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CUBIC DIFFERENTIAL SYSTEMS WITH AFFINE REAL INVARIANT STRAIGHT LINES OF TOTAL PARALLEL MULTIPLICITY SIX AND CONFIGURATIONS $\big(3(m),1,1,1\big)$

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Abstract. We classify all cubic differential systems with exactly six affine real invariant straight lines (taking into account their parallel multiplicity) of four slopes. One invariant strait line of the first slope has parallel multiplicity m, $m = 1, 2, 3$. We proove that there are five distinct classes of such systems. For every class we carried out the qualitative investigation on the Poincaré disk.

Keywords: Cubic differential system, invariant straight line, phase portrait.
2010 Mathematics Subject Classification: 34C05

SISTEMELE DIFERENŢIALE CUBICE CU DREPTE INVARIANTE AFINE REALE DE MULTIPLICITATE PARALELĂ TOTALĂ ŞASE ŞI DE CONFIGURAŢIA $\big(3(m),1,1,1\big)$

Rezumat. Sunt clasificate sistemele diferenţiale cubice cu exact şase drepte afine reale invariante (ţinându-se cont de multiplicitatea paralelă) de patru pante. O dreaptă de prima pantă are multiplicitatea paralelă m, $m = 1, 2, 3$. Se arată că există cinci clase distincte de astfel de sisteme. Fiecare clasă este studiată din punct de vedere calitativ şi pe discul Poincaré sunt construite portretele de fază.

Cuvinte-cheie: Sistem diferenţial cubic, dreaptă invariantă, portret de fază.

1. Introduction and statement of main results

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y),\ \ \gcd(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{2}$$

associated to system (1).

Denote $n = \max\big\{\deg(P), \deg(Q)\big\}$. If $n = 2$ ($n = 3$) then system (1) is called quadratic (cubic).

An algebraic curve $f(x,y) = 0,\ f \in \mathbb{C}[x,y]$ $\big($a function $f = \exp(\frac{g}{h}),\ g,h \in \mathbb{C}[x,y]\big)$ is called invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial
$K_f \in \mathbb{C}[x,y]$, $\deg(K_f) \le n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y),\ \ (x,y) \in \mathbb{R}^2$ holds. In particular, a straight line $l \equiv \alpha x + \beta y + \gamma = 0,\ \alpha,\beta,\gamma \in \mathbb{C}$ is invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x,y]$ such that the identity

$$\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma)K_l(x,y),\ (x,y) \in \mathbb{R}^2 \tag{3}$$

holds. The polynomial $K_f(x,y)$ is called cofactor of the invariant algebraic curve (exponential factor) f. If m is the greatest natural number such that l^m divides $\mathbb{X}(l)$ then we say that l has parallel multiplicity m. In the case of cubic systems we have $m \in \{1,2,3\}$. If l has the parallel multiplicity m, then $f_1 = \exp(\frac{1}{l}), ..., f_{m-1} = \exp(\frac{1}{l^{m-1}})$ are exponential factors.

Let $f_1, \ldots, f_r$ $\big(f_{r+1} = \exp(g_{r+1}/h_{r+1}), \ldots, f_s = \exp(g_s/h_s)\big)$ are invariant algebraic curves (exponential factors) of (1) with cofactors $K_{f_1}(x,y), ..., K_{f_s}(x,y)$, respectively. The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $F = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$,
$\alpha_j \in \mathbb{C}$, $j = \overline{1,s}$, such that either F is a first integral or F is an integrating factor for (1) (about the theory of Darboux, presented in the context of planar polynomial differential systems on the affine plane, see [23]). The function of the form

$$f_1^{\alpha_1} \cdots f_s^{\alpha_s}, \tag{4}$$

where $\alpha_j \in \mathbb{C}$, $|\alpha_1| + \cdots + |\alpha_s| \neq 0$, is a first integral (an integrating factor) for (1) if and only if in x and y the identity

$$\alpha_1 K_{f_1}(x,y) + \alpha_2 K_{f_2}(x,y) + ... + \alpha_s K_{f_s}(x,y) \equiv 0 \tag{5}$$

$$\left(\sum_{j=1}^{s} \alpha_j K_{f_j}(x,y) \equiv -\frac{\partial P(x,y)}{\partial x} - \frac{\partial Q(x,y)}{\partial y} \right) \tag{6}$$

holds.

By present a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have was considered in [2]; the problem of coexistence of the invariant straight lines and limit cycles was examined in $\{[22] : n = 2\}$, $\{[11], n = 3\}$, [10].

The classification of all cubic systems with the maximum number of invariant straight lines, including the line at infinity, and taking into account their geometric multiplicities, is given in [13].

In [2] it was proved that the non-degenerate cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with exactly eight and exactly seven distinct affine invariant straight lines have been studied in [13], [15]; with invariant straight lines of total geometric (parallel) multiplicity eight (seven) - in [3], [4], [5] $\big($[19], [30]$\big)$, and with six real invariant straight lines along two (three) directions - in [17], [18]. The cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity six and total parallel multiplicity five were investigated in [20], [27], [28]. In [31] it was shown that in the class of cubic differential systems the maximal $\big($algebraic, geometric, integrable or infinitesimal, see [6]$\big)$ multiplicity of an affine real straight line (of the line at infinity) is seven. In [32] the cubic systems with two affine real non-parallel invariant straight lines of maximal multiplicity are classified.

In this paper a qualitative investigation of real cubic systems of the form

$$\begin{cases} \dot{x} = P_0 + P_1(x,y) + P_2(x,y) + P_3(x,y) \equiv P(x,y), \\ \dot{y} = Q_0 + Q_1(x,y) + Q_2(x,y) + Q_3(x,y) \equiv Q(x,y), \ \gcd(P,Q) = 1, \end{cases} \tag{7}$$

where $P_k = \sum\limits_{j+l=k} a_{jl} x^j x^l$, $Q_k = \sum\limits_{j+l=k} b_{jl} x^j x^l$ $(k = \overline{0,3})$ and $|P_3(x,y)| + |Q_3(x,y)| \not\equiv 0$, with affine real invariant straight lines of total parallel multiplicity six and of four distinct slopes, is given. Only one invariant straight line from these lines can have the parallel multiplicity greater or equal two. Our main result is the following one:

Theorem 1.1. *Assuming that a cubic system* (7) *possesses affine real invariant straight lines of total parallel multiplicity six with four distinct directions and at least three of these lines have multiplicity one. Then via an affine transformation and time rescaling this system can be brought to one of the five systems* (8)−(12) *given in Table 1.1. Also, in this table for each system* (8)−(12) *the invariant straight lines, Darboux first integral* $F(x,y)$ $\big($*or integrating factor* $\mu(x,y)\big)$ *and phase portrait in the Poincaré disk are given.*

Table 1.1. Canonical forms and qualitative investigation of the cubic systems with invariant straight lines of configurations $\big(3,1,1,1\big)$, $\big(3(2),1,1,1\big)$ and $\big(3(3),1,1,1\big)$

Systems, invariant straight lines l_j, first integral (F) or integrating factor (μ)	Fig./ Tab.
Configuration (3,1,1,1). (8) $\begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = (y-1)\big(ay + (1-b)x^2 + (a-1)bxy + aby^2\big), \\ (b-1)(a+b+ab)(1+b+ab) \neq 0,\ a>0,\ b\in\mathbb{R}, \end{cases}$ $l_1 = x,\ l_2 = x+1,\ l_3 = x-a,\ l_4 = y-1,\ l_5 = x-ay,\ l_6 = x+y;$ $\mu(x,y) = x^{\alpha_1}(x+1)^{\alpha_2}(x-a)^{\alpha_3}(y-1)^{\alpha_4}(x-ay)^{\alpha_5}(x+y)^{\alpha_6}$ where $\alpha_4 = (1-b)\alpha_1 = \dfrac{1-b}{b},\ \alpha_2 = a\alpha_3 = -\dfrac{a}{b(a+1)},\ \alpha_5 = (a+b+ab)\alpha_3,$ $\alpha_6 = (1+b+ab)\alpha_3$ if $b\neq 0$; $F_1(x,y) = \dfrac{(x+1)(x-ay)}{x(y-1)}$ if $b=0$;	1.1/ 4.1
Configuration (3,1,1,1). (9) $\begin{cases} \dot{x} = x(x+1)(x-a),\ -1<a\le 1, a\neq 0,\ b>0,\ c\in\mathbb{R}^*, \\ \dot{y} = y\big(-a + (1-a)x + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ \big(\lvert a+b+ab\rvert + \lvert ac-(a+1)^2\rvert\big)\big(\lvert 1+a+ab\rvert + \lvert c-a\rvert\big) \neq 0, \text{ if } -1<a<0, \\ \text{and } \big(\lvert b-a\rvert + \lvert ac-1\rvert\big)\big(\lvert c-a\rvert + \lvert ab-1\rvert\big)\neq 0, \text{ if } 0<a\le 1, \end{cases}$ $l_1 = x,\ l_2 = x+1,\ l_3 = x-a,\ l_4 = y,\ l_5 = y-x,\ l_6 = y+bx;$ $F_2(x,y) = (x+1)^{-\frac{(b+1)bc}{a+1}}(x-a)^{-\frac{(b+1)abc}{a+1}}y^{-(b+1)}(y-x)^b(y+bx);$	1.2/ 4.2
Configuration $\big($3(2),1,1,1$\big)$. (10) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(x + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b\in\mathbb{R}^*_+,\ c\in\mathbb{R}^*, \end{cases}$ $l_{1,2} = x,\ l_3 = x+1,\ l_4 = y,\ l_5 = y-x,\ l_6 = y+bx;$ $F_3(x,y) = (x+1)^{-(b+1)bc}y^{-(b+1)}(y-x)^b(y+bx);$	1.3

Systems, invariant straight lines l_j, first integral (F) or integrating factor (μ)	Fig./ Tab.
Configuration $\big(3(2),1,1,1\big)$. (11) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(-bc - 2bcx + (b-1)cy + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b \in \mathbb{R}_+^*,\ c \in \mathbb{R}^*, \end{cases}$ $l_{1,2} = x,\ l_3 = x+1,\ l_4 = y,\ l_5 = y-x-1,\ l_6 = y + b(x+1);$ $F_4(x,y) = x^{-(b+1)bc} e^{(b+1)bc/x} y^{-(b+1)} (y-x-1)^b \big(y + b(x+1)\big);$	1.4
Configuration $\big(3(3),1,1,1\big)$. (12) $\begin{cases} \dot{x} = x^3, \\ \dot{y} = y\big((1-bc)x^2 + (b-1)cxy + cy^2\big), \\ c(bc-1)(bc+c+1)(b^2+bc+1) \neq 0,\ b > 0,\ c \in \mathbb{R}, \end{cases}$ $l_{1,2,3} = x,\ l_4 = y,\ l_5 = y - x,\ l_6 = y + bx;$ $F_5(x,y) = x^{-(b+1)bc} y^{-(b+1)} (y-x)^b (y+bx).$	1.5

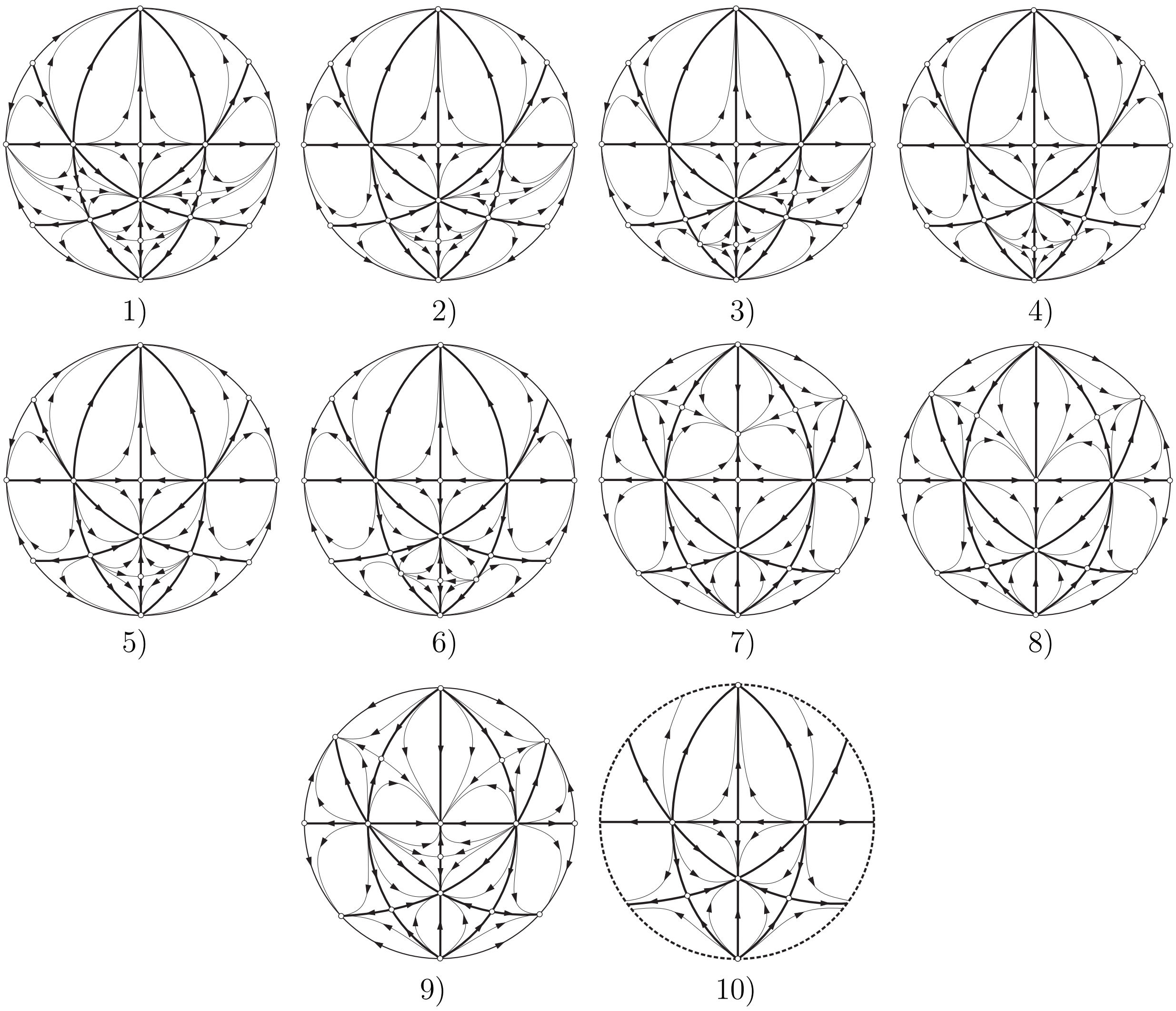
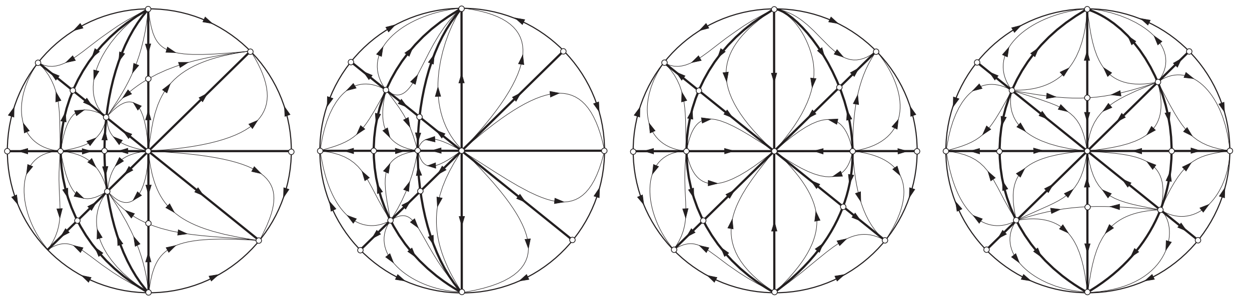
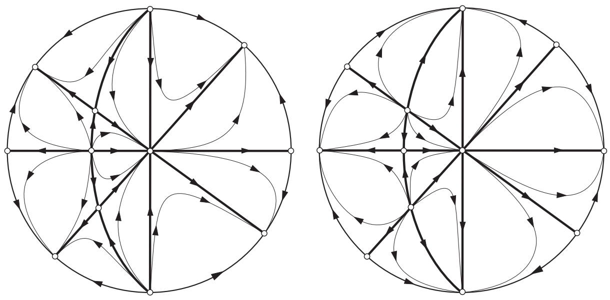


Fig. 1.1. Phase portraits of the system (8)



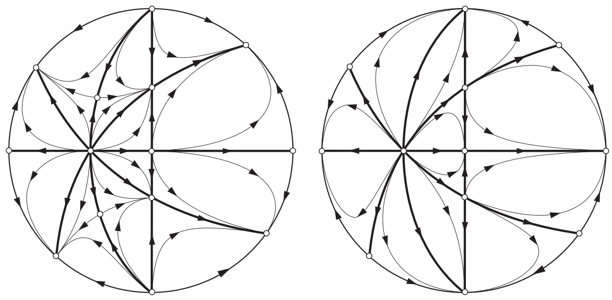
1) 2) 3) 4)

Fig. 1.2. Phase portraits of the system (9)



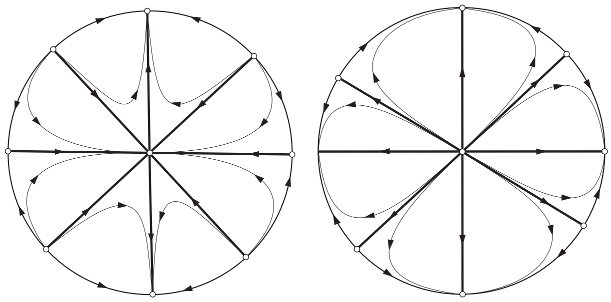
1) 2)

Fig. 1.3. Phase portraits of the system (10)



1) 2)

Fig. 1.4. Phase portraits of the system (11)



1) 2)

Fig. 1.5. Phase portraits of the system (12)

2. Some properties of the cubic systems with invariant straight lines

By a *straight lines configuration of invariant straight lines* of a cubic system we understand the set of all its invariant affine straight lines, each endowed with its own parallel multiplicity.

The goal of this section is to enumerate such properties for invariant straight lines that will allow the construction of configurations of straight lines realizable for (7). Some of these properties are obvious or easy to prove and others were proved in [29].

Properties:

2.1) *In the finite part of the phase plane each system* (7) *has at most nine singular points.*

2.2) *In the finite part of the phase plane, on any straight line there are at most three singular points of the system* (7).

2.3) *The system* (7) *has no more than eight affine invariant straight lines* $\big([2]\big)$.

2.4) *At infinity the system* (7) *has at most four distinct singular points if* $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$. *In the case* $yP_3(x,y) - xQ_3(x,y) \equiv 0$ *the infinity is degenerate, i.e. consists only of singular points.*

2.5) *If $yP_3(x, y) - xQ_3(x, y) \not\equiv 0$, then the infinity represents for* (7) *a non-singular invariant straight line, i.e. a line that is not filled up with singularities.*

2.6) *Through one point cannot pass more than four distinct invariant straight lines of the system* (7).

We say that the straight lines $l_j \equiv \alpha_j x + \beta_j y + \gamma_j \in \mathbb{C}[x, y]$, $(\alpha_j, \beta_j) \neq (0, 0), j = 1, 2$, are *parallel* if $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$. Otherwise the straight lines are called *concurrent.* If an affine invariant straight line l has the parallel multiplicity equal to m, then we will consider that we have m parallel invariant straight lines identical with l.

2.7) *The intersection point (x_0, y_0) of two concurrent invariant straight lines l_1 and l_2 of the system* (7) *is a singular point for this system.*

By a triplet of parallel affine invariant straight lines we shall mean a set of parallel affine invariant straight lines of total parallel multiplicity 3.

2.8) *If the cubic system* (7) *has a triplet of parallel affine invariant straight lines, then all its finite singular points lie on these straight lines.*

2.9) *The parallel multiplicity of an affine invariant straight line of the cubic system* (7) *is at most three.*

2.10) *If the cubic system* (7) *has two concurrent affine invariant straight lines l_1, l_2 and l_1 has the parallel multiplicity equal to m, $1 \leq m \leq 3$, then this system cannot have more than $3 - m$ singular points on $l_2 \setminus l_1$.*

We say that three affine straight lines are in generic position if no pair of these lines are parallel and no more that two lines are passing through the same point.

2.11) *For the cubic system* (7) *the total parallel multiplicity of three affine invariant straight lines in generic position is at most four.*

Proposition 2.1. *If $l \equiv \alpha x + \beta y + \gamma = 0$, $\alpha \neq 0$ $(\beta \neq 0)$ is a real invariant straight line of the system* (7) *then the transformation $X = \alpha x + \beta y + \gamma$, $Y = y$ $\big(X = \alpha x + \beta y + \gamma$, $Y = x\big)$ reduce* (7) *to a system of the form*

$$\begin{cases} \dot{X} = X(a_0 + a_1X + a_2Y + a_3X^2 + a_4XY + a_5Y^2), \\ \dot{Y} = b_0 + b_1X + b_2Y + b_3X^2 + b_4XY + b_5Y^2 + \\ \qquad +b_6X^3 + b_7X^2Y + b_8XY^2 + b_9Y^3. \end{cases} \tag{13}$$

Indeed, in the case $\alpha \neq 0$ $(\beta \neq 0)$, from (7) and (3), we have:

$$\begin{array}{l} \dot{X} = \alpha\dot{x} + \beta\dot{y} = (\alpha x + \beta y + \gamma)K_l(x, y) = X \cdot K_l\big((X - \beta Y - \gamma)/\alpha,\ Y\big), \\ \dot{Y} = \dot{y} = Q(x, y) = Q\big((X - \beta Y - \gamma)/\alpha,\ Y\big) \end{array}$$

$$\left(\begin{array}{l} \dot{X} = \alpha\dot{x} + \beta\dot{y} = (\alpha x + \beta y + \gamma)K_l(x, y) = X \cdot K_l\big(Y, (X - \alpha Y - \gamma)/\beta\big), \\ \dot{Y} = \dot{y} = Q(x, y) = Q\big(Y, (X - \alpha Y - \gamma)/\beta\big) \end{array}\right).$$

Denote that the polynomial $K_l(x, y)$ has degree less or equal to two and, consequently, $K_l\big((X - \beta Y - \gamma)/\alpha,\ Y\big)$ has the same degree. □

Proposition 2.2. *If $l_j \equiv \alpha_j x + \beta_j y + \gamma_j = 0$, $j = 1, 2$, $\Delta \equiv \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ are two real invariant straight lines of the system* (7) *then the transformation $X = \alpha_1 x + \beta_1 y + \gamma_1$,*

$Y = \alpha_2 x + \beta_2 y + \gamma_2$ *reduce* (7) *to a system of the form*

$$\begin{cases} \dot{X} = X(a_0 + a_1 X + a_2 Y + a_3 X^2 + a_4 XY + a_5 Y^2), \\ \dot{Y} = Y(b_0 + b_1 X + b_2 Y + b_3 X^2 + b_4 XY + b_5 Y^2). \end{cases} \tag{14}$$

Indeed,

$$\begin{aligned} \dot{X} &= \alpha_1 \dot{x} + \beta_1 \dot{y} = (\alpha_1 x + \beta_1 y + \gamma_1) K_{l_1}(x, y) = \\ &= X \cdot K_{l_1}\big((\beta_2 X - \beta_1 Y + \beta_1\gamma_2 - \beta_2\gamma_1)/\Delta,\ (-\alpha_2 X + \alpha_1 Y + \alpha_2\gamma_1 - \alpha_1\gamma_2)/\Delta\big), \\ \dot{Y} &= \alpha_2 \dot{x} + \beta_2 \dot{y} = (\alpha_2 x + \beta_2 y + \gamma_2) K_{l_2}(x, y) = \\ &= Y \cdot K_{l_2}\big((\beta_2 X - \beta_1 Y + \beta_1\gamma_2 - \beta_2\gamma_1)/\Delta,\ (-\alpha_2 X + \alpha_1 Y + \alpha_2\gamma_1 - \alpha_1\gamma_2)/\Delta\big). \ \square \end{aligned}$$

3. Canonical forms

Let the system (7) hase a triplet $\{l_1,\, l_2,\, l_3\}$ of parallel invariant straight lines. Then:

3.1) l_j, $j = 1, 2, 3$ are distinct and $l_1 \parallel l_2 \parallel l_3$, or

3.2) l_1 has parallel multiplicity two, $l_2 \equiv l_1 \not\equiv l_3$ and $l_1 \parallel l_3$, or

3.3) $l_1 \equiv l_2 \equiv l_3$ and l_1 has parallel multiplicity three.

Along four directions there are only three possible configurations of six invariant straight lines, three of which form a triplet of parallel invariant straight lines:

1) $(3, 1, 1, 1)$, **2)** $\big(3(2), 1, 1, 1\big)$, **3)** $\big(3(3), 1, 1, 1\big)$.

Notation $(3, 1, 1, 1)$ means that there are six distinct real invariant straight lines of four directions and three of these lines form a triplet of parallel straight lines (the case 3.1)). Configurations $\big(3(2), 1, 1, 1\big)$ and $\big(3(3), 1, 1, 1\big)$ correspond to the cases 3.2) and 3.3), respectively.

3.1. Configuration $(3, 1, 1, 1)$. Without loss of generality we can consider that one straight line of these six is parallel with to Ox axis and the straight lines from triplet are parallel with to Oy axis of coordinates. Taking into account the properties **2.2), 2.7)** and **2.8)** from Section 2, the straight lines can have (up to some affine transformations) one of the following three positions given in Fig. 3.1.

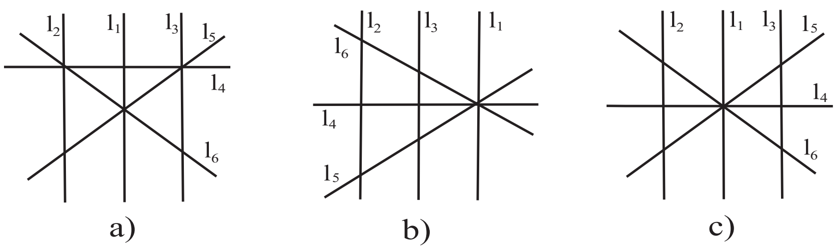


Fig. 3.1. Configurations of the type (3,1,1,1)

It is clear that the set of cubic systems which have the invariant straight lines of configuration $(3, 1, 1, 1)$ is a subset of the set of all cubic systems which have invariant straight lines of configuration $(3, 1)$.

In the case a) of Fig. 3.1 we can consider $l_1 = x$, $l_1 \cap l_5 \cap l_6 = (0, 0)$, $l_2 = x + 1$, $l_3 = x - a$, $a > 0$, $l_4 = y - 1$. Then, using an affine transformation and time rescaling, the cubic system for which $(0, 0)$ is a singular point and l_j, $j = 1, 2, 3, 4$ are invariant straight lines can be written in the form

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y),\ a>0, \\ \dot{y} = (y-1)(b_1x+b_2y+b_3x^2+b_4xy+b_5y^2) \equiv Q(x,y),\ \gcd(P,Q)=1. \end{cases} \tag{15}$$

Note that the straight line $l \equiv y - Ax - B = 0,\ A, B \in \mathbb{R}$ is invariant for polynomial differential system (1) if and only if the polynomial in x:

$$\Psi_l(x) = Q(x, Ax+B) - A \cdot P(x, Ax+B)$$

is identically zero. According to [16] if the straight line $l \equiv y - Ax - B = 0,\ A, B \in \mathbb{R}$ is invariant for (1) then l divides

$$E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P), i.e.$$

$$E(\mathbb{X}) = P(x,y)\Big(P(x,y)\tfrac{\partial Q(x,y)}{\partial x} + Q(x,y)\tfrac{\partial Q(x,y)}{\partial y}\Big) - Q(x,y)\Big(P(x,y)\tfrac{\partial P(x,y)}{\partial x} + Q(x,y)\tfrac{\partial P(x,y)}{\partial y}\Big).$$

The polynomial $E(\mathbb{X})$ has in x and y the degree $3(n-1)+2$. In particular, in the case of cubic systems we have $\deg\big(E(\mathbb{X})\big) = 8$. Let $l_1, ..., l_6$ be the invariant straight lines of (1) and $l = y - Ax - B$. Suppose that the lines $l,\ l_j,\ j = 1, ..., 6$ are distinct. Denote $E_l(x) = \big(E(\mathbb{X})/(l_1 \cdots l_6)\big)|_{y=Ax+B}$. The straight line $l = y - Ax - B$ is invariant for (1) if and only if in the same time the identities $\Psi_l(x) \equiv 0$ and $E_l(x) \equiv 0$ take place.

The straight line l_5 (l_6) passes through the singular points $(0,0)$ and $(a,1)$ $\big((-1,1)\big)$, therefore it is described by equation $x - ay = 0$ $(x + y = 0)$. The lines l_5 and l_6 are invariant if

$$\begin{cases} \Psi_{l_5}(x) = x(a-x)\big(a(b_2 - ab_1 - a) + (b_5 + ab_4 + a^2b_3 - a^2)x\big) \equiv 0, \\ \Psi_{l_6}(x) = x(x+1)\big(b_2 - b_1 - a + (1 - b_3 + b_4 - b_5)x\big) \equiv 0, \end{cases}$$

i.e. if the following series of conditions is satisfied: $b_1 = 0,\ b_2 = a,\ b_4 = b(a-1),\ b_5 = ab$, where $b = 1 - b_3$. In these conditions the system (15) looks as

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y),\ a>0, \\ \dot{y} = (y-1)\big(ay + (1-b)x^2 + (a-1)bxy + aby^2\big) \equiv Q(x,y),\ \gcd(P,Q)=1, \end{cases} \tag{16}$$

i.e. we obtain the system (8) from Table 1.1.

Let $l = y - Ax - B$. For (16) we have

$$\begin{aligned} E_l(x) = &-\big(a(1+bB)(2-2b+3bB) + b(a-1+5aA+b-ab-2aAb-2bB+2abB+ \\ &+6aAbB)x + b(1-b-2Ab+2aAb+3aA^2b)x^2\big), \\ \Psi_l(x) = &\ aB(B-1)(1+bB) + B(2aA+b-ab-2aAb-bB+abB+3aAbB)x+ \\ &+\big((1-b)(1+A)(aA-1) + B(1-b-2Ab+2aAb+3aA^2b)\big)x^2+ \\ &+bA(1+A)(aA-1)x^3. \end{aligned}$$

In conditions $a > 0$ and $\deg\big(\gcd(P,Q)\big) = 0$ the identities $\big\{\Psi_l(x) \equiv 0,\ E_l(x) \equiv 0\big\}$ hold if $(b-1)\big(a+b(a+1)\big)\big(1+b(a+1)\big) = 0$. In this case (15) has more than six invariant straight lines. Indeed, in the case $b = 1$ $\big($respectively, $a + b(a+1) = 0;\ 1 + b(a+1) = 0\big)$ the system (15) has the invariant straight line $l_7 = y$ $\big($respectively, $l_7 = x - ay + a + 1;\ l_7 = 1 + b(x+y)\big)$.

In the case b) and c) of Fig. 3.1 we can consider $l_1 = x,\ l_2 = x+1,\ l_3 = x-a$ and $l_4 = y$. It is clear that in the case b) $\big($c)$\big)$ of Fig. 3.1 we have $-1 < a < 0$ $(a > 0)$. Moreover, in the case c) we can consider $0 < a \le 1$. The cubic system for which l_j, $j = 1, 2, 3, 4$ are invariant straight lines looks as

$$\begin{cases} \dot{x} = x(x+1)(x-a), \ (-1 < a < 0 \text{ or } 0 < a \le 1), \\ \dot{y} = y(b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2). \end{cases} \tag{17}$$

The straight lines $l_{5,6}$ pass through the singular point $(0,0)$. Therefore, they are described by an equation of the form $y - bx = 0$, $b \in \mathbb{R} \setminus \{0\}$. Using the transformation $x \to x$, $y \to \alpha y$, $\alpha > 0$ we can choose $l_5 = y - x$. Then, $l_6 = y + bx$, $b > 0$. Solving the system of identities

$$\begin{cases} \Psi_{l_5}(x) = x(a + b_0 + (b_1 + b_2 + a - 1)x + (b_3 + b_4 + b_5 - 1)x^2) \equiv 0, \\ \Psi_{l_6}(x) = -bx(a + b_0 - (b \cdot b_2 - b_1 - a + 1)x + (b^2 \cdot b_5 - b \cdot b_4 + b_3 - 1)x^2) \equiv 0 \end{cases}$$

we obtain that the straight lines $l_{5,6}$ are invariant for (17) if $b_0 = -a$, $b_1 = 1 - a$, $b_2 = 0$, $b_3 = 1 - bc$, $b_4 = c(b-1)$, where $c = b_5$, i.e. if the system (17) has the form

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y), \ -1 < a \le 1, \ a \ne 0, \ b > 0, \\ \dot{y} = y\big[-a + (1-a)x + (1-bc)x^2 + c(b-1)xy + cy^2\big] \equiv Q(x,y). \end{cases} \tag{18}$$

Let $l = y - Ax - B$. For (18) we have

$$E_l(x) = c(3(cB^2 - a) + 2(1 - a - cB + 3cAB + bcB)x + (1 - 2cA + 3cA^2 - bc + 2cAb)x^2),$$
$$\Psi_l(x) = B(B^2c - a) + B(1 - a - cB + 3cAB + bcB)x + B(1 - 2cA + 3cA^2 - bc + 2bcA)x^2 + \\ + cA(A-1)(A+b)x^3.$$

If $c = 0$, then (18) is degenerate, i.e. $\deg\big(\gcd(P,Q)\big) > 0$. Let $c \ne 0$. Then, the system of identities $\big\{E_l(x) \equiv 0, \ \Psi_l(x) \equiv 0\big\}$ is equivalent to the system of equalities $\big\{A(A-1)(A+b) = 0, \ cB^2 - a = 0, \ 1 - a - cB + 3cAB + bcB = 0, \ 1 - 2cA + 3cA^2 - bc + 2cAb = 0\big\}$.

In the case $A = 0$ we obtain $b - a = ac - 1 = 0$, $B = 1/a$ or $c - a = ab - 1 = 0$, $B = -1$. Therefore, if $0 < a \le 1$ then the system (18) has the seventh invariant straight line $l_7 \equiv y - a = 0$ if $b - a = ac - 1 = 0$, $B = 1/a$ and $l_7 \equiv y + 1 = 0$ if $c - a = ab - 1 = 0$, $B = -1$. Let $\big(|b - a| + |ac - 1|\big)\big(|c - a| + |ab - 1|\big) \ne 0$ and $A = 1$. Then $\{c - a = ab + a + 1 = 0, \ B = 1\}$ $\Rightarrow -1 < a < 0$ and we have the invariant straight line $l_7 \equiv y - x - 1 = 0$. At last, if $A = -b$ then $a + b(a+1) = ac - (a+1)^2 = 0, A = B = a/(a+1)$. Taking into account that $b > 0$ these equalities imply $-1 < a < 0$. Thus, if $-1 < a < 0$ then the system (18) has exactly six distinct invariant straight lines if and only if the following inequality $\big(|c - a| + |ab + a + 1|\big)\big(|a + b(a+1)| + |ac - (a+1)^2|\big) \ne 0$ holds.

The above description leads us to the system (9) from Table 1.1 and to the inequalities associated with it.

3.2. Configuration $\big(3(2), 1, 1, 1\big)$. Let the system (7) have six invariant straight lines of the considered configuration of which l_1 has parallel multiplicity two, $l_2 \equiv l - 1$, and $l_3 \parallel l_{1,2}$. Taking into account Properties **2.8)** and **2.10)** the invariant straight lines l_j, $j = 1, ..., 6$ have (up to some affine transformations) one of the following two positions given in Fig. 3.2.

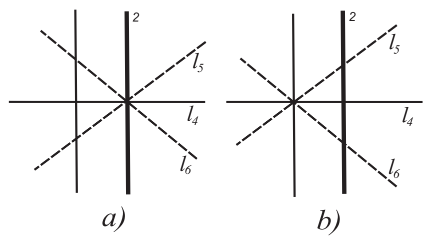


Fig. 3.2. Configuration $\big(3(2),1,1,1\big)$

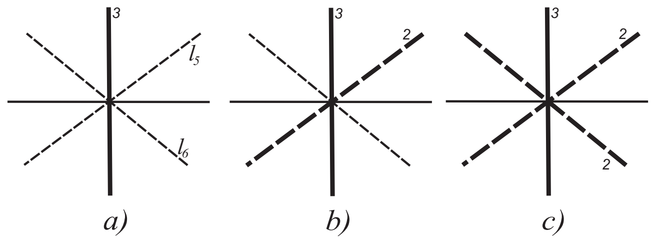


Fig. 3.3. Configuration $\big(3(3),1,1,1\big)$

Without loss of generality we can consider that $l_{1,2} = x$, $l_3 = x+1$, $l_4 = y$. The cubic system for which these lines are invariant looks as

$$\begin{cases} \dot{x} = x^2(x+1) \equiv P(x,y), \\ \dot{y} = y(b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2) \equiv Q(x,y), \ \gcd(P,Q) = 1. \end{cases} \tag{19}$$

In the case $a)$ $\big(b)\big)$ of Fig. 3.2 via the transformation $x \to x$, $y \to \gamma y$, $\gamma \neq 0$ we make the line l_5 to be described by the equation $y - x = 0$ $\ (y - x - 1 = 0)$. The equation of l_6 has the form $y = -bx$ $(y = -bx - b)$, $b > 0$. In this case, i.e. $a)$ $\big(b)\big)$ of Fig. 3.2, the straight lines $l_{5,6}$ are invariant for (19) if the identities hold:

$$\begin{cases} \Psi_{l_5} = x\big[b_0 + (b_1 + b_2 - 1)x + (b_3 + b_4 + b_5 - 1)x^2\big] \equiv 0, \\ \Psi_{l_6} = bx\big[-b_0 + (bb_2 - b_1 + 1)x + (bb_4 - b^2b_5 - b_3 + 1)x^2\big] \equiv 0 \end{cases}$$

$$\left(\begin{cases} \Psi_{l_5} = b_0 + b_2 + b_5 + (b_0 + b_1 + 2b_2 + b_4 + 3b_5)x + (b_1 + b_2 + b_3 + 2b_4 + 3b_5 - 1)x^2 + \\ \qquad +(b_3 + b_4 + b_5 - 1)x^3 \equiv 0, \\ \Psi_{l_6} = b\big[-b_0 - b^2b_5 + bb_2 + (-3b^2b_5 + 2bb_2 + bb_4 - b_0 - b_1)x + (-3b^2b_5 + bb_2 + \\ \qquad +2bb_4 - b_1 - b_3 + 1)x^2 + (-b^2b_5 + bb_4 - b_3 + 1)x^3\big] \equiv 0 \end{cases}\right).$$

These identities give us

$$b_0 = b_2 = 0,\ b_1 = 1,\ b_3 = 1 - bc,\ b_4 = c(b-1)$$

$$\big(b_0 = -bc,\ b_1 = -2bc,\ b_2 = b_4 = c(b-1),\ b_3 = 1 - bc\big),$$

where $c = b_5$. We obtained the system (10) $\big((11)\big)$ from Table 1.1. For both systems the equality $c = 0$ is in contradiction with the condition $\gcd(P,Q) = 1$.

3.3. Configuration $\big(3(3),1,1,1\big)$. For the first step, without loss of generality, we consider the system

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y(b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2). \end{cases} \tag{20}$$

The system (20) has the invariant straight lines: $l_{1,2,3} = x$ and $l_4 = y$. The other invariant straight lines l_5 and l_6 of (20) (if exist) must pass through singular point $(0,0)$. Moreover, we can consider that l_5 $\ (l_6)$ is described by the equation $y - x = 0$ $\ (y + bx = 0,\ b > 0)$. The identities

$$\Psi_{l_5} = x\big[b_0 + (b_1 + b_2)x + (b_3 + b_4 + b_5 - 1)x^2\big] \equiv 0,$$
$$\Psi_{l_6} = -bx\big[b_0 + (b_1 - bb_2)x + (b^2b_5 - bb_4 + b_3 - 1)x^2\big] \equiv 0$$

have the solution

$$b_0 = b_1 = b_2 = 0,\ b_3 = 1 - bc,\ b_4 = c(b-1), \tag{21}$$

where $c = b_5$.

In the conditions $\{(21), b_5 = c\}$ the straight lines $l_{5,6}$ are invariant for the system (20). The cofactors of lines l_4, l_5, l_6 are respectively: $K_4(x,y) = (1-bc)x^2 + c(b-1)xy + cy^2$, $K_5(x,y) = x^2 + bcxy + cy^2$ and $K_6(x,y) = x^2 - cxy + cy^2$. From these, $K_4(x,0) = (1-bc)x^2$, $K_5(x,x) = (bc+c+1)x^2$ and $K_6(x,-bx) = (b^2c+bc+1)x^2$. Therefore, if $(bc-1)(bc+c+1)(b^2c+bc+1) = 0$ then at least one of the invariant straight lines has parallel multiplicity greater than one but this is not allowed in the examined configuration. If in the system $\{(21),(21), b_5 = c\}$ the parameter c vanishes then the condition $\gcd(P,Q) = 1$ is not met. Thus, the system (12) from Table 1.1 of Theorem 1.1 and its associated conditions are obtained.

4. Darboux integrability

In this section we construct the first integrals (F) or the integrating factors (μ) for systems (8)−(12).

4.1. Integrability of the system (8):

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y), \\ \dot{y} = (y-1)\big(ay + (1-b)x^2 + (a-1)bxy + aby^2\big) \equiv Q(x,y), \\ (b-1)(a+b+ab)(1+b+ab) \neq 0,\ a > 0,\ b \in \mathbb{R}. \end{cases}$$

The cofactors of the invariant straight lines: $l_1 = x$, $l_2 = x+1$, $l_3 = x-a$, $l_4 = y-1$, $l_5 = x - ay$, $l_6 = x+y$ of this system are, respectively:

$$K_{l_1}(x,y) = (x+1)(x-a),\ K_{l_2}(x,y) = x(x-a),\ K_{l_3}(x,y) = x(x+1),$$
$$K_{l_4}(x,y) = ay + (1-b)x^2 + (a-1)bxy + aby^2,$$
$$K_{l_5}(x,y) = -a + (1-ab)x + a(1-b)y + x^2 + abxy + aby^2,$$
$$K_{l_6}(x,y) = -a + (b-a)x + a(1-b)y + x^2 - bxy + aby^2.$$

Putting $s = 6$, $f \equiv l$ and $K_{l_j}(x,y)$, $j = \overline{1,6}$ in (6) and identifying the coefficients near the same powers of x and y, we get the system

$$\begin{cases} \alpha_1 + \alpha_5 + \alpha_6 = -2, \\ (1-a)\alpha_1 - a\alpha_2 + \alpha_3 + (1-ab)\alpha_5 + (b-a)\alpha_6 = (a-1)(b+2), \\ \alpha_4 + (1-b)(\alpha_5 + \alpha_6) = 2(b-1), \\ \alpha_1 + \alpha_2 + \alpha_3 + (1-b)\alpha_4 + \alpha_5 + \alpha_6 = b-4, \\ b\big(2a - 2 + (a-1)\alpha_4 + a\alpha_5 - \alpha_6\big) = 0, \\ b(3 + \alpha_4 + \alpha_5 + \alpha_6) = 0. \end{cases}$$

If $b \neq 0$ then this system has the following solution in $\alpha_1, ..., \alpha_6$:

$$\alpha_1 = \frac{1}{b},\ \alpha_2 = -\frac{a}{(a+1)b},\ \alpha_3 = -\frac{1}{(a+1)b},\ \alpha_4 = \frac{1-b}{b},\ \alpha_5 = -\frac{a+(a+1)b}{(a+1)b},\ \alpha_6 = -\frac{1+(a+1)b}{(a+1)b}.$$

Therefore,

$$\mu(x,y) = x^{\frac{1}{b}}(x+1)^{-\frac{a}{(a+1)b}}(x-a)^{-\frac{1}{(a+1)b}}(y-1)^{\frac{1-b}{b}}(x-ay)^{-\frac{a+(a+1)b}{(a+1)b}}(x+y)^{-\frac{1+(a+1)b}{(a+1)b}}$$

is a Darboux integrating factor of the system (8) $\big($see, (4)$\big)$.

For these cofactors in the case $b \neq 0$, the identity (5) takes place if and only if $\alpha_1 = 0, ..., \alpha_6 = 0$. If $b = 0$, then the identity (5) is equivalent to the system

$$\begin{cases} \alpha_1+\alpha_5+\alpha_6=0, \\ (1-a)\alpha_1-a\alpha_2+\alpha_3+\alpha_5+a\alpha_6=0, \\ \alpha_4+\alpha_5+\alpha_6=0, \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6=0; \end{cases} \Leftrightarrow \begin{cases} \alpha_1=-(\alpha_5+\alpha_6), \\ \alpha_2=\alpha_5, \\ \alpha_3=\alpha_6, \\ \alpha_4=-(\alpha_5+\alpha_6). \end{cases}$$

This system has the solution $\alpha_1=\alpha_4=-\alpha_2=-\alpha_5=-1$. Thus,

$$F(x,y)=\frac{(x+1)(x-ay)}{x(y-1)}$$

is a first integral of the system $\{(8),\ b=0\}$.

4.2. Integrability of the system (9):

$$\begin{cases} \dot{x}=x(x+1)(x-a),\ a>-1,\ b>0,\ c\in\mathbb{R}^*, \\ \dot{y}=y\big(-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2\big), \\ \big(|a+b+ab|+|ac-(a+1)^2|\big)\big(|1+a+ab|+|c-a|\big)\neq 0,\ \text{if }\ -1<a<0, \\ \text{and }\big(|b-a|+|ac-1|\big)\big(|c-a|+|ab-1|\big)\neq 0,\ \text{if } 0<a\leq 1. \end{cases}$$

The invariant straight lines: $l_1=x$, $l_2=x+1$, $l_3=x-a$, $l_4=y$, $l_5=y-x$, $l_6=y+bx$ of (9) have the cofactors, respectively:

$$K_{l_1}(x,y)=(x+1)(x-a),\ K_{l_2}(x,y)=x(x-a),\ K_{l_3}(x,y)=x(x+1),$$
$$K_{l_4}(x,y)=-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2,$$
$$K_{l_5}(x,y)=-a+(1-a)x+x^2+bcxy+cy^2,\ K_{l_6}(x,y)=-a+(1-a)x+x^2-cxy+cy^2.$$

Putting $K_{l_i}(x,y)$, $i=\overline{1,6}$ in the identity (5) we obtain in α_i, $i=\overline{1,6}$ the system:

$$\begin{cases} \alpha_1+\alpha_4+\alpha_5+\alpha_6=0, \\ (1-a)(\alpha_1+\alpha_4+\alpha_5+\alpha_6)-a\alpha_2+\alpha_3=0, \\ \alpha_1+\alpha_2+\alpha_3+(1-bc)\alpha_4+\alpha_5+\alpha_6=0, \\ (b-1)\alpha_4+b\alpha_5-\alpha_6=0, \\ \alpha_4+\alpha_5+\alpha_6=0; \end{cases} \Leftrightarrow \begin{cases} \alpha_1=0, \\ \alpha_2=-(b+1)bc\alpha_6/(a+1), \\ \alpha_3=-(b+1)abc\alpha_6/(a+1), \\ \alpha_4=-(b+1)\alpha_6, \\ \alpha_5=b\alpha_6. \end{cases}$$

Considering $\alpha_6=1$, the solution of this system gives us the following first integral of (9):

$$F(x,y)=(x+1)^{-\frac{(b+1)bc}{a+1}}(x-a)^{-\frac{(b+1)abc}{a+1}}y^{-(b+1)}(y-x)^b(y+bx).$$

4.3. Integrability of the systems (10)−(12).

Similarly to subsections **4.1** and **4.2** for each system (10)−(12) we calculate the cofactors $K_{l_j}(x,y)$, $j=\overline{1,6}$ (see, (3)) of invariant straight lines and the exponents α_j, $j=\overline{1,6}$ (see, (5)) of the first integrals $F(x,y)$ of the form (4). The obtained results are given in Table 4.1.

Table 4.1. First integrals of systems (10) − (12)

Syst.	l_i, $i=\overline{1,6}$	$K_i(x,y)$, $i=\overline{1,6}$	α_i, $i=\overline{1,6}$	F
(10)	$l_1=x$, $l_2=e^{1/x}$, $l_3=x+1$, $l_4=y$, $l_5=y-x$, $l_6=y+bx$,	$K_{l_1}=x(x+1)$, $K_{l_2}=-x-1$, $K_{l_3}=x^2$, $K_{l_4}=x+(1-bc)x^2+(b-1)cxy+cy^2$, $K_{l_5}=x+x^2+bcxy+cy^2$, $K_{l_6}=x+x^2-cxy+cy^2$,	$\alpha_1=0$, $\alpha_2=0$, $\alpha_3=-(b+1)bc\alpha_6$, $\alpha_4=-(b+1)\alpha_6$, $\alpha_5=b\alpha_6$;	F_3

Table 4.1 (continued)

Syst.	l_i, $i=\overline{1,6}$	$K_i(x,y)$, $i=\overline{1,6}$	α_i, $i=\overline{1,6}$	F
(11)	$l_1=x$, $l_2=e^{1/x}$, $l_3=x+1$, $l_4=y$, $l_5=y-x-1$, $l_6=y+bx+b$,	$K_{l_1}=x(x+1)$, $K_{l_2}=-x-1$, $K_{l_3}=x^2$, $K_{l_4}=bc(y-2x-1)-cy+(1-bc)x^2+$ $+(b-1)cxy+cy^2$, $K_{l_5}=bcy+x^2+bcxy+cy^2$, $K_{l_6}=-cy+x^2-cxy+cy^2$,	$\alpha_1=-(b+1)bc\alpha_6$, $\alpha_2=(b+1)bc\alpha_6$, $\alpha_3=0$, $\alpha_4=-(b+1)\alpha_6$, $\alpha_5=b\alpha_6$;	F_4
(12)	$l_1=x$, $l_2=e^{1/x}$, $l_3=e^{1/x^2}$, $l_4=y$, $l_5=y-x$, $l_6=y+bx$,	$K_{l_1}=x^2$, $K_{l_2}=-x$, $K_{l_3}=-2$, $K_{l_4}=(1-bc)x^2+(b-1)cxy+cy^2$, $K_{l_5}=x^2+bcxy+cy^2$, $K_{l_6}=x^2-cxy+cy^2$,	$\alpha_1=-(b+1)bc\alpha_6$, $\alpha_2=0$, $\alpha_3=0$, $\alpha_4=-(b+1)\alpha_6$, $\alpha_5=b\alpha_6$.	F_5

5. Qualitative investigation of the systems (8)−(12)

In this section, the qualitative study of the systems (8)−(12) from Theorem 1.1 will be done. For this purpose, in order to determine the topological behavior of trajectories, the singular points in the finite and infinite part of the phase plane will be examined. This information and the information provided by the existence of invariant straight lines will be taken into account when the phase portraits of the system (8)−(12) on the Poincaré disk will be constructed.

We set the abbrevation SP for a singular point and use here the following notations: λ_1 and λ_2 for eigenvalues of SP; S for a saddle ($\lambda_1\lambda_2<0$); N^s for a stable node ($\lambda_1,\lambda_2<0$); N^u for a unstable node ($\lambda_1,\lambda_2>0$); $S-N^{s(u)}$ for a saddle-node with a stable (unstable) parabolic sector, $P^{s(u)}$ for a stable (unstable) parabolic sector, H for a hyperbolic sector.

5.1. System (8) (configuration (3,1,1,1)), i.e. the system

$$\begin{cases} \dot{x}=x(x+1)(x-a)\equiv P(x,y),\\ \dot{y}=(y-1)\big(ay+(1-b)x^2+(a-1)bxy+aby^2\big)\equiv Q(x,y),\\ (b-1)(a+b+ab)(1+b+ab)\neq 0,\ a>0,\ b\in\mathbb{R}, \end{cases}$$

which has the invariant straight lines: $l_1=x$, $l_2=x+1$, $l_3=x-a$, $l_4=y-1$, $l_5=x-ay$ and $l_6=x+y$. This system has in the finite part of the phase plane nine singular points if $b\neq 0$ and six if $b=0$. The semi-plane of parameters $a,b;\ a>0$ is divided in thirteen sectors I_j by straight lines $a=0$, $b=0$, $b=\pm 1$ and the hyperbolas $(a+1)b=\pm 1$, $(a+1)b=\pm a$ (see, Fig. 5.1).

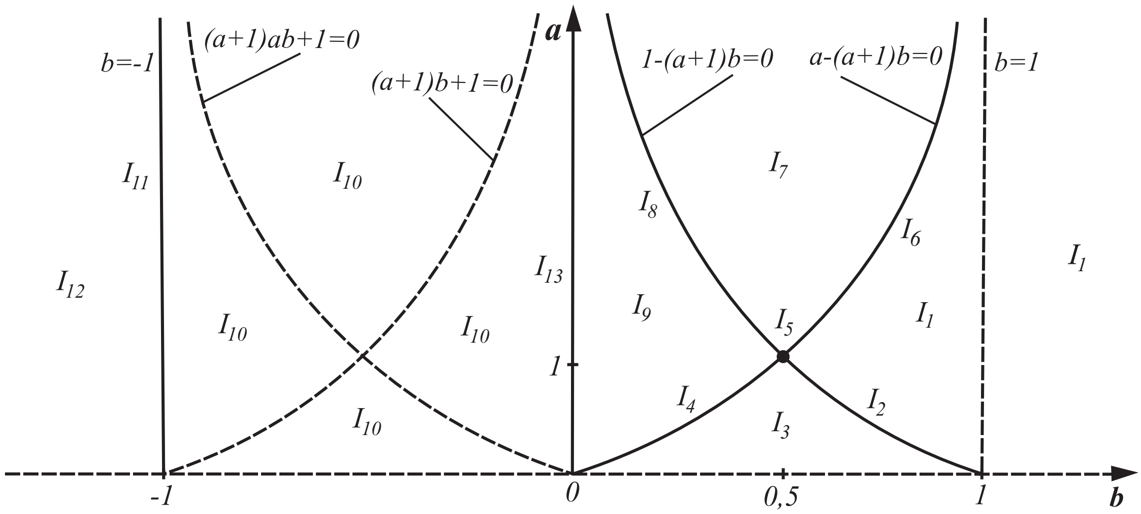


Fig. 5.1. Bifurcation diagram of the system (8)

In each of these sectors we calculate the eigenvalues of singular points and bring them in Table 5.1. In Table 5.1 we used the notations: $\alpha = a + 1$ and $\beta = a(b + 1)$.

Taking into account that $a > 0$, from Table 5.1 it is easy to see that the singular points O_1 and O_3 (respectively, point O_5) of the system (8) are unstable nodes (is a stable nod). If $b < -1$ or $b > 0$ then the O_9 is a saddle, and if $-1 < b < 0$ then O_9 is a stable node and so on.

Further we will study non-hyperbolic singular points of the system (8): O_2 in the sector I_{11}; O_4 in the sectors $I_4 - I_6$ and O_6 in I_2, I_5 and I_8. In the other cases the singular points are hyperbolic.

Table 5.1. System (8): singular points, eigenvalues and types of SP

SP	$\lambda_1;\ \lambda_2$	$I_1/I_2/I_3$	$I_4/I_5/I_6$	$I_7/I_8/I_9$	$I_{10}/I_{11}/I_{12}$	I_{13}
$O_1(-1,1)$	$\alpha;\ \alpha$	N^i				
$O_2(0,1)$	$-a;\ \beta$	S	S	S	$S/S{-}N^s/N^s$	S
$O_3(a,1)$	$a\alpha;\ a\alpha$	N^u				
$O_4(-1,-\frac{1}{a})$	$\alpha;\ \frac{\alpha(\alpha b-a)}{a}$	N^u	$S{-}N^u$	S	S	S
$O_5(0,0)$	$-a;\ -a$	N^s				
$O_6(a,-a)$	$a\alpha;$ $a\alpha(\alpha b-1)$	$N^u/S{-}N^u/S$	$S/S{-}N^u/N^u$	$N^u/S{-}N^i/S$	S	S
$O_7(-1,\frac{b-1}{b})$	$\alpha;\ \frac{a-\alpha b}{b}$	S	$-$	N^u	S	$-$
$O_8(a,\frac{b-1}{b})$	$a\alpha;\ \frac{a-\alpha ab}{b}$	$S/-/N^u$	$N^u/-/S$	$S/-/N^u$	S	$-$
$O_9(0,-\frac{1}{b})$	$-a;\ \frac{\beta}{b}$	S	S	S	$N^s/-/S$	$-$
Fig. 1.1:		1)/2)/3)	4)/5)/2)	3)/4)/6)	7)/8)/9)	10)

1) *Singular point* $O_2(0,1)$. Sector I_{11} is the semi-straight line of the semi-plane bOa, $a > 0$ given by equation $b = -1$. On I_{11} the eigenvalues of O_2 are $\lambda_1 = -a$ and $\lambda_2 = 0$, therefore it is a semi-hyperbolic singular point. The transformation $(x, y) \to (x, y-1)$ translate O_2 in the origin of the system of coordinates xOy. Then, changing x by y and y by x, i.e. $x = Y$,

$y = X$ and rescaling the time $\tau = -at$, the system $\{(8),\ b = -1\}$ takes the form

$$\begin{cases} \dot{X} = \frac{1}{a}X\big(aX + (a-1)Y + aX^2 + (a-1)XY - 2Y\big) = P(X,Y), \\ \dot{Y} = Y - \frac{1-a}{a}Y^2 - \frac{1}{a}Y^3 = Y + Q(X,Y). \end{cases}$$

The function $Y = \varphi(X) = \sum_{i\geq 1} c_i X^i$ is an analitic solution of equation $Y + Q(X,Y) = 0$ if and only if it vanishes $Y = \varphi(X) \equiv 0$. Putting $Y = 0$ in $P(X,Y)$ we obtain $\psi(X) = X^2 + X^3$. According to [1] the singular point $O_2(0,1)$ is a stable saddle-node.

2) *Singular point* $O_4\big(-1, -\frac{1}{a}\big)$. In this case the sectors I_4, I_5 and I_6 are placed on the hyperbola $a - (1+a)b = 0$, i.e. $b = \frac{a}{a+1}$, where $a > 0$. The eigenvalues of O_4 are $\lambda_1 = 1 + a$ and $\lambda_2 = 0$, therefore O_4 is a semi-hyperbolic singular point. Translating O_4 in the origin $\big((x,y) \to (x+1, y+1/a)\big)$ and putting $b = \frac{a}{a+1}$ in (8) we obtain

$$\dot{x} = x(x-1)(x-a-1),\ \dot{y} = (ay-a-1)\big(-(a+1)x + x^2 + a(a-1)xy + a^2y^2\big)/(a^2+a).$$

The nondegenerate transformation $(x,y) \to \big(Y, X + Y/a\big)$ and the time rescaling $\tau = (a+1)t$ reduce the last system to the form

$$\begin{cases} \dot{X} = -\frac{1}{a+1}X\big(aX + (a+2)Y - \frac{a^2}{a+1}X^2 - \frac{(a+2)a}{a+1}XY - 2Y^2\big) = P(X,Y), \\ \dot{Y} = Y - \frac{a+2}{a+1}Y^2 + \frac{1}{a+1}Y^3 = Y + Q(X,Y). \end{cases}$$

From the equation $Y + Q(X,Y) = 0$ we find $Y = \varphi(X) = 0$. Putting $Y = 0$ in $P(X,Y)$ we obtain $\psi(X) = -\frac{a}{a+1}X^2 + \frac{a^2}{(a+1)^2}X^3$. According to [1], the singular point $O_4\big(-1, -\frac{1}{a}\big)$ is an unstable saddle-node.

3) *Singular point* $O_6(a, -a)$, $a > 0$. The sectors I_2, I_5 and I_6 are placed on the hyperbola $(a+1)b = 1$, i.e. $b = \frac{1}{a+1}$. The eigenvalues of O_6 are $\lambda_1 = (1+a)a$ and $\lambda_2 = 0$, thus the singular point O_6 is semi-hyperbolic. Proceeding in the same way as in the case 2) for O_6 we obtain $\psi(X) = -\frac{1}{a+1}X^2 + \frac{1}{(a+1)^2}X^3$. According to [1] the point O_6 is of saddle-node type.

Proposition 5.1. At infinity the system (8) has the following singular points:

a) $X_{1\infty}(1,0,0)$ – saddle; $X_{2\infty}(1,-1,0)$, $X_{3\infty}\big(1, \frac{1}{a}, 0\big)$ – stable nodes and $Y_\infty(0,1,0)$ – unstable node, if $b < 0$;

b) $X_{1\infty}(1,0,0)$ – stable node; $X_{2\infty}(1,-1,0)$, $X_{3\infty}\big(1, \frac{1}{a}, 0\big)$ – saddles and $Y_\infty(0,1,0)$ – stable node, if $b > 0$;

c) if $b = 0$ then the infinity is degenerate for (8), i.e. consists only of singular points. The singular points situated at the ends of the Oy axis are nodes. Through each of every other singular point at the infinity passes only one trajectory.

Proof. In the case $b \neq 0$ $(b = 0)$ the first Poincaré transformation $x = 1/z$, $y = u/z$ and the time rescaling $\tau = t/z^2$ $(\tau = t/z)$ reduce (8) to the system

$$\dot{z} = z(z+1)(az-1),\ \dot{u} = (u+1)(au-1)\big(bu + (1-b)z\big)$$

$$\big(\dot{z} = (z+1)(az-1),\ \dot{u} = (u+1)(au-1)\big),$$

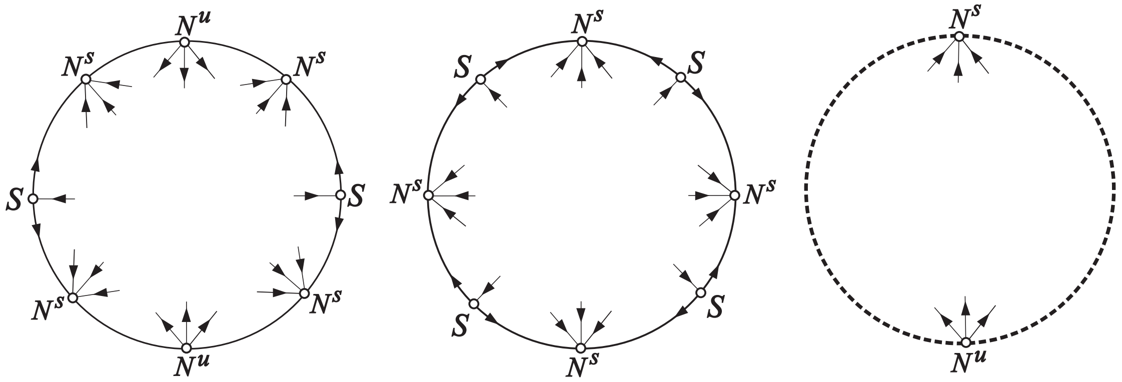
and the second transformation: $x = v/z$, $y = 1/z$ and $\tau = t/z^2$ $(\tau = zt)$ give us

$$\dot{v} = v(v+1)(v-a)\big(b + (1-b)z\big),\ \dot{z} = z(z-1)\big(ab + (a-1)bv + az + (1-b)v^2\big)$$

$$\big(\dot v = v(v+1)(v-a),\ \dot z = (z-1)(az+v^2)\big).$$

Putting $z=0$ in the right-hand sides of these systems and equaling them with zero we obtain the following singular points, respectively: $X_{1_\infty}(1,0,0):\{\lambda_1=-1,\lambda_2=-b\}$, $X_{2_\infty}(1,-1,0):\{\lambda_1=-1,\lambda_2=b(a+1)\}$, $X_{3_\infty}\big(1,\frac{1}{a},0\big):\big\{\lambda_1=-1,\lambda_2=\frac{b(a+1)}{a}\big\}$ and $Y_\infty(0,1,0):\{\lambda_1=\lambda_2=-ab\}$ $\big(Y_\infty(0,1,0):\{\lambda_1=\lambda_2=-a\}\big)$. The types of these singular points are completely determined by their eigenvalues: λ_1 and λ_2. □

In Fig. 5.2 are illustrated the singular points from Proposition 5.1.



a) $b<0$ b) $b>0$ c) $b=0$

Fig. 5.2. Singular points at the infinity of the system (8)

The qualitative study in the finite part of phase plane and at the infinity leads us to the portraits given in Fig. 1.1.

5.2. System (9) (configuration (3,1,1,1)):

$$\begin{cases} \dot x = x(x+1)(x-a),\ a>-1,\ b>0,\ c\in\mathbb{R}^*,\\ \dot y = y\big(-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2\big),\\ \big(|a+b+ab|+|ac-(a+1)^2|\big)\big(|1+a+ab|+|c-a|\big)\neq 0,\ \text{if } -1<a<0,\\ \text{and } \big(|b-a|+|ac-1|\big)\big(|c-a|+|ab-1|\big)\neq 0,\ \text{if } 0<a\le 1. \end{cases}$$

For this system the straight lines: $l_1=x$, $l_2=x+1$, $l_3=x-a$, $l_4=y$, $l_5=y-x$, $l_6=y+bx$ are invariant. At the infinity it has four singular points and in the finite part of the phase plan it has nine (seven). All singular point are hyperbolic. Their eigenvalues and their types are given in Table 5.2. The information from the Table 5.2 are sufficiently to sketch phase portraits (see Fig. 1.2).

In the Table 5.2 we used the notations: $\alpha=a+1$ and $\beta=b+1$.

Table 5.2. System (9): singular points, eigenvalues and types of SP

SP	$\lambda_1; \lambda_2$	$-1 < a < 0$		$0 < a \leq 1$	
		$c < 0$	$c > 0$	$c < 0$	$c > 0$
$O_1(-1, 0)$	$\alpha;\ -bc$	N^i	S	N^i	S
$O_2(0, 0)$	$-a;\ -a$	N^i	N^i	N^s	N^s
$O_3(a, 0)$	$a\alpha;\ -a^2bc$	S	N^s	N^i	S
$O_4(-1, -1)$	$\alpha;\ c\beta$	S	N^i	S	N^i
$O_5(a, a)$	$a\alpha;\ a^2c\beta$	N^s	S	S	N^i
$O_6(-1, b)$	$\alpha;\ bc\beta$	S	N^i	S	N^i
$O_7(a, -ab)$	$a\alpha;\ a^2bc\beta$	N^s	S	S	N^i
$O_{8,9}\left(0, \pm\sqrt{\frac{a}{c}}\right)$	$-a;\ 2a$	S	$-$	$-$	S
$X_{1_\infty}(1, 0, 0)$	$-1;\ -bc$	S	N^s	S	N^s
$X_{2_\infty}(1, 1, 0)$	$-1;\ c\beta$	N^s	S	N^s	S
$X_{3_\infty}(1, -b, 0)$	$-1;\ bc\beta$	N^s	S	N^s	S
$Y_\infty(0, 1, 0)$	$-c;\ -c$	N^i	N^s	N^i	N^s
see Fig. 1.2:		1)	2)	3)	4)

5.3. System (10) (configuration $(3(2),1,1,1)$):

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(x + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b \in \mathbb{R}_+^*, \ \ c \in \mathbb{R}^*. \end{cases}$$

The straight lines: $l_{1,2} = x$, $l_3 = x+1$, $l_4 = y$, $l_5 = y - x$ şi $l_6 = y + bx$ are invariant for (10).

The lines l_1, l_5, l_4 and l_6 divide the neighborhood of $O(0,0)$ in eight sectors. We enumerate these sectors from positive Ox semi-axis in counterclockwise direction. The notation $P^uH4P^sHP^u$ means that the first sector is unstable parabolic, the second sector is of hyperbolic type, the 3,4,5,6 sectors are stable parabolic, the 7 sector is hyperbolic and the 8 sector is unstable parabolic.

Proposition 5.2. In the finite part of the phase plane the system (10) has the following singular points:

1) $O_1(0,0) - P^uH4P^sHP^u$ if $c < 0$, and $2P^uH2P^sH2P^u$, if $c > 0$;

2) $O_2(-1,0)$ $-$ unstable node if $c < 0$, and saddle if $c > 0$;

3) $O_{3,4}(-1,-1)$ $-$ saddle if $c < 0$, and unstable node if $c > 0$.

Proof. We will examine separately every singular point $O_1 - O_4$.

a) *Singular point* $O_1(0,0)$. Both eigenvalues of the point O_1 are null. We will study the behavior of the trajectories in a neighborhood of this point using blow-up method. First we apply in (10) the transformation $x = X$, $y = XY$:

$$\begin{cases} \dot{X} = \dot{x} = x^2(x+1) = X^2(X+1), \\ \dot{Y} = \dot{y}/x - y\dot{x}/x^2 = bX^2Y(Y-1)(Y+a). \end{cases}$$

Then, rescaling the time $\tau = X^2 t$ and using the substitution $(X, Y) \to (X+1, Y)$, the last system takes the form:

$$\dot{X} = X,\ \dot{Y} = bY(Y-1)(Y+a). \tag{22}$$

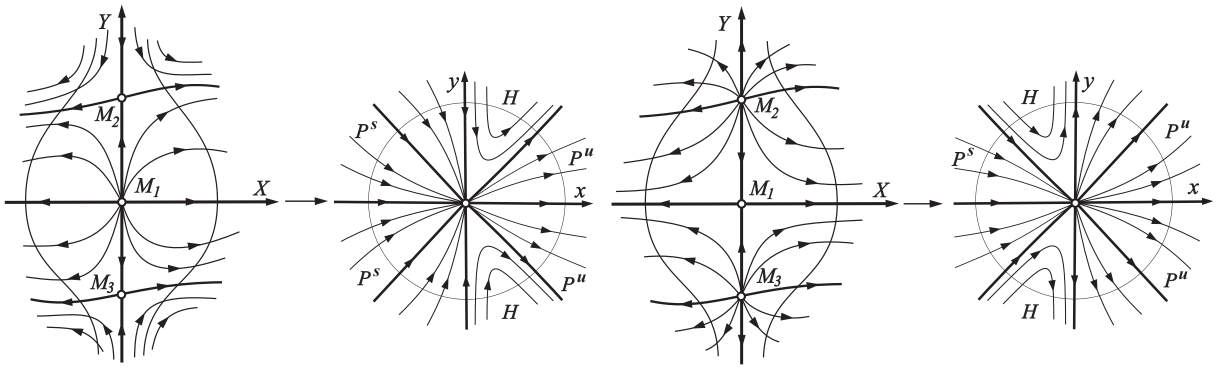
The singular points of the system (22) and their eigenvalues are:

$\{M_1(0,0) : \lambda_1 = 1, \lambda_2 = -bc\}$ – unstable node if $c < 0$, and saddle if $c > 0$;

$\{M_2(0,1) : \lambda_1 = 1, \lambda_2 = (b+1)c\}$ – saddle if $c < 0$, and unstable node if $c > 0$;

$\{M_3(0,-b) : \lambda_1 = 1, \lambda_2 = (b+1)bc\}$ – saddle if $c < 0$, and unstable node if $c > 0$.

The behavior of the trajectories near the points: M_1, M_2, and $(0,0)$ is illustrated in Fig. 5.3a (Fig. 5.3b) if $c < 0$ $(c > 0)$.



a) $c < 0$ b) $c > 0$

Fig. 5.3. System (10): the type of the singular point $(0,0)$

b) *Singular points* $O_2(-1,0)$, $O_3(-1,-1)$ *and* $O_4(-1,0)$. These points have the eigenvalues respectively:

O_2: $\lambda^2 + (bc-1)\lambda - bc = 0$; $\lambda_1 = 1$; $\lambda_2 = -bc$;

O_3: $\lambda^2 - (1+(b+1)c)\lambda + (b+1)c = 0$; $\lambda_1 = 1$; $\lambda_2 = (b+1)c$;

O_4: $\lambda^2 - (1+(b+1)bc)\lambda + (b+1)bc = 0$; $\lambda_1 = 1$; $\lambda_2 = (b+1)bc$.

Each of the point O_2, O_3 and O_4 are hyperbolic and is not difficult to determine their types. □

Because the cubic nonlinearities of (9) and (10) coincide, these systems have the same singular points at the infinity: $X_{1_\infty}(1,0,0)$, $X_{2_\infty}(1,1,0)$, $X_{3_\infty}(1,-b,0)$, $Y_\infty(0,1,0)$. Moreover, for both systems the eigenvalues λ_1, λ_2 are the same, respectively, and their types are completely determined by the value of parameter c (see, Tab. 5.2).

The arguments outlined above are enough to be able to draw the phases portraits of the system (10)

(see, Fig. 1.3,1) if $c < 0$ and Fig. 1.3,2) if $c > 0$.)

5.4. System (11) (configuration (3(2),1,1,1)):

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(-bc - 2bcx + (b-1)cy + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b \in \mathbb{R}^*_+,\ c \in \mathbb{R}^*. \end{cases}$$

For the system (11) the straight lines: $l_{1,2} = x$, $l_3 = x+1$, $l_4 = y$, $l_5 = y - x - 1$ and $l_6 = y + b(x+1)$ are invariant.

Proposition 5.3. If $c < 0$ $(c > 0)$, then the system (11) has in the finite parte of the phase plane the following six (four) singular points:

1) $O_1(0,0)$, $O_2(0,1)$, $O_3(0,-b)$ – saddle-nodes;

2) $O_4(-1,0)$ – unstable node;

3) $O_{5,6}\left(-1, \pm\frac{1}{\sqrt{-c}}\right)$ – saddles.

Proof. a) *Singular point* $O_1(0,0)$. This point has the eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = -bc$. Therefore, O_1 is a semi-hyperbolic. Rescaling in (11) the time $\tau = -bct$ we obtain the system

$$\begin{cases} \dot{x} = -\frac{1}{bc}x^2(x+1) = P(x,y), \\ \dot{y} = y - \frac{b-1}{b}y^2 + 2xy + \frac{bc-1}{bc}x^2y + \frac{1-b}{b}xy^2 - \frac{1}{b}y^3 = y + Q(x,y). \end{cases}$$

The equation $\{y + Q(x,y) = 0,\ y(0) = 0\}$ has the solution $y = 0$. Putting $y = 0$ in $P(x,y)$ we have $\psi(x) = P(x,0) = -\frac{1}{bc}(x^2 + x^3)$. According to [1], the singular point $O_1(0,0)$ is of saddle-node type.

b) *Singular point* $O_2(0,1)$ has the eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = (b+1)c$, i.e. O_2 is semi-hyperbolic. At the beginning, via substitution $(x,y) \to (x, y-1)$ we translate O_2 in origin, then rescaling in (11) the time $\tau = (b+1)ct$, we obtain the system

$$\begin{cases} \dot{x} = \frac{1}{(b+1)c}x^2(x+1) = P(x,y), \\ \dot{y} = y + \frac{2}{(b+1)c}x^2(1-x) + 2xy + \frac{b+2}{b+1}y^2(x+1) + \frac{1+(b+1)c}{(b+1)c}x^2y + \frac{1}{b+1}y^3 = y + Q(x,y). \end{cases}$$

The solution $y = \varphi(x) = \sum_{i\geq 1} c_i x^i$ of the equation $y + Q(x,y) = 0$ has the form $\varphi(x) = -\frac{2}{(b+1)c}x^2 + \frac{2}{(b+1)c}x^3 + \cdots$. Putting $\varphi(x)$ in $P(x,\varphi(x))$ we come to the function $\psi(x) = \frac{1}{(b+1)c}(x^2 + x^3)$. Therefore, the singular point $O_2(0,1)$ is of saddle-node type (see, [1]).

c) *Singular point* $O_3(0,-b)$. Similarly as in b), for $O_3(0,-b)$ we get $\varphi = \frac{2}{(b+1)c}x^2 + \cdots$ and $\psi(x) = \frac{1}{(b+1)bc}(x^2 + x^3)$. Thus, O_3 is of saddle-node type $\big([1]\big)$.

d) *Singular points* $O_4(-1,0)$ *and* $O_{5,6}(0, \pm 1/\sqrt{-c})$. The eigenvalues of O_4 $(O_{5,6})$ are $\lambda_1 = \lambda_2 = 1$ ($\lambda_1 = -2$ and $\lambda_2 = 1$). Therefore, O_4 $(O_{5,6})$ is (are) unstable node (saddles). □

Because the systems (9) and (11) have the same cubic non-linearities, their singular points at the infinity coincide. The qualitative characteristics of these points are given in Tab. 5.2.

The investigations allowed us to draw the phase portraits of the system (11) (see, Fig. 1.4).

5.5. System (12) (configuration (3(3),1,1,1)):

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y\big((1-bc)x^2 + (b-1)cxy + cy^2\big), \\ c(bc-1)\big((b+1)c+1\big)\big((b+1)bc+1\big) \neq 0,\ b > 0,\ bc \in \mathbb{R}. \end{cases}$$

This system has the following invariant straight lines: $l_{1,2,3} = x$, $l_4 = y$, $l_5 = y - x$ and $l_6 = y + bx$.

Proposition 5.4. *If $c < 0$ $(c > 0)$ then in the finite parte of the phase plane the system* (12) *has only one singular point* $(0,0)$ *which is of the type* $P^u 2H2P^u 2HP^u$ *(unstable topological node) if $c < 0$ $(c > 0)$.*

Proof. Both eigenvalues of the singular point $O(0,0)$ are null. Therefore, $O(0,0)$ is nilpotent. We will study the behavior of the trajectories in a neighborhood of this point using blow-up method. In the polar coordinates $x = \rho cos\theta$, $y = \rho sin\theta$ the system (12) takes the form:

$$\begin{cases} \frac{d\rho}{d\tau} = \rho\big(\cos^4\theta + b\sin^4\theta + (1-bc)\sin^2\theta\cos^2\theta + c(b-1)\sin^3\theta\cos\theta\big), \\ \frac{d\theta}{d\tau} = c\sin\theta\cos\theta(\sin\theta - \cos\theta)(\sin\theta + b\cos\theta), \end{cases} \tag{23}$$

where $\tau = \rho^2 t$. Taking into account that the system (12) is symmetric with respect to the origin, it is sufficient to consider $\theta \in [0, \pi)$. The singular points of the system (23) with first coordinate $\rho = 0$ and the second $\theta \in [0, \pi)$, their eigenvalues and types respectively are:

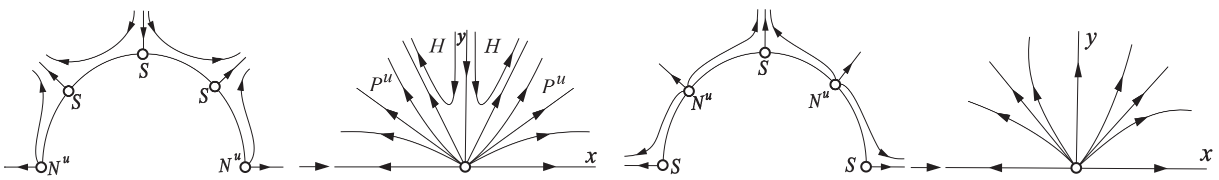
$M_1(0,0)$: $\{\lambda_1 = 1, \lambda_2 = -bc\}-$ unstable node, if $c < 0$, and saddle, if $c > 0$;

$M_2(0, \pi/2)$: $\{\lambda_{1,2} = \pm c\}-$ saddle;

$M_3(0, \pi/4)$: $\Big\{\lambda_1 = \dfrac{1}{2}, \lambda_2 = \dfrac{(b+1)c}{2}\Big\}-$ saddle, if $c < 0$, and unstable node, if $c > 0$;

$M_4(0, -\mathrm{arctg}\, b)$: $\Big\{\lambda_1 = \dfrac{1}{b^2+1}, \lambda_2 = \dfrac{(b+1)bc}{b^2+1}\Big\}-$ saddle, if $c < 0$, and unstable node, if $c > 0$.

We obtain Fig. 5.4a), if $c < 0$, and Fig. 5.4b), if $c > 0$. In the case $c < 0$ we have the following partition in sectors of the neighborhood of the origin: $P^u 2H2P^u 2HP^u$ and in the case $c > 0$ the neighborhood of the origin is an unstable topological node. □



a) $c < 0$ b) $c > 0$

Fig. 5.4. System (12): the type of the singular point $(0,0)$

The systems (9) and (12) have the same qualitative characteristic at the infinity.

The phase portraits of the system (12) are given in Fig. 1.5.

The results obtained in the Sections **3** − **5** prove the Theorem 1.1.

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