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THE PROJECTIVE SERIES OF PENCILS OF CONICS

Cezar PORT
Alexandru Ioan Cuza University, Iasi, Romania
Sergiu PORT, PhD, associate professor
Department of Informatics and Mathematics
"Ion Creangă" State Pedagogical University

Abstract. In this paper there are discussed some results which will be of help in the future, to classify and prove certain theorems of the cubic curves in the projective plane**.**

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PROIECTIVITATEA FASCICOLELOR DE SECȚIUNI CONICE

Abstract. În această lucrare sunt discutate câteva rezultate care vor fi de ajutor în viitor, de a clasifica și de a demonstra anumite teoreme ale curbelor cubice în planul proiectiv.

Cuvinte cheie: plan proiectiv, secțiuni conice, serii proiective, fascicole de secțiuni conice.

We are working in the projective plane.

Definition 1. *A series* (*or range*) *is a bijective function which has as an image a line from the plane.*

Definition 2. *Let* f, g *be two series and* r *a bijective function, such that* $Dom(r) = Dom(g)$ *and* $Im(r) = Dom(r)$. *Then we define the series* f *and* g (*in this order*) *to be* r *- projective, written simply as* $f \wedge_r g$, *if and only if for any distinct points* $\{A, B, C, D\} \subset Im(g)$;

$$(A, B; C, D) = (frg^{-1}(A), frg^{-1}(B); frg^{-1}(C), frg^{-1}(D))$$ *- as cross-ratios* [1, p. 33].

Because of bijectivity if the above equality is true, then also

$$(A, B; C, D) = (grf^{-1}(A), grf^{-1}(B); grf^{-1}(C), grf^{-1}(D))$$

is true. Hence $f \wedge_r g \to g \wedge_r f$. *Similarly* $g \wedge_r f \to f \wedge_r g$. *Therefore the order does not matter, and we will simply denote* $f \wedge_r g \to g \wedge_r f$ *to mean that* f *and* g *are* r *- projective.*

Definition 3. *Let* A, B, C, D *be four distinct any three non-collinear points in the projective plane.* P_{ABCD} *is the set that contains all the conics that pass through* A, B, C *and* D *also named a pencil of conics. Let* x *be a line that passes through only one of the points* A, B, C *or* D. *Suppose it passes through* A (*the same procedure is undertaken for the other points*). *Then any conic from the pencil* P_{ABCD} *intersects the line* x *in another second point, let it be* X. X *is different from* A *in all cases except the case when the conic is tangent* x, *and* $A = X$ *will be a double point. Now, for any* $X \in x$ *there is, respectively, the conic* $XABCD \in P_{ABCD}$, *the conic that passes through the points* X, A, B, C *and* D *when* $X = A$ *it will be the conic from the pencil tangent to* x.

This establishes a bijective correspondence between points $X \in x$ *and conics from* P_{ABCD}, *in particular a function* $f : P_{ABCD} \to x$. *This series will be denoted by* $s_{x,A,B,C,D}$ *or simply* s_x, *when there is no confusion*.

Before going forward with the main theorem, we need a lemma, which is a well-known result in projective plane geometry.

Lemma 1. *Let* A *and* B *be two points,* a_i *and* b_i *will represent lines passing through* A *and respectively* B, $i \in \mathbb{N}$.

1. *If* $(a_0,a_1;a_2,a_3)=(b_0,b_1;b_2,b_3)$ (*this is the cross-ratio of lines*), $(a_0,a_1;a_2,a_4)= =(b_0,b_1;b_2,b_4)$, $(a_0,a_1;a_2,a_5)=(b_0,b_1;b_2,b_5)$ *and finally* $(a_0,a_1;a_2,a_6)=(b_0,b_1;b_2,b_6)$, *then* $(a_3,a_4;a_5,a_6)=(b_3,b_4;b_5,b_6)$.

2. *In this part every line passes through* A. *If* $(a,a';n,m)=(b,b';n,m)= =(c,c';n,m)=(d,d';n,m)$ *then* $(a,b;c,d)=(a',b';c',d')$.

Theorem 1. *Let* A,B,C,D *be four distinct non-collinear points in the projective plane, see Figure* 1. *Let* x,y *be lines that pass through only one of the points* A,B,C *or* D. *Then* $s_x \wedge_{id} s_y$ *where* id *is the identity function on* P_{ABCD}.

Proof.

Let $X \in x$ and $Y = XABCD \cap y$, where Y is the second point of intersection on line y. There are two cases, either the lines pass through the same point or through two different points.

First case. Suppose, without loss of generality, that $x \cap y = A$. Then

$$A(X,Y;D,C) = B(X,Y;D,C)$$

by the conic's general properties. As X varies on x, the cross-ratio of $A(X,Y;D,C)$ is constant, as the lines x,y are fixed, results that the cross-ratio of $B(X,Y;D,C)$ also must be constant. So as X varies on x, Y moves accordingly on y. Because $B(X,Y;D,C)$ is constant for any $X \in x$, by the lemma (here $n = BD$, $m = BC$) from above, we have for X_1, X_2, X_3, X_4 (distinct points on x) and their corresponding Y_1,Y_2,Y_3,Y_4 on y, that

$$B(X_1, X_2; X_3, X_4) = B(Y_1,Y_2;Y_3,Y_4)$$

which means exactly

$$(X_1, X_2; X_3, X_4) = (Y_1,Y_2;Y_3,Y_4)$$

therefore $s_x \wedge_{id} s_y$.

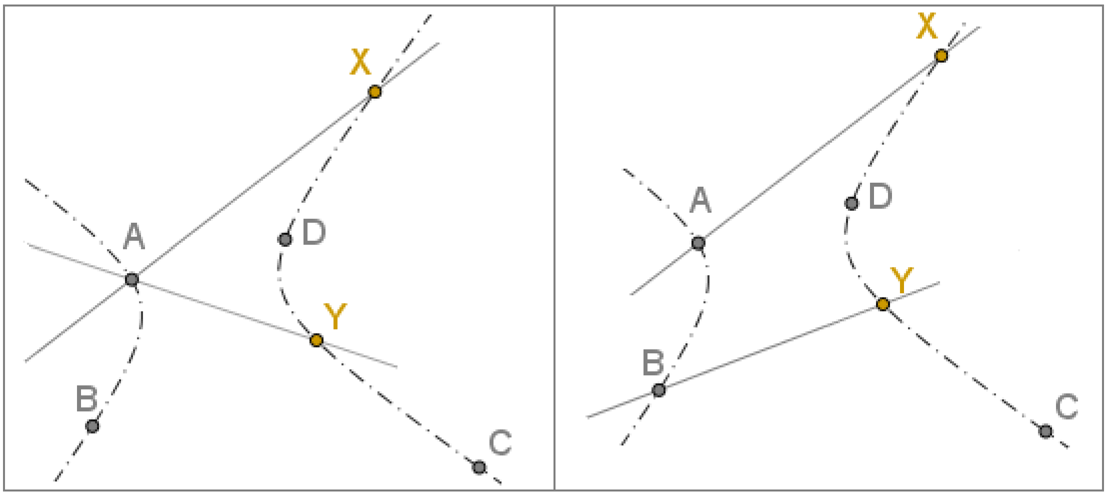

Figure 1. The two cases

Second case. Suppose, without loss of generality, that $A \in x$, $B \in y$. Then

$$A(X,Y;D,C) = B(X,Y;D,C).$$

Furthermore, the cross-ratio of $A(X,Y;D,C)$, depends only on Y, as the lines AX, AD, AC are fixed. Same way, the cross-ratio of $B(X,Y;D,C)$ depends only on X. So as X is varies on x, Y moves accordingly on y. By the cross-ratio properties, we have also that

$$A(D,C;X,Y) = B(X,Y;D,C).$$

By the lemma (here $a_0 = AD$, $a_1 = AC$, $a_2 = AX$ and $b_0 = BC$, $b_1 = BD$, $b_2 = BY$), we have for X_1, X_2, X_3, X_4 (distinct points on x) and their corresponding Y_1, Y_2, Y_3, Y_4 on y, that

$$A(X_1, X_2; X_3, X_4) = B(Y_1, Y_2; Y_3, Y_4)$$

which means exactly

$$(X_1, X_2; X_3, X_4) = (Y_1, Y_2; Y_3, Y_4)$$

therefore $s_x \wedge_{id} s_y$.

This theorem shows that it does not matter which line x (as in the theorem) is chosen, the series is projectively "invariant". In conclusion, any pencil of conics gives a unique projective series.

Bibliography

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