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CANONICAL FORMS OF CUBIC DIFFERENTIAL SYSTEMS WITH REAL INVARIANT STRAIGHT LINES OF TOTAL MULTIPLICITY SEVEN ALONG ONE DIRECTION

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Abstract. Consider the general cubic differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, where $P,Q \in \mathbf{R}[x,y]$, $\max\{\deg P, \deg Q\} = 3$, $GCD(P,Q) = 1$. If this system has enough invariant straight lines considered with their multiplicities, then, according to [1], we can construct a Darboux first integral. In this paper we obtain 26 canonical forms for cubic differential systems which possess real invariant straight lines along one direction of total multiplicity seven including the straight line at the infinity.
Keywords: cubic differential system, invariant straight line, Darboux integrability.
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FORMELE CANONICE ALE SISTEMELOR DIFERENȚIALE CUBICE CE POSEDĂ DREPTE INVARIANTE REALE DE-A LUNGUL UNEI DIRECȚII A CĂROR MULTIPLICITATE TOTALĂ ESTE EGALĂ CU ȘAPTE

Rezumat. Fie sistemul diferențial cubic general $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, unde $P,Q \in \mathbf{R}[x,y]$, $\max\{\deg P, \deg Q\} = 3$, $GCD(P,Q) = 1$. Conform [1], pentru un sistem diferențial cubic se poate de construit o integrală primă de tip Darboux, dacă sistemul dat posedă un număr suficient de drepte invariante considerate cu multiplicitățile lor. În această lucrare se obțin 26 sisteme ce reprezintă formele canonice ale sistemelor diferențiale cubice ce posedă drepte invariante reale de-a lungul unei direcții și a căror multiplicitate totală este egală cu șapte împreună cu dreapta de la infinit.
Cuvinte cheie: sistem diferențial cubic, dreaptă invariantă, integrabilitate Darboux.

1. Introduction

We consider the real polynomial system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases}, \qquad GCD(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{2}$$

associated to system (1). Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ ($n = 3$), then the system (1) is called a quadratic (cubic) system.

Definition 1. An algebraic curve $f(x,y) = 0$, $f \in \mathbf{C}[x,y]$, is called *invariant algebraic curve* for the system (1), if there exists a polynomial $K_f \in \mathbf{C}[x,y]$, such that the identity

$$\mathbb{X}(f) = f(x,y)K_f(x,y) \tag{3}$$

holds.

The invariant algebraic curves $f(x,y)=\alpha x+\beta y+\gamma$ of the order one of system (1) are called *invariant straight lines* of the system (1).

System (1) is called *Darboux integrable* if there exists a non-constant function of the form $F=f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdot\ldots\cdot f_s^{\lambda_s}$, where f_j is an invariant algebraic curve and $\lambda_j\in\mathbf{C}$, $j=\overline{1,s}$, such that F is a first integral or F is an integrating factor for (1). The function $F=f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdot\ldots\cdot f_s^{\lambda_s}$ is called a *Darboux first integral*. If a polynomial differential system has enough invariant straight lines (including their multiplicity), then, according to [1], a Darboux first integral can be constructed for this system.

In the theory of dynamic systems, the investigation of polynomial differential systems with invariant straight lines is done using different types of multiplicities of these invariant straight lines, for example: parallel multiplicity, geometric multiplicity; algebraic multiplicity; etc [2]. In this paper we will use the notion of algebraic multiplicity of an invariant straight line.

Definition 2. Let $\mathbf{C}_m[x]$ be the $\mathbf{C}$-vector space of polynomials in $\mathbf{C}[x]$ of degree at most m. Then it has dimension $R=C_{n+m}^n$. Let $v_1,v_2,\ldots,v_R$ be a base of $\mathbf{C}_m[x]$. If k is the greatest positive integer such that the k-th power of $f(x,y)$ divides $\det M_R$, where

$$M_R=\begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \ldots & \mathbb{X}(v_R) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \ldots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic curve f of degree m of the vector field $\mathbb{X}$ has algebraic multiplicity k.

In the above definition, the expression $\mathbb{X}^{R-1}(v_1)$ means that the operator $\mathbb{X}$ is applied $R-1$ times on vector v_1, i.e. $\mathbb{X}^{k+1}(v_i)=\mathbb{X}\left(\mathbb{X}^k(v_i)\right)$.

There are a great number of articles dedicated to the investigation of polynomial differential systems with invariant straight lines. In [3] the authors estimate the number of invariant straight lines that a polynomial differential system can have. The problem of coexistence of invariant straight lines and limit cycles has been studied in [4,5], and the problem of coexistence of invariant straight lines and singular points of the center type for cubic system has been studied in [6,7]. The classification of all cubic systems which have the maximum number of invariant straight lines including their multiplicities was performed in [8,9]. In [10] were studied the cubic systems with exactly eight invariant straight lines. The cubic systems with six real invariant straight lines along two and three directions were studied in [11,12].

In this paper we obtain all canonical forms of cubic differential systems with real invariants straight lines along one direction with their total multiplicity equal to seven including the multiplicity of the invariant straight line at the infinity.

2. The algebraic multiplicity of invariant straight lines

We will study the following cubic differential system

$$\begin{cases} \dot{x} = a_0 + P_1(x,y) + P_2(x,y) + P_3(x,y), \\ \dot{y} = b_0 + Q_1(x,y) + Q_2(x,y) + Q_3(x,y), \end{cases} \tag{4}$$

where $P_i(x,y), Q_i(x,y), i=\overline{1,3}$ are homogenous polynomials of degree i, and the coefficients are arbitrary parameters $P_i(x,y) = \sum_{j=0}^{i} a_{i-j,j} x^{i-j} y^j$, $Q_i(x,y) = \sum_{j=0}^{i} b_{i-j,j} x^{i-j} y^j$, $i=\overline{1,3}$.

When the system (4) has an invariant straight line of the form $\alpha x + \beta y + \gamma = 0$, we can bring this straight line to the form $\overline{x} = 0$ using the affine transformation $\overline{x} = \alpha x + \beta y + \gamma, \overline{y} = y$. It is obvious that the conditions for the existence of invariant straight line $x = 0$ for system (4) are simpler than the conditions for the existence of invariant straight line $\alpha x + \beta y + \gamma = 0$ for the same system.

Besides the existence of invariant straight lines, we are interested the invariant straight lines to have a certain algebraic multiplicity. According to Definition 2, for the cubic differential system with invariant straight lines we have $R = C_3^2 = 3$. As the basis of the vector space of polynomials $\mathbf{C}_1[x]$ we can choose $v_1 = 1, v_2 = x, v_3 = y$. Then the matrix M_R has the form

$$M_R = \begin{pmatrix} 1 & x & y \\ 0 & P(x,y) & Q(x,y) \\ 0 & \mathbb{X}(P) & \mathbb{X}(Q) \end{pmatrix}.$$

In this case, the polynomial $\det M_R$ looks $\det M_R = P\mathbb{X}(Q) - Q\mathbb{X}(P)$ and is a polynomial of degree 8 with respect to x and y. According to Definition 2, the straight line $x = 0$ is invariant if and only if the polynomial $\det M_R$ can be written as $\det M_R = x\sum_{i=0}^{7} A_i(y) x^i$, where $\deg\{A_i(y)\} = 7 - i$. Moreover, if the polynomials $A_0(y), A_1(y), ..., A_k(y)$, $k \in \overline{0,6}$, are identically zero, then the straight line $x = 0$ has the algebraic multiplicity $k+2$.

To study the multiplicity of an invariant straight line at infinity we carry out the Poincaré transformation $x = \frac{1}{\overline{x}}$, $y = \frac{\overline{y}}{\overline{x}}$. The multiplicity of an invariant straight line at infinity is equal to the multiplicity of the invariant straight line $\overline{x} = 0$ of the following system

$$\begin{cases} \dot{\bar{x}} = \overline{yx}^3 P\left(\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}}\right) - \bar{x}^3 Q\left(\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}}\right), \\ \dot{\bar{y}} = \bar{x}^4 P\left(\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}}\right). \end{cases}$$

3. Obtaining the canonical forms of cubic differential systems

We emphasize that the calculations used in determination of canonical forms are quite large, so we will show in detail only formulas used for obtaining a single canonical form, and the rest will be omitted. Let us note by $\left(d_1(m_1)+d_2(m_2)+d_3\right)$ a configuration of invariant straight lines, where d_i is the number of straight lines, and m_i is their corresponding multiplicity. If $m_i = 1$, then m_i is not written. For example, the notation (1(4)+2) indicates that there are three parallel invariant straight lines, where one of them has the multiplicity equal to four and the other two have the multiplicity equal to one. Depending on the multiplicity of the invariant straight lines at infinity, we divide the investigation into seven cases.

Case 1: The straight line at infinity has multiplicity equal to 1. The real invariant straight lines from the finite plane can have the following configurations: **a)** (1(6)); **b)** (1(4)+2); **c)** (1(4)+1(2)); **d)** (1(3)+1(3)); **e)** (1(3)+1(2)+1); **f)** (1(2)+1(2)+1(2)).

1.a) (1(6)) Conditioning the system (4) to have the invariant straight line $x = 0$ and applying Definition 1, we obtain the following conditions on the parameters of the system (4):

$$a_{00} = 0; a_{01} = 0; a_{02} = 0; a_{03} = 0.$$

According to Definition 2, the condition for the invariant straight line $x = 0$ to have algebraic multiplicity equal to two is equivalent with the condition $A_0(y) = 0$, i.e. the following system of equations hold

$$\begin{cases} b_{00}\left(a_{10} + a_{11}b_{00} - a_{10}b_{01}\right) = 0; \\ -2a_{10}a_{11}b_{00} - 2a_{12}b_{00}^2 - a_{10}^2 b_{01} - a_{11}b_{00}b_{01} + a_{10}b_{01}^2 + 2a_{10}b_{00}b_{02} = 0; \\ -a_{11}^2 b_{00} - 2a_{10}a_{12}b_{00} - 2a_{10}a_{11}b_{01} - 3a_{12}b_{00}b_{01} - a_{10}^2 b_{02} + 3a_{10}b_{01}b_{02} + 3a_{10}b_{00}b_{03} = 0; \\ -2a_{11}a_{12}b_{00} - a_{11}^2 b_{01} - 2a_{10}a_{12}b_{01} - a_{12}b_{01}^2 - 2a_{10}a_{11}b_{02} - 2a_{12}b_{00}b_{02} + a_{11}b_{01}b_{02} + 2a_{10}b_{02}^2 - a_{10}^2 b_{03} + a_{11}b_{00}b_{03} + 4a_{10}b_{01}b_{03} = 0; \\ -a_{12}^2 b_{00} - 2a_{11}a_{12}b_{01} - a_{11}^2 b_{02} - 2a_{10}a_{12}b_{02} - a_{12}b_{01}b_{02} + a_{11}b_{02}^2 - 2a_{10}a_{11}b_{03} - a_{12}b_{00}b_{03} + 2a_{11}b_{01}b_{03} + 5a_{10}b_{02}b_{03} = 0; \\ -a_{12}^2 b_{01} - 2a_{11}a_{12}b_{02} - a_{11}^2 b_{03} - 2a_{10}a_{12}b_{03} + 3a_{11}b_{02}b_{03} + 3a_{10}b_{03}^2 = 0; \\ \left(a_{12} - b_{03}\right)\left(a_{12}b_{02} + 2a_{11}b_{03}\right) = 0; \\ a_{12}b_{03}\left(a_{12} - b_{03}\right) = 0. \end{cases}$$

By solving this system, we get four sets of conditions, i.e. the system (4) with the invariant straight line $x = 0$ implies four cubic differential systems which have this invariant straight line of algebraic multiplicity equal to two.

By asking the invariant straight line $x = 0$ of the system (4) to have algebraic multiplicity equal to three, i.e. the condition $A_1(y) = 0$ must hold for each one of those four systems, we obtain eight differential systems.

By asking the invariant straight line $x=0$ of the system (4) to have algebraic multiplicity equal to four, i.e. the condition $A_2(y)=0$ to be realized for each one of those eight systems, we obtain 11 cubic differential systems.

By asking the invariant straight line $x=0$ of the system (4) to have algebraic multiplicity equal to five, i.e. the condition $A_3(y)=0$ must hold for each one of those 11 systems, we obtain two differential systems.

Finally, by conditioning the invariant straight line $x=0$ to have multiplicity equal to six for the system (4), we obtain one set of conditions, i.e. the system satisfying these conditions has the form

$$\begin{cases} \dot{x} = a_{30}x^3, b_{00} \neq 0, \\ \dot{y} = b_{00} + b_{10}x + b_{20}x^2 + b_{30}x^3 + 3a_{30}x^2 y. \end{cases}$$

Carrying out the transformations $x \to x, y \to \dfrac{-b_{30}x + 2b_{20}y}{2a_{30}b_{00}b_{20}} - \dfrac{b_{20}}{3a_{30}}$, $t = \dfrac{\tau}{a_{30}}$ and using the notation $b_{10} = ab_{00}$, we obtain the canonical form of the cubic differential system with invariant straight lines of total algebraic multiplicity equal to seven including the straight line at infinity:

$$\begin{cases} \dot{x} = x^3, \mathrm{a} \geq 0, \\ \dot{y} = 1 + ax + 3x^2 y, \end{cases} \qquad \textbf{(s1)}$$

where $a \geq 0$, as the transformation $x \to -x, a \to -a$ doesn't change the system **(s1)**.

1.b) (1(4)+2) There are 11 systems with the invariant straight line $x=0$ of total multiplicity equal to 4, but only one of them can have the invariant straight lines $x+1=0$ and $x-a=0$. This system can be brought to the form:

$$\begin{cases} \dot{x} = x(x+1)(x-a), a \geq 1, \\ \dot{y} = -ay + (1-a)xy + x^3 + x^2 y. \end{cases} \qquad \textbf{(s2)}$$

1.c) (1(4)+1(2)) We have established that 11 systems have the invariant straight line $x=0$ with total multiplicity equal to 4. Asking that the straight line $x+1=0$ to be invariant with multiplicity equal to 2, we obtain only one system, which can be brought to the form:

$$\begin{cases} \dot{x} = x^2(x+1), a \in \mathbf{R}, \\ \dot{y} = 1 + ax^2 + 2xy + 3x^2 y. \end{cases} \qquad \textbf{(s3)}$$

1.d) (1(3)+1(3)) There are 8 systems that have the invariant straight line $x=0$ with algebraic multiplicity equal to 3. Asking that the straight line $x+1=0$ to be invariant with multiplicity equal to 3, we obtain only one system, which can be brought to the form:

$$\begin{cases} \dot{x} = x(x+1)^2, a \in \mathbf{R} \setminus \{1\}, \\ \dot{y} = y + ax^2 + 2xy + x^3 + x^2 y. \end{cases} \qquad \textbf{(s4)}$$

1.e) (1(3)+1(2)+1) There are 8 systems that have the invariant straight line $x=0$ with multiplicity equal to 3. Asking that the straight line $x+1=0$ to be invariant with multiplicity equal to 2 and the straight line $x-a=0$ to be invariant, we obtain only one system, which can be brought to the form:

$$\begin{cases} \dot{x}=x(x+1)(x-a), a\in\mathbf{R}\setminus\{-1;0\}, \\ \dot{y}=-ay+(a+1)x^2+(1-a)xy+(a+1)x^2y. \end{cases} \quad \textbf{(s5)}$$

1.f) (1(2)+1(2)+1(2)) There are 4 systems that have the invariant straight line $x=0$ with multiplicity equal to 2. Asking that the straight lines $x+1=0$ and $x-a=0$ to be invariant with multiplicity equal to 2, we obtain only one differential system that can be brought to the form:

$$\begin{cases} \dot{x}=x(x+1)(x-a), a>1, b\in\mathbf{R}, \\ \dot{y}=bx-ay+x^2+2(1-a)xy+3x^2y. \end{cases} \quad \textbf{(s6)}$$

Case 2: The straight line at infinity has multiplicity equal to 2. Asking that the invariant straight line at infinity of the system (4) to have multiplicity equal to two, we obtain 5 sets of conditions, i.e. there are 5 cubic differential systems that satisfy this condition. For the total algebraic multiplicity to be equal to 7, we must search for real planar invariant straight lines with total multiplicity equal to 5. The planar invariant straight lines can have the following configurations: **a)** (1(5)); **b)** (1(4)+1); **c)** (1(3)+1(2)); **d)** (1(3)+1+1); **e)** (1(2)+1(2)+1).

2.a) (1(5)) For the five differential systems which have the invariant straight line at infinity with multiplicity equal to two we require that the invariant straight line $x=0$ to be invariant with multiplicity five. As a result, we obtain the following system:

$$\begin{cases} \dot{x}=x^3, \mathrm{a}\neq 0, \\ \dot{y}=a+x-x^2. \end{cases} \quad \textbf{(s7)}$$

2.b) (1(4)+1) In this case we obtain two differential systems:

$$\begin{cases} \dot{x}=x(x+1), \\ \dot{y}=y+xy+x^3. \end{cases} \quad \textbf{(s8)} \qquad \begin{cases} \dot{x}=x^2(x+1), \mathrm{a}\neq 0, \\ \dot{y}=a+x^2+2xy. \end{cases} \quad \textbf{(s9)}$$

2.c) (1(3)+1(2)) In this case we obtain three differential systems:

$$\begin{cases} \dot{x}=x(x+1), \\ \dot{y}=y+ax^2+xy-x^2y. \end{cases} \textbf{(s10)} \quad \begin{cases} \dot{x}=x^2(x+1), \mathrm{a}\in\mathbf{R}, \\ \dot{y}=1+ax^2-xy. \end{cases} \textbf{(s11)} \quad \begin{cases} \dot{x}=x(x+1)^2, \mathrm{a}\in\mathbf{R}, \\ \dot{y}=y+ax^2+2xy. \end{cases} \textbf{(s12)}$$

2.d) (1(3)+1+1) This configuration corresponds to the system

$$\begin{cases} \dot{x}=x(x+1)(a-x), |a|>1, \\ \dot{y}=ay+x^2+(a-1)xy. \end{cases} \quad \textbf{(s13)}$$

2.e) (1(2)+1(2)+1) In this case we obtain the system

$$\begin{cases} \dot{x}=x(x+1)(a-x), \mathrm{a}\in(-1;+\infty)\setminus\{0\}, b\in\mathbf{R}, \\ \dot{y}=bx+ay+x^2+(1+2a)xy. \end{cases} \quad \textbf{(s14)}$$

Case 3: The straight line at the infinity has multiplicity equal to 3. There are 10 cubic differential systems that have the invariant straight line at infinity with the multiplicity equal to 3. It follows that the real invariant straight lines must have total algebraic multiplicity equal with four, therefore we can have the following configurations:

a) (1(4)); **b)** (1(3)+1); **c)** (1(2)+1(2)); **d)** (1(2)+1+1).

3.a) (1(4)) Asking the straight line $x=0$ to be invariant with algebraic multiplicity equal to four, we establish that only one of those 10 systems satisfies this condition and it can be brought to the form

$$\begin{cases} \dot{x} = x^2, \mathrm{a} \neq 0, \\ \dot{y} = a + x^2 + 2xy + x^3. \end{cases} \quad \textbf{(s15)}$$

3.b) (1(3)+1) In this case we obtain the system

$$\begin{cases} \dot{x} = x(x+1), \mathrm{a} \in \mathbf{R}, \\ \dot{y} = y + ax^2 + xy + x^3. \end{cases} \quad \textbf{(s16)}$$

3.c) (1(2)+1(2)) By solving the remaining system of algebraic equations, we will obtain several sets of condition. By performing affine transformations and time rescaling, we can bring the obtained systems to one of the following two canonical forms:

$$\begin{cases} \dot{x} = x(x+1), \mathrm{a} \in \mathbf{R}, \\ \dot{y} = ax + y + 2xy + x^3. \end{cases} \quad \textbf{(s17)} \qquad \begin{cases} \dot{x} = x(x+1)^2, \\ \dot{y} = x + y. \end{cases} \quad \textbf{(s18)}$$

3.d) (1(2)+1+1) In this case we obtain the following canonical form:

$$\begin{cases} \dot{x} = x(x+1)(a-x), \\ \dot{y} = x + ay, a > 1. \end{cases} \quad \textbf{(s19)}$$

Case 4: The straight line at infinity has multiplicity equal to 4. By asking the invariant straight line at infinity of the system (4) to have multiplicity equal to four, we obtain 13 cubic differential systems. Therefore, the real planar invariant straight lines must have total multiplicity equal to three, so they can have the following configurations:

a) (1(3)); **b)** (1(2)+1); **c)** (1+1+1).

4.a) (1(3)) By asking the straight line $x=0$ to be invariant with multiplicity equal to three, we obtain the following two systems:

$$\begin{cases} \dot{x} = x^2, \mathrm{a} > 0, \\ \dot{y} = a - xy + x^3. \end{cases} \quad \textbf{(s20)} \qquad \begin{cases} \dot{x} = x, \\ \dot{y} = y + x^2 + x^3. \end{cases} \quad \textbf{(s21)}$$

4.b) (1(2)+1) If the straight line $x=0$ is invariant with multiplicity equal to two and the straight line $x+1=0$ is invariant, then only one system from those 13 systems satisfies these conditions, and he can be brought it to the following form:

$$\begin{cases} \dot{x} = x(x+1), \mathrm{a} \in \mathbf{R}, \\ \dot{y} = ax + y - xy + x^3. \end{cases} \quad \textbf{(s22)}$$

4.c) (1+1+1) In this case, for each one of these 13 systems we determine three invariant straight lines of the form $x=0$, $x+1=0$ and $x-a=0, a>1$. Only one system satisfy these conditions and it can be brought to the form:

$$\begin{cases} \dot{x} = x(x+1)(a-x), \\ \dot{y} = 1, a>1. \end{cases} \qquad \textbf{(s23)}$$

Case 5: The straight line at infinity has multiplicity equal to 5. In this case we have seven cubic differential systems. It follows that the real planar invariant straight lines must have total algebraic multiplicity equal to two, therefore they can have one of the following two configurations: **a)** (1(2)); **b)** (1+1).

5.a) (1(2)) For these 7 systems, we obtain that only one system can be brought to the canonical form:

$$\begin{cases} \dot{x} = x, \mathrm{a} \in \mathbf{R}, \\ \dot{y} = ax + y + x^2 + x^3. \end{cases} \qquad \textbf{(s24)}$$

5.b) (1+1) By asking the straight lines $x=0$ and $x+1=0$ to be invariant for the cubic differential systems with an invariant straight line at infinity which have algebraic multiplicity equal to five, we obtain that there are no parameter values satisfying these conditions. Therefore, there are no cubic differential systems of such configuration.

Case 6: The straight line at infinity has multiplicity equal to 6. By asking the invariant straight line at infinity of the system (4) to have multiplicity equal to six, we obtain three cubic differential systems. We can have only one real planar invariant straight line. Therefore, for each of these systems we condition the straight line $x=0$ to be invariant. Thus, we obtain a single system, which can be brought to the following form:

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -2y + x^2 + x^3. \end{cases} \qquad \textbf{(s25)}$$

Case 7: The straight line at infinity has multiplicity equal to 7. By asking the invariant straight line at infinity of the system (4) to have multiplicity equal to seven, we get the system:

$$\begin{cases} \dot{x} = 1, \mathrm{a} \in \mathbf{R}, \\ \dot{y} = x(a + x^2). \end{cases} \qquad \textbf{(s26)}$$

According to the above obtained results, we have proved the following theorem:

Theorem. Any cubic differential system with real invariant straight lines along one direction with total algebraic multiplicity equal to seven, including the invariant straight line at the infinity, by an affine transformation and time rescaling can be brought to one of the systems **(s1)** – (**s26)**.

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