}L_{3,4}P_{1,4}+L_{2,4}P_{2,4}+L_{1,4}P_{3,4}+P_{4,4}\right), \tag{45}$$

where $P_{j,4}$ $(j=\overline{1,4})$ are from (6) and $L_{i,4}$ $(i=\overline{1,4})$ are from (8).

Remark 7. The comitant $\Phi_{4,4}$ for the system (28) has the form $\Phi_{4,4}=-\lambda\varsigma_2$, where ς_2 is from (42).

Using the Lyapunov's theorem [6], the theorems 6-7 and remarks 6-7, we obtain

Theorem 7. [8] Assume for the system (1) with $\tilde{K}_{1,4} \equiv 0$ and $\bar{R}_{6,4} \neq 0$ under center-affine invariant conditions (18), the comitant (45) is not identically zero. Then the system has a periodic solution containing an arbitrary constant, and varying this constant one can obtain a continuous sequence of periodic motions, which comprises the studied unperturbed motion. This motion is stable and any perturbed motion, sufficiently close to the unperturbed motion, will tend asymptotically to one of the periodic motions.

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