THE COMMON HILBERT SERIES FOR SOME DIFFERENTIAL SYSTEMS WITH HOMOGENEOUS NONLINEARITIES OF ODD DEGREE

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Abstract. The Hilbert series for Sibirsky graded algebras of differential systems still now were examined using a generalized Sylvester method. These series have a special importance for some problems of qualitative theory of differential systems. For example, a problem related to the Hilbert series of differential systems is to determine relationships between them. Before finding some relations between Hilbert series, generalized or ordinary, it is necessary to build these Hilbert series. The article proposes the construction of Hilbert series of Sibirsky graded algebras using the residue method.
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SERII HILBERT OBIȘNUITE PENTRU UNELE SISTEME DIFERENȚIALE CU NELINIARITĂȚI IMPARE

Rezumat. Seriile Hilbert pentru algebrele graduate Sibirschi ale sistemelor diferențiale până în prezent au fost examinate utilizând metoda generalizată a lui Sylvester. Aceste serii au o importanță deosebită pentru unele probleme ale teoriei calitative ale sistemelor diferențiale. De exemplu, o problemă legată de seriile Hilbert corespunzătoare sistemelor diferențiale este determinarea unor relații între ele. Pentru a obține relații între serii Hilbert atât generalizate cât și obișnuite este nevoie de a construi aceste serii Hilbert. În articol se propune construirea seriilor Hilbert ale algebrelor graduate Sibirschi prin metoda reziduurilor.
Cuvinte-cheie: Serii Hilbert, algebre graduate Sibirschi, sisteme diferențiale.

1. Introduction

A problem related to the Hilbert series of differential systems is to determine the relationships between them. Some relations between generalized Hilbert series of differential systems with homogeneous nonlinearities of odd degree were found in [1].

Lemma 1 [1]. *The following relation*

$$H(SI_{1,3},b,d)=H(SI_1,b)H(S_3,u,d)|_{u^2=b} \tag{1}$$

exists between the generalized Hilbert series of algebras SI_1, S_3 *and* $SI_{1,3}$.

Lemma 2 [1]. *The following relation*

$$H(SI_{1,5},b,f)=H(SI_1,b)H(S_5,u,f)|_{u^2=b} \tag{2}$$

exists between the generalized Hilbert series of algebras SI_1, S_5 *and* $SI_{1,5}$.

According to (1) and (2) we can assume that between generalized Hilbert series of algebras SI_1, S_{2k+1} and $SI_{1,2k+1}$ there exists the next relation

$$H(SI_{1,2k+1},b,z)=H(SI_1,b)H(S_{2k+1},u,z)|_{u^2=b} \tag{3}$$

for any $k \geq 1$.

Before finding some relations between Hilbert series, generalized or ordinary, it is necessary to build these Hilbert series.

The construction of Hilbert series with generalized Sylvester method [2] is not always simple. The method of computing ordinary Hilbert series for invariants rings using the residues it is known from [3].

2. Hilbert series

Definition 1 [3]. *For a graded vector space* $V = \bigoplus_{d=k}^{\infty} V_d$ *with* V_d *finite dimensional for all d we define the Hilbert series of V as a formal Laurent series*

$$H(v,t) = \sum_{d=k}^{\infty} \dim(V_d)t^d.$$

Let G be a linearly reductive group over an algebraically closed field K and V be a n – dimensional rational representation. Through $H(K[V]^G, t)$ is denoted the Hilbert series of invariants ring $K[V]^G$ [3].

Theorem 1 (Molien's formula [3]). *Let* G *be a finite group acting on a finite dimensional vector space* V *over a field* K *of characteristic not dividing* $|G|$. *Then*

$$H(K[V]^G, t) = \frac{1}{|G|}\sum_{\sigma\in G}\frac{1}{\det_V^0(1-t\sigma)}.$$

If K *has characteristic* 0, *then* $\det_V^0(1-t\sigma)$ *can be taken as* $\det_V(1-t\sigma)$.

Suppose that $char(K) = 0$. In Theorem 1 we have seen that for a finite group the Hilbert series of invariant ring can easily be computed. If G is a finite group and V is a finite dimensional representation, then according to [3] we have

$$H(K[V]^G, t) = \frac{1}{|G|}\sum_{\sigma\in G}\frac{1}{\det_V(1-t\sigma)}. \tag{4}$$

This idea can be generalized to arbitrary reductive groups. Let us assume that K is the complex numbers $\mathbb{C}$. We can choose a Haar measure $d\mu$ on C and normalize it such that $\int_C d\mu = 1$. Let V be a finite dimensional rational representation of G. The proper generalization of (4) is given in [3]

$$H(\mathbb{C}[V]^G, t) = \int_C \frac{d\mu}{\det_V(1-t\sigma)}. \tag{5}$$

We mention that the Hilbert series $H(\mathbb{C}[V]^G, t)$ converges for $|t| < 1$ because it is a rational function with poles only at $t = 1$. Since C is compact, there exist constants $A > 0$ such that for every $\sigma \in \mathbb{C}$ and every eigenvalue λ of σ we have $|\lambda| \leq A$. Since λ^ℓ is an eigenvalue of σ^{ℓ}, it follows that $|\lambda^{\ell}| \le A$ for all ℓ, so $|\lambda| \le 1$. It is clear that the integral on the right-hand side of (5) is also defined for $|t| < 1$ [3].

Assume that G is also connected. Let T be a maximal torus of G, and let D be a maximal compact subgroup of T. We may assume that C contains D. The torus can be identified with $(\mathbb{C}^*)^r$, where r is the rank of G, and D can be identified with the subgroup $(S^1)^r$ of $(\mathbb{C}^*)^r$, where $S^1 \subset \mathbb{C}^*$ is the unit circle. We can choose a Haar measure $d\mu$ on D such that $\int_D d\mu = 1$ [3].

Suppose that f is a continuous class function on C. An integral like $\int_C f(\sigma)d\mu$ can be viewed as an integral over D, since f is constant on conjugacy classes. More precisely, there exists a weight function $\varphi : D \to \mathbb{R}$, such that for every continuous class function f we have $\int_C f(\sigma)d\mu = \int_D \varphi(\sigma) f(\sigma)d\nu$.

So, from [3], we have

$$H(\mathbb{C}[V]^G, t) = \int_C \frac{d\mu}{\det_V(1 - t\sigma)} = \int_D \frac{\varphi(\sigma)d\nu}{\det_V(1 - t\sigma)}. \qquad (6)$$

3. The Residue Theorem

We recall the Residue Theorem in complex function theory. This theorem can be applied to compute the Hilbert series of invariant rings [3].

Suppose that $f(z)$ is a meromorphic function on $\mathbb{C}$. If $a \in \mathbb{C}$, then f can be written as a Laurent series around $z = a$

$$f(z) = \sum_{k=-d}^{\infty} c_k (z-a)^k.$$

If $d > 0$ and $c_{-d} \neq 0$, then f has a pole at $z = a$ and the pole order is d.

The residue of f at $z = a$ is denoted by $Res(f, a)$ and defined by

$$Res(f, a) = c_{-1}.$$

If the pole order of f at $z = a$ is $k \ge 1$, then the residue can be computed by

$$Res(f, a) = \frac{1}{(k-1)!} \lim_{z \to a} \frac{d^{k-1}}{dz^{k-1}} ((z-a)^k f(z)).$$

Suppose that $\gamma : [0,\ 1] \to \mathbb{C}$ is a smooth curve. The integral over the curve γ is defined by

$$\int_{\gamma} f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

Theorem 2 (The Residue Theorem [3]). *Suppose that* D *is a connected, simply connected compact region in* $\mathbb{C}$ *whose border is* ∂D, *and* $\gamma:[0,\ 1]\to\mathbb{C}$ *is a smooth curve such that* $\gamma([0,\ 1])=\partial D$, $\gamma(0)=\gamma(1)$ *and* γ *circles around* D *exactly once in counterclockwise direction. Assume that* f *is a meromorphic function on* $\mathbb{C}$ *with no poles in* ∂D. *Then we have*

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz=\sum_{a\in D} Res(f,a).$$

There are only finitely many points in the compact region D such that f has non-zero residue there. So we have

Theorem 3 [4].

$$H(K[V]^{G},t)=\frac{1}{2\pi i}\int_{S^1}\frac{1}{\det(I-t_{\rho V}(z))}\frac{dz}{z}, \qquad (7)$$

where $S^1\subset\mathbb{C}$ *is the unit circle* $\{z:|z|=1\}$.

4. Applications of the Residue Theorem to compute Hilbert series of Sibirsky graded algebras of differential systems

Using the Residue Theorem and corresponding generating function [2] the formula (7) can be adapted for computing ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems [5].

Theorem 4. *The ordinary Hilbert series for Sibirsky graded algebras of invariants of differential systems can be calculated using the formula*

$$H_{SI_\Gamma}(t)=\frac{1}{2\pi i}\int_{S^1}\frac{\varphi_\Gamma^{(0)}(z)}{z}dz, \qquad (8)$$

where $S^1\subset\mathbb{C}$ *is the unit circle* $\{z:|z|=1\}$, *and* $\varphi_\Gamma^{(0)}(z)$ *is the corresponding generating function* [2],

$$\varphi_\Gamma^{(0)}=(1-z^{-2})\psi_{m_0}^{(0)}(z)\psi_{m_1}^{(0)}(z)...\psi_{m_\ell}^{(0)}(z),$$

$$\psi_{m_i}^{(0)}(z)=\begin{cases}\dfrac{1}{(1-zt)(1-z^{-1}t)}, & \text{for } m_i=0,\\[2ex] \dfrac{1}{(1-z^{m_i+1}t)(1-z^{-m_i-1}t)\prod_{k=1}^{m_i}(1-z^{m_i-2k+1}t)^2}, & \text{for } m_i\neq 0,\end{cases}$$

$\Gamma=\{m_i\}_{i=0}^{\ell}$ *and consists of a finite number* $(\ell<\infty)$ *of distinct natural numbers*.

We mention that this method of computing ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants for differential systems was verified for the following known Hilbert series H_{SI_1}, H_{S_2}, H_{SI_2}, $H_{SI_{0,2}}$, $H_{SI_{1,2}}$, $H_{SI_{1,3}}$, $H_{SI_{2,3}}$, H_{S_5} from [2] and $H_{S_{1,5}}$, $H_{SI_{1,5}}$ from [1].

Remark 1. *The ordinary Hilbert series of Sibirsky graded algebra of comitants are obtained from the ordinary Hilbert series of algebra of invariants in the following way*: $H_{S_\Gamma}(t) = H_{SI_{\Gamma\cup\{0\}}}(t)$, *where* $\Gamma = \{m_1, m_2, ..., m_\ell\} \not\ni \{0\}$.

From the paper [3] it is known the method of computing ordinary Hilbert series for invariants rings using the residues. This method was adapted for ordinary Hilbert series of Sibirsky graded algebras of comitants and invariants of differential systems. In contrast to the construction methods of these series, exposed in [2], with the help of residues [3], of the primary generating functions [2], we obtained the ordinary Hilbert series for Sibirsky graded algebras of the differential systems $s(7)$, $s(1,7)$, $s(1,2,3)$, $s(1,3,5)$, $s(1,3,7)$, $s(1,3,5,7)$.

Theorem 5. *For differential system* $s(7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* S_7 *and invariants* SI_7 *were obtained*

$$H_{S_7}(t) = \frac{1}{(1+t)^5\,(1-t)\,(1-t^3)^4(1-t^4)^2\,(1-t^5)^4\,(1-t^7)^3(1-t^9)}(1+4t+7t^2+6t^3+6t^4+$$
$$+ 28t^5 + 112t^6 + 325t^7 + 788t^8 + 1719t^9 + 3499t^{10} + 6716t^{11} + 12225t^{12} + 21205t^{13} +$$
$$+ 35194t^{14} + 56030t^{15} + 85698t^{16} + 126023t^{17} + 178425t^{18} + 243697t^{19} + 321789t^{20} +$$
$$+ 411501t^{21} + 510260t^{22} + 613944t^{23} + 717118t^{24} + 813553t^{25} + 896906t^{26} + 961309t^{27} +$$
$$+ 1002042t^{28} + 1015982t^{29} + 1002042t^{30} + 961309t^{31} + 896906t^{32} +$$
$$+ 813553t^{33} + 717118t^{34} + 613944t^{35} + 510260t^{36} + 411501t^{37} + 321789t^{38} +$$
$$+ 243697t^{39} + 178425t^{40} + 126023t^{41} +$$
$$+ 85698t^{42} + 56030t^{43} + 35194t^{44} + 21205t^{45} + 12225t^{46} + 6716t^{47} + 3499t^{48} + 1719t^{49} +$$
$$+ 788t^{50} + 325t^{51} + 112t^{52} + 28t^{53} + 6t^{54} + 6t^{55} + 7t^{56} + 4t^{57} + t^{58}),$$

$$H_{SI_7}(t) = \frac{1}{(1+t)^4\,(1-t)\,(1-t^3)^4(1-t^4)^3\,(1-t^5)^3\,(1-t^7)^2}(1+4t+4t^2+2t^3+2t^4+15t^5+$$
$$+ 59t^6 + 150t^7 + 312t^8 + 578t^9 + 1011t^{10} + 1673t^{11} + 2631t^{12} + 3917t^{13} + 5541t^{14} + 7450t^{15}$$
$$+ + 9551t^{16} + 11651t^{17} + 13543t^{18} + 15011t^{19} + 15933t^{20} + 16238t^{21} + 15933t^{22} +$$
$$+ 15011t^{23} + 13543t^{24} + 11651t^{25} + 9551t^{26} + 7450t^{27} + 5541t^{28} + 3917t^{29} + 2631t^{30} +$$
$$1673t^{31} + 1011t^{32} + 578t^{33} + 312t^{34} + 150t^{35} + 59t^{36} + 15t^{37} + 2t^{38} + 2t^{39} + 4t^{40} + 3t^{41} + t^{42}).$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra S_7 (respectively SI_7) is equal to 15 (respectively 13).

Theorem 6. *For differential system* $s(1,7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,7}$ *and invariants* $SI_{1,7}$ *were obtained*

$$H_{S_{1,7}}(t) = \frac{1}{(1+t)^3(1-t^2)^3(1-t^3)^5(1-t^4)^3(1-t^5)^4(1-t^7)^3(1-t^9)}(1+4t+7t^2+7t^3+17t^4+$$
$$+ 85t^5 + 331t^6 + 1009t^7 + 2657t^8 + 6368t^9 + 14278t^{10} + 30208t^{11} + 60574t^{12} + 115441t^{13} +$$
$$+ 209688t^{14} + 363888t^{15} + 604838t^{16} + 965096t^{17} + 1481667t^{18} + 2193216\,t^{19} +$$
$$+ 3135942t^{20} + 4337738t^{21} + 5811835t^{22} + 7550176t^{23} + 9518852t^{24} + 11655892t^{25} +$$

$+ 13872730t^{26} + 16058633t^{27} + 18089130t^{28} + 19836497t^{29} + 21182751t^{30} +$
$+ 22032184t^{31} + 22322579t^{32} + 22032184t^{33} + 21182751t^{34} + 19836497t^{35} +$
$+ 18089130t^{36} + 16058633t^{37} + 13872730t^{38} + 11655892t^{39} + 9518852t^{40} + 7550176t^{41} +$
$+ 5811835t^{42} + 4337738t^{43} + 3135942t^{44} + 2193216t^{45} + 1481667t^{46} + 965096t^{47} +$
$+ 604838t^{48} + 363888t^{49} + 209688t^{50} + 115441t^{51} + 60574t^{52} + 30208t^{53} + 14278t^{54} +$
$+ 6368t^{55} + 2657t^{56} + 1009t^{57} + 331t^{58} + 85t^{59} + 17t^{60} + 7t^{61} + 7t^{62} + 4t^{63} + t^{64}),$

$$H_{SI_{1,7}}(t) = \frac{1}{(1+t)^5(1-t)^3(1-t^3)^5(1-t^4)^4(1-t^5)^3(1-t^7)^2}(1+3t+4t^2+2t^3+9t^4+53t^5+$$
$$+ 196t^6 + 525t^7 + 1214t^8 + 2558t^9 + 5097t^{10} + 9569t^{11} + 16975t^{12} + 28396t^{13} + 44981t^{14} +$$
$$+ 67577t^{15} + 96665t^{16} + 131839t^{17} + 171920t^{18} + 214631t^{19} + 257063t^{20} + 295599t^{21} +$$
$$+ 326684t^{22} + 346880t^{23} + 353937t^{24} + 346880t^{25} + 326684t^{26} + 295599t^{27} + 257063t^{28} +$$
$$+ 214631t^{29} + 171920t^{30} + 131839t^{31} + 96665t^{32} + 67577t^{33} + 44981t^{34} + 28396t^{35} +$$
$$+ 16975t^{36} + 9569t^{37} + 5097t^{38} + 2558t^{39} + 1214t^{40} + 525t^{41} + 196t^{42} + 53t^{43} +$$
$$+ 9t^{44} + 2t^{45} + 4t^{46} + 3t^{47} + t^{48}).$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,7}$ (respectively $SI_{1,7}$) is equal to 19 (respectively 17).

Theorem 7. *For differential system $s(1,2,3)$ the following ordinary Hilbert series of the Sibirsky graded algebras of comitants $S_{1,2,3}$ and invariants $SI_{1,2,3}$ were obtained*

$$H_{S_{1,2,3}}(t) = \frac{1}{(1-t)^2(1-t^2)^2(1-t^3)^6(1-t^4)^3(1-t^5)^3(1-t^7)}(1-t+3t^2+9t^3+36t^4+90t^5+$$
$$+ 220t^6 + 459t^7 + 946t^8 + 1748t^9 + 3032t^{10} + 4845t^{11} + 7302t^{12} + 10268t^{13} + 13749t^{14} +$$
$$+ 17327t^{15} + 20781t^{16} + 23565t^{17} + 25460t^{18} + 26051t^{19} + 25460t^{20} + 23565t^{21} +$$
$$+20781t^{22} + 17327t^{23} + 13749t^{24} + 10268t^{25} + 7302t^{26} + 4845t^{27} + 3032t^{28} + 1748t^{29} +$$
$$+ 946t^{30} + 459t^{31} + 220t^{32} + 90t^{33} + 36t^{34} + 9t^{35} + 3t^{36} - t^{37} + t^{38}),$$

$$H_{SI_{1,2,3}}(t) = \frac{1}{(1-t)(1-t^2)^3(1-t^3)^5(1-t^4)^2(1-t^5)^3(1-t^7)}(1+t^2+6t^3+24t^4+57t^5+128t^6+$$
$$+ 244t^7 + 447t^8 + 756t^9 + 1203t^{10} + 1760t^{11} + 2433t^{12} + 3124t^{13} + 3800t^{14} + 4351t^{15} +$$
$$+ 4736t^{16} + 4854t^{17} + 4736t^{18} + 4351t^{19} + 3800t^{20} + 3124t^{21} + 2433t^{22} + 1760t^{23} +$$
$$+1203t^{24} + 756t^{25} + 447t^{26} + 244t^{27} + 128t^{28} + 57t^{29} + 24t^{30} + 6t^{31} + t^{32} + t^{34}).$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,2,3}$ (respectively $SI_{1,2,3}$) is equal to 17 (respectively 15).

Theorem 8. *For ifferential system $s(1,3,5)$ the following ordinary Hilbert series of the Sibirsky graded algebras of comitants $S_{1,3,5}$ and invariants $SI_{1,3,5}$ were obtained*

$$H_{S_{1,3,5}}(t) = \frac{1}{(1+t)^7(1-t)^6(1-t^3)^8(1-t^4)^4(1-t^5)^4(1-t^7)}(1+2t+3t^2+17t^3+102t^4+393t^5+$$
$$+ 1295t^6 + 3788t^7 + 10229t^8 + 25559t^9 + 59435t^{10} + 128624t^{11} + 260754t^{12} + 497142t^{13} +$$
$$+ 895543t^{14} + 1528784t^{15} + 2480535t^{16} + 3832821t^{17} + 5651535t^{18} + 7964888t^{19} +$$

$+ 10746190t^{20} + 13897132t^{21} + 17246232t^{22} + 20554573t^{23} + 23544429t^{24} +$
$+ 25932413t^{25} + 27476107t^{26} + 28009657t^{27} + 27476107t^{28} + 25932413t^{29} +$
$+ 23544429t^{30} + 20554573t^{31} + 17246232t^{32} + 13897132t^{33} + 10746190t^{34} +$
$+ 7964888t^{35} + 5651535t^{36} +497142t^{41} + 260754t^{42} + 3832821t^{37} + 2480535t^{38} +$
$+ 1528784t^{39} + 895543t^{40} + 128624t^{43} + 59435t^{44} + + 25559t^{45} + 10229t^{46} +$
$+ 3788t^{47} + 1295t^{48} + 393t^{49} + 102t^{50} + 17t^{51} + 3t^{52} + 2t^{53} +t^{54}),$

$$H_{SI_{1,3,5}}(t) \;=\; \frac{1}{(1+t)^6(1-t)^6(1-t^3)^7(1-t^4)^5(1-t^5)^3}(1+t+t^2+14t^3+77t^4+253t^5+781t^6+$$

$+ 2077t^7 + 5160t^8 + 11689t^9 + 24616t^{10} + 47739t^{11} + 86576t^{12} + 146479t^{13} + 233075t^{14} +$
$+ 348813t^{15} + 493340t^{16} + 659032t^{17} + 834212t^{18} + 1000116t^{19} + 1138132t^{20} +$
$+1228974t^{21} + 1261281t^{22} + 1228974t^{23} + 1138132t^{24} + 1000116t^{25} + 834212t^{26} +$
$+659032t^{27} + 493340t^{28} + 348813t^{29} + 233075t^{30} + 146479t^{31} + 86576t^{32} + 47739t^{33} +$
$+24616t^{34} +11689t^{35} + 5160t^{36} + 2077t^{37} + 781t^{38} + 253t^{39} + 77t^{40} +$
$+ 14t^{41} + 2t^{42} +t^{43} + t^{44}).$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,3,5}$ (respectively $SI_{1,3,5}$) is equal to 23 (respectively 21).

Theorem 9. *For differential system* $s(1,3,7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,3,7}$ *and invariants* $SI_{1,3,7}$ *were obtained*

$$H_{S_{1,3,7}}(t) = \frac{1}{(1+t)^3(1-t^2)^5(1-t^3)^8(1-t^4)^5(1-t^5)^5(1-t^7)^3(1-t^9)}(1+4t+8t^2+20t^3+119t^4+$$

$+ 630t^5 + 2704t^6 + 10022t^7 + 33698t^8 + 104818t^9 + 304181t^{10} + 826655t^{11} + 2112616t^{12} +$
$+ 5098405t^{13} + 11666106t^{14} + 25400587t^{15} +52790206t^{16} + 105011044t^{17} +$
$+ 200416900t^{18} + 367773321t^{19} + 650140950t^{20} + 1109089748t^{21} + 1828673257t^{22} +$
$+2918286116t^{23} + 4513317434t^{24} + 6772373326t^{25} + 9869976204t^{26} +$
$+ 13983988556t^{27} + 19277729149t^{28} + 25877612329t^{29} + 33848259389t^{30} +$
$+ 43167949995t^{31} + 53708076135t^{32} + 65220413010t^{33} + 77335714909t^{34} +$
$+ 89575940034t^{35} + 101380841773t^{36} + 112147463549t^{37} + 121279087722t^{38} +$
$+ 128238286339t^{39} + 132597788686t^{40} + 34082589969t^{41} + 132597788686t^{42} +$
$+ 128238286339t^{43} + 121279087722t^{44} + 112147463549t^{45} + 101380841773t^{46} +$
$+ 89575940034t^{47} + 77335714909t^{48} + 65220413010t^{49} + 53708076135t^{50} +$
$+ 43167949995t^{51} + 33848259389t^{52} + 25877612329t^{53} + 19277729149t^{54} +$
$+ 13983988556t^{55} + 9869976204t^{56} + 6772373326t^{57} + 4513317434t^{58} + 2918286116t^{59} +$
$+ 1828673257t^{60} + 1109089748t^{61} + 650140950t^{62} + 367773321t^{63} + 200416900t^{64} +$
$+ 105011044t^{65} + 52790206t^{66} + 25400587t^{67} + 11666106t^{68} + 5098405t^{69} +$
$+ 2112616t^{70} + 826655t^{71} + 304181t^{72} + 104818t^{73} + 33698t^{74} + 10022t^{75} + 2704t^{76} +$
$+ 630t^{77} + 119t^{78} + 20t^{79} + 8t^{80} + 4t^{81} +t^{82}),$

$$H_{SI_{1,3,7}}(t) \;=\; \frac{1}{(1+t)^3(1-t^2)^4(1-t^3)^8(1-t^4)^6(1-t^5)^5(1-t^7)^2}(1+4t+9t^2+22t^3+114t^4+$$

$+ 576t^5 + 2433t^6 + 8812t^7 + 28787t^8 + 86580t^9 + 242349t^{10} + 633691t^{11} + 1554313t^{12} +$
$+ 3589873t^{13} + 7838767t^{14} + 16239174t^{15} + 32018338t^{16} + 60242752t^{17} + 108417618t^{18} +$
$+ 187010583t^{19} + 309738539t^{20} + 493386952t^{21} + 756961044t^{22} + 1119980967t^{23} +$
$+ 1599914185t^{24} + 2208870842t^{25} + 2949986298t^{26} + 3814040685t^{27} + 4777086279t^{28} +$
$+ 5799732655t^{29} + 6828681083t^{30} + 7800621224t^{31} + 8648294432t^{32} + 9307907390t^{33} +$
$+ 9726879111t^{34} + 9870564527t^{35} + 9726879111t^{36} + 9307907390t^{37} + 8648294432t^{38} +$
$+ 7800621224t^{39} + 6828681083t^{40} + 5799732655t^{41} + 4777086279t^{42} + 3814040685t^{43} +$
$+ 2949986298t^{44} + 2208870842t^{45} + 1599914185t^{46} + 1119980967t^{47} + 756961044t^{48} +$
$+ 493386952t^{49} + 309738539t^{50} + 187010583t^{51} + 108417618t^{52} + 60242752t^{53} +$
$+ 32018338t^{54} + 16239174t^{55} + 7838767t^{56} + 3589873t^{57} + 1554313t^{58} + 633691t^{59} +$
$+ 242349t^{60} + 86580t^{61} + 28787t^{62} + 8812t^{63} + 2433t^{64} + 576t^{65} + 114t^{66} + 22t^{67} + 9t^{68} +$
$+ 4t^{69} + t^{70})$.

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,3,7}$ (respectively $SI_{1,3,7}$) is equal to 27 (respectively 25).

Theorem 10. *For differential system* $s(1,3,5,7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,3,5,7}$ *and invariants* $SI_{1,3,5,7}$ *were obtained*

$$H_{S_{1,3,5,7}}(t) = \frac{U(t) + 2298270315\ 143980746\ t^{60} + t^{120}U(t^{-1})}{(1+t)^{19}(1+t^2)^8(1-t)^{14}(1-t^3)^{12}(1-t^5)^8(1-t^7)^4(1-t^9)},$$

where $U(t)=1 + 6t + 20t^2 + 87t^3 + 642t^4 + 4481t^5 + 26793t^6 + 141973t^7 + 684115t^8 +$
$+ 3033350t^9 + 12465139t^{10} + 47749507t^{11} + 171414077t^{12} + 579433144t^{13} +$
$+ 1852114710t^{14} + 5618767624t^{15} + 16230539293t^{16} + 44770726947t^{17} +$
$+ 118233818156t^{18} + 299625404135t^{19} + 730145608913t^{20} + 1714167261299t^{21} +$
$+ 3883773551652t^{22} + 8505306230645t^{23} + 18029418149708t^{24} + 37042309655531t^{25} +$
$+ 73851959357894t^{26} + 143039363140182t^{27} + 269416219454043t^{28} +$
$+ 493944596168225t^{29} + 882268074320900t^{30} + 1536543007952396t^{31} +$
$+ 2611196867637156t^{32} + 4333024660344442t^{33} + 7025611335473678t^{34} +$
$+ 11137398421309529t^{35} + 17271787147116907t^{36} + 26216525599773850t^{37} +$
$+ 38968364210329669t^{38} + 56747752371861786t^{39} + 80997424826732157t^{40} +$
$+ 113358368681589288t^{41} + 155617153462411693t^{42} + 209620178940739772t^{43} +$
$+ 277153165150321324t^{44} + 359788117447054402t^{45} + 458704770582751394t^{46} +$
$+ 574498645384155800t^{47} + 706992640391687667t^{48} + 855072713288320920t^{49} +$
$+ 1016569872742669961t^{50} + 1188209740459545784t^{51} + 1365646993055807450t^{52} +$
$+ 1543595104982837472t^{53} + 1716052512321252802t^{54} + 1876615582976246945t^{55} +$
$+ 2018857942986265569t^{56} + 2136746272693569424t^{57} + 2225056091622875140t^{58} +$
$+ 2279748435060291614t^{59}$,

$$H_{SI_{1,3,5,7}}(t) = \frac{V(t) + 3293350250\ 5147932\ t^{52} + t^{104}V(t^{-1})}{(1+t)^{19}(1-t)^{15}(1+t^2)^9(1-t^3)^{12}(1-t^5)^7(1-t^7)^3},$$

where $V(t)=1 + 5t + 15t^2 + 70t^3 + 546t^4 + 3691t^5 + 21211t^6 + 108097t^7 + 501215t^8 +$

$$+ 2135708t^9 + 8420376t^{10} + 30894213t^{11} + 106057925t^{12} + 342316946t^{13} +$$
$$+ 1043225615t^{14} + 3012988906t^{15} + 8273667765t^{16} + 21663519624t^{17} +$$
$$+ 54225659702t^{18} + 130054129145t^{19} + 299492368986t^{20} + 663439513913t^{21} +$$
$$+ 1416140486098t^{22} + 2917219852903t^{23} + 5807630254373t^{24} + 11187994444298t^{25} +$$
$$+ 20880385856690t^{26} + 37794195363608t^{27} + 66411190209119t^{28} +$$
$$+ 113391841520052t^{29} + 188282608991333t^{30} + 304271520124478t^{31} +$$
$$+ 478898737877115t^{32} + 734584562409596t^{33} + 1098797608776741t^{34} +$$
$$+ 1603661779481979t^{35} + 2284804664001899t^{36} + 3179293473234493t^{37} +$$
$$+ 4322594520474429t^{38} + 5744627532607767t^{39} + 7465155325802975t^{40} +$$
$$+ 9488929831214829t^{41} + 11801175204390804t^{42} + 14364091127469868t^{43} +$$
$$+ 17115070624832596t^{44} + 19967223601230372t^{45} + 22812575427180540t^{46} +$$
$$+ 25527987499683011t^{47} + 27983465544664079t^{48} + 30052140716959960t^{49} +$$
$$+ 31620895669339212t^{50} + 32600424240909358t^{51}.$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,3,5,7}$ (respectively $SI_{1,3,5,7}$) is equal to 39 (respectively 37).

Remark 2. *The Theorem* 5 – 10 *are published for the first time into the papers* [6-11].

We note that the Krull dimension plays an important role in solving the center-focus problem for differential systems [12].

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