MAXIMAL MULTIPLICITY OF THE LINE AT INFINITY FOR QUARTIC DIFFERENTIAL SYSTEMS

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Abstract. In this work we show that in the class of quartic differential systems the maximal algebraic multiplicity of the line at infinity is 10.

Keywords: quartic differential system, invariant straight line, algebraic multiplicity.

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MULTIPLICITATEA MAXIMALĂ A LINIEI DE LA INFINIT PENTRU SISTEMELE DIFERENŢIALE DE GRADUL PATRU

Rezumat. În această lucrare se arată că în clasa sistemelor diferenţiale de gradul patru multiplicitatea algebrică maximală a liniei de la infinit este egală cu 10.

Cuvinte-cheie: sistem diferenţial de gradul patru, dreaptă invariantă, multiplicitate algebrică.

1. Introduction and the statement of main result

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y). \tag{1}$$

Denote $n = max\{deg(P), \deg(Q)\}$. If $n = 4$ then system (1) is called quartic.

At present, a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines. The problem of the estimation of the number of invariant straight lines which can have a polynomial differential system was considered in [1].

In [2] it is given the estimation $3n-2 \leq M_a(n) \leq 3n-1$ of maximal algebraic multiplicity $M_a(n)$ of an invariant straight line for the class of two-dimensional polynomial differential systems of degree $n \geq 2$ and it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven.

In this paper we show that in the class of quartic differential systems the maximal algebraic multiplicity of the line at infinity is equal to 10.

Theorem. *For quartic differential systems the algebraic multiplicity of the line at infinity is at most ten. Any quartic system having the line at infinite of multiplicity 10 via affine transformations and time rescaling can be written in the form*

$$\dot{x} = -x, \ \dot{y} = x^4 + 3y. \tag{2}$$

2. The proof of the Theorem

We consider the real quartic system of differential equations

$$\dot{x} = \sum_{j=0}^{4} p_j(x,y) \equiv p(x,y), \quad \dot{y} = \sum_{j=0}^{4} q_j(x,y) \equiv q(x,y), \tag{3}$$

where $p_{0=}a_0$, $p_1(x,y)=a_1x+a_2y$, $p_2(x,y)=a_3x^2+a_4xy+a_5y^2$, $p_3(x,y)=a_6x^3+a_7x^2y+a_8xy^2+a_9y^3$, $p_4(x,y)=a_{10}x^4+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}y^4$, $q_{0=}b_0$ $q_1(x,y)=b_1x+b_2y$, $q_2(x,y)=b_3x^2+b_4xy+b_5y^2$, $q_3(x,y)=b_6x^3+b_7x^2y+b_8xy^2+b_9y^3$, $q_4(x,y)=b_{10}x^4+b_{11}x^3y+b_{12}x^2y^2+b_{13}xy^3+b_{14}y^4$.

Suppose that the right-hand sides of (3) do not have the common divisors of degree greatest than 0, i.e.

$$gcd(p,q)=1 \text{ and } yp_4(x,y)-xq_4(x,y)\not\equiv 0, \tag{4}$$

i.e. at infinity the system (3) has at most five distinct singular points.

The homogeneous system associated to the system (3) has the form

$$\dot{x}=\sum_{j=0}^{4}p_j(x,y)Z^{4-j}\equiv P(x,y,Z),\quad \dot{y}=\sum_{j=0}^{4}q_j(x,y)Z^{4-j}\equiv Q(x,y,Z). \tag{5}$$

Denote $\mathbb{X}=P(x,y,Z)\frac{\partial}{\partial x}+Q\,(x,y,Z)\frac{\partial}{\partial y}$.

We say that the line at infinity $Z=0$ has *algebraic multiplicity* $m+1$ if m is the greatest positive integer such that Z^m divides $\mathbb{E}_\infty=P\cdot\mathbb{X}(Q)-Q\cdot\mathbb{X}(P)$ (see [3]).

In this section, for quartic system (3) we determine the maximal algebraic multiplicity of the line at infinity $Z=0$.

Because $p_4^2(x,y)+q_4^2(x,y)$ is not identically zero, by a centro-affine transformation and time rescaling we can make $b_{10}\neq 0$, and more that, $b_{10}=1$.

For the homogenized system (4) we calculate the determinant $\mathbb{E}_\infty$ from the definition of the algebraic multiplicity. $\mathbb{E}_\infty$ is a polynomial of degree 11 in x,y,Z. We write it in the form:

$$\begin{aligned}\mathbb{E}_\infty&=A_0(x,y)+A_1(x,y)Z+A_2(x,y)Z^2+A_3(x,y)Z^3+\\&+A_4(x,y)Z^4+A_5(x,y)Z^5+A_6(x,y)Z^6+A_7(x,y)Z^7+\\&+A_8(x,y)Z^8+A_9(x,y)Z^9+A_{10}(x,y)Z^{10}+A_{11}(x,y)Z^{11}\end{aligned} \tag{6}$$

where $A_i(x,y)$, $i=0,\ldots,11$, are polynomials in x and y.

The algebraic multiplicity of the line at infinity is $m_\infty\in N^*$ if m_∞ is the maximal number such that $Z^{m_\infty-1}$ divides $\mathbb{E}_\infty$.

The algebraic multiplicity m_∞ of the line at infinity is at least two if the identity $A_0(x,y)\equiv 0$ holds.

The polynomial $A_0(x,y)$ looks as: $A_0(x,y)=A_{01}(x,y)A_{02}(x,y)$ where

$$\begin{aligned}A_{01}(x,y)&=-x^5+(a_{10}-b_{11})\,x^4y+(a_{11}-b_{12})\,x^3y^2+(a_{12}-\,b_{13})\,x^2y^3+\\&+(a_{13}-b_{14})x\,y^4+a_{14}y^5,\end{aligned}$$

$$\begin{aligned}A_{02}(x,y)&=(a_{11}-a_{10}b_{11})x^6+2(a_{12}-a_{10}b_{12})x^5y\;+(3a_{13}+a_{12}b_{11}-a_{11}b_{12}-\\&-3\,a_{10}b_{13})\,x^4y^2+2(2\,a_{14}+\,a_{13}b_{11}-\,a_{11}b_{13}-2\,a_{10}b_{14})\,x^3y^3+\\&+(3\,a_{14}b_{11}+a_{13}\cdot b_{12}-a_{12}b_{13}-3\,a_{11}b_{14})x^2y^4+\\&+2\,(a_{14}b_{12}-a_{12}b_{14})\,xy^5\;+(a_{14}b_{13}\,-\,a_{13}b_{14})y^6.\end{aligned}$$

As $A_{01}(x,y)\not\equiv\;0$ (see (4)), we require $A_{02}(x,y)$ to be identically equal to zero. The identity $A_{02}(x,y)\equiv 0$ holds if the following conditions

$a_{11} = a_{10}\, b_{11},\ a_{12} = a_{10}\, b_{12},\ a_{13} = a_{10}\, b_{13},\ a_{14} = a_{10}\, b_{14}$ are satisfied.

The algebraic multiplicity m_∞ of the line at infinity is at least three if $A_1(x,y) \equiv 0$.

Unde the above conditions we have $A_1(x,y) = -A_{11}(x,y)A_{12}(x,y)$, where

$$A_{11}(x,y) = x^4 + b_{11}\, x^3\, y + b_{12}\, x^2\, y^2 + b_{13}\, x\, y^3 + b_{14}\, y^4 \not\equiv\ 0,$$

$$\begin{gathered}A_{12}\,(x,y) = (a_7 - a_{10}a_6\ - a_6 b_{11} + a_{10}^2\, b_6 + a_{10}b_{11}b_6 - a_{10}b_7)x^6 + \\ +(2a_8 - 2a_{10}\cdot a_7 - 2\, a_6\, b_{12}\ + 2\, a_{10}\, b_{12}\, b_6 + 2\, a_{10}^2 b_7 - 2\, a_{10}b_8)x^5 y + \\ +(3a_9 - 3a_{10}a_8 - a_{10}a_7 b_{11} + a_8\cdot b_{11} + a_{10}\, a_6 b_{12} - a_7\, b_{12} - 3\, a_6 b_{13} - \\ -a_{10}^2 b_{12} b_6 + 3\, a_{10} b_{13}\, b_6 + a_{10}^2\, b_{11} b_7\ + a_{10} b_{12} b_7\ + 3\, a_{10}^2 b_8\ - \\ -a_{10}\, b_{11} b_8 - 3\, a_{10} b_9)x^4 y^2 + \\ +(-4\, a_{10}\, a_9\ -\ 2\, a_{10} a_8 b_{11}\ + 2\, a_9\, b_{11}\ + 2\, a_{10}\, a_6\cdot\ b_{13}\ -\ 2\, a_7\, b_{13} - \\ -4\, a_6\, b_{14}\ - 2\, a_{10}^2\, b_{13} b_6\ +\ 4\, a_{10} b_{14} b_6\ + 2\, a_{10}\, b_{13}\, b_7\ \ + 2\, a_{10}^2\, b_{11} b_8 + \\ +4\, a_{10}^2\, b_9 - 2\, a_{10}\, b_{11}\, b_9)\, x^3\, y^3 + \\ +(-\ 3\, a_{10} a_9 b_{11} - a_{10} a_8 b_{12} +\ a_9 b_{12} + a_{10} a_7 b_{13} - a_8 b_{13} +\ 3\, a_{10} a_6 b_{14} - \\ -3\, a_7 b_{14} - 3\, a_{10}^2 b_{14} b_6 - a_{10}^2\, b_{13}\, b_7 + 3\, a_{10}\, b_{14}\, b_7 + a_{10}^2 b_{12}\, b_8\ + \\ +a_{10} b_{13} b_8 +\ 3\, a_{10}^2 b_{11} b_9 - a_{10} b_{12} b_9)\, x^2\, y^4 + \\ +(-2\, a_{10}\, a_9\, b_{12} + 2\, a_{10}\, a_7\, b_{14} - 2\, a_8\, b_{14} - 2\, a_{10}^2\, b_{14}\, b_7\ +\ 2a_{10} b_{14} b_8 + \\ +2\, a_{10}^2 b_{12} b_9)x\, y^5 + \\ +(-a_{10} a_9\, b_{13} + a_{10} a_8 b_{14} - a_9 b_{14} - a_{10}^2 b_{14} b_8 + a_{10}^2 b_{13} b_9 +\ a_{10}\, b_{14}\, b_9)\, y^6.\end{gathered}$$

If $A_{12}\,(x,y) \equiv 0$ then we obtain the following two series of conditions:

1) $a_6 =\ a_{10} b_6,\ a_7 =\ a_{10}\, b_7,\ a_8 = a_{10}\, b_8,\ a_9 = a_{10}\, b_9;$

2) $a_7\ =\ a_{10}\, b_7\ -\ a_{10}\alpha -\ b_{11}\alpha,\ \ a_8 = a_{10} b_8\ -\ a_{10}^2\, \alpha\ -\ a_{10}\, b_{11}\alpha\ -$
$b_{12}\, \alpha,\ \ \ a_9 =\ a_{10}\, b_9\ -\ a_{10}^3\alpha -\ a_{10}^2\, b_{11}\, \alpha - a_{10}\, b_{12}\, \alpha\ -\ \ b_{13}\, \alpha,\ b_{14} =$
$-a_{10}\,(a_{10}^3\ + a_{10}^2\, b_{11} +\ a_{10} b_{12}\ +\ b_{13}),\ \ \alpha = a_{10}\, b_6\ - a_6,\ \alpha\ \neq\ 0.$

In the conditions 1) we have $A_2(x,y) = -A_{11}(x,y)A_{21}(x,y)$, where

$$\begin{gathered}A_{21}(x,y) = (a_4 - 2a_{10}a_3 - a_3\, b_{11} + 2a_{10}^2\, b_3 + a_{10}b_{11}b_3\ - a_{10}b_4\,)x^5 + \\ +(2a_5 - 3a_{10}a_4 - a_{10}\, a_3\, b_{11} -\ 2\, a_3 b_{12} +\ a_{10}^2\, b_{11}\, b_3 + 2\, a_{10}\, b_{12}\, b_3 + \\ +3\, a_{10}^2 b_4 - 2\, a_{10} b_5)\, x^4\, y - (4\, a_{10} a_5 +\ 2a_{10}\, a_4\, b_{11}\ -\ a_5\, b_{11}\ +\ a_4\, b_{12} + \\ +3\, a_3\, b_{13}\ -\ 3\, a_{10}\, b_{13}\, b_3\ - 2\, a_{10}^2\, b_{11}\, b_4 -\ a_{10}\, b_{12}\, b_4 -\ 4\, a_{10}^2 b_5 + \\ +a_{10} b_{11} b_5)\ \ x^3\, y^2 - (3a_{10} a_5 b_{11} + a_{10} a_4 b_{12} -\ a_{10} a_3 b_{13} +\ 2a_4 b_{13} +\ 4\, a_3 b_{14} + \\ +a_{10}^2\, b_{13} b_3 -\ 4\, a_{10}\, b_{14}\, b_3 - a_{10}^2\, b_{12}\, b_4 - 2\, a_{10}\, b_{13}\, b_4 - 3\, a_{10}^2\, b_{11} b_5)\, x^2 y^3 - \\ -(2\, a_{10} a_5 b_{12} + a_5 b_{13} - 2\, a_{10} a_3 b_{14} + 3\, a_4 b_{14} + 2\, a_{10}^2 b_{14} b_3 - 3a_{10} b_{14} b_4 - \\ -2a_{10}^2 b_{12} b_5 -\ a_{10} b_{13} b_5)\, x y^4 -\ (a_{10}\, a_5 b_{13}\ - a_{10}\, a_4\, b_{14} + 2\, a_5\, b_{14} + a_{10}^2\, b_{14}\, b_4 - \\ -a_{10}^2\, b_{13}\, b_5\ -\ 2\, a_{10} b_{14}\, b_5)\, y^5\,.\end{gathered}$$

If the identity $A_{21}(x,y) \equiv 0$ holds, then the multiplicity m_∞ is at least four. The identity $A_{21}(x,y) \equiv 0$ leads us to the following two series of conditions:

1.1) $a_3 = a_{10}\, b_3,\quad a_4 = a_{10}\, b_4\,,\quad a_5 = a_{10}\, b_5;$

1.2) $a_4 = a_{10}\, b_4 + 2\, a_{10}\beta + b_{11}\beta,\ a_5 = a_{10}\, b_5 + 3\, a_{10}^2\beta + 2\, a_{10}b_{11}\beta + b_{12}\beta,\quad b_{13} = -a_{10}\,(4a_{10}^2 + 3\, a_{10}\, b_{11} + 2b_{12}),\quad b_{14} = a_{10}^2\,(3\, a_{10}^2 + 2\, a_{10}\, b_{11} + b_{12}),\ \ \beta = a_3 - a_{10}b_3,\ \beta \neq 0;$

In the conditions 1.1) we have $A_3(x,y) = A_{11}(x,y)A_{31}(x,y)$, where

$$A_{31}(x,y) = (3\, a_1\, a_{10} - a_2 - 3a_{10}^2\, b_1 + a_1\, b_{11} - a_{10}\, b_1\, b_{11} + a_{10}b_2)\, x^4 + \\ +(4\, a_{10}\, a_2 + 2\, a_1\, a_{10}\, b_{11} - 2a_{10}^2 b_1\, b_{11} + 2\, a_1\, b_{12} - 2\, a_{10}\, b_1 b_{12} - 4\, a_{10}^2\, b_2)x^3 y + \\ +(3\, a_{10}\, a_2\, b_{11} + a_1 a_{10} b_{12} + a_2 b_{12} - a_{10}^2\, b_1\, b_{12} + 3\, a_1 b_{13} - 3\, a_{10}\, b_1\, b_{13} - \\ -3a_{10}^2\, b_{11}\, b_2 - a_{10}\, b_{12}\, b_2)\cdot x^2 y^2 + (2a_{10}a_2 b_{12} + 2a_2 b_{13} + 4a_1 b_{14} - 4a_{10} b_1 b_{14} - \\ -2a_{10}^2 b_{12} b_2 - 2a_{10} b_{13} b_2)\, x\, y^3 + (\, a_{10}\, a_2\, b_{13} - a_1\, a_{10}\, b_{14} + 3\, a_2\, b_{14} + a_{10}^2\, b_1\, b_{14} - \\ -a_{10}^2\, b_{13}\, b_2 - 3\, a_{10}\, b_{14}\, b_2)\, y^4.$$

The identity $A_{31}(x,y) \equiv 0$ holds if one of the following two sets of conditions is satisfied:

1.1.1) $a_1 = a_{10}\, b_1,\ a_2 = a_{10}\, b_2;$

1.1.2) $a_2 = a_{10}\, b_2 + 3\, a_{10}\gamma + b_{11}\gamma,\ \ b_{12} = -3\, a_{10}\,(2\, a_{10} + b_{11}),\ \ b_{13} = a_{10}^2\,(8\, a_{10} + 3\, b_{11}),\ \ b_{14} = -a_{10}^3\,(3\, a_{10} + b_{11}),\ \gamma = a_1 - a_{10}\, b_1,\ \gamma \neq 0.$

If one of the conditions 1.1.1) or 1.1.2) is satisfied, then the multiplicity $m_\infty \geq 5$.

In the conditions 1.1.1) we have $A_4(x,y) = \delta A_{11}(x,y)\cdot A_{41}(x,y)$, where $\delta = a_0 - a_{10}\, b_0$ and $A_{41}(x,y) = 4\, a_{10}\, x^3 + b_{11}\, x^3 + 3\, a_{10}\, b_{11}\, x^2\, y + 2b_{12}\, x^2\, y + 2\, a_{10}\, b_{12}\, x\, y^2 + 3\, b_{13}\, x\, y^2 + a_{10}b_{13}\, y^3 + 4\, b_{14}\, y^3.$

If $\delta = 0$, then $deg(gcd(P,Q)) > 0$ (see (4)). Let $\delta \neq 0$ and $A_{41}(x,y) \equiv 0 \Rightarrow b_{11} = -4\, a_{10},\ \ b_{12} = 6a_{10}^2,\ b_{13} = -4\, a_{10}^3,\ \ b_{14} = a_{10}^4,$ then $A_5(x,y) = \delta A_{11}(x,y)\cdot A_{51}(x,y)$, where $A_{51}(x,y) = 3\, a_{10}b_6\, x^2 + b_7 x^2 + 2a_{10}b_7\, xy + 2b_8\, x\, y + a_{10}b_8\, y^2 + 3b_9 y^2.$

The identity $A_{51}(x,y) \equiv 0$ holds if $b_7 = -3\, a_{10}\, b_6,\ b_8 = 3\, a_{10}^2\, b_6,\ b_9 = -a_{10}^3\, b_6.$ In these conditions $A_6(x,y) = \delta\, A_{11}(x,y)(2\, a_{10}b_3\, x + b_4\, x + a_{10}b_4\, y + 2\, b_5\, y) \equiv 0 \ \Rightarrow\ b_4 = -2\, a_{10}b_3,\ b_5 = a_{10}^2 b_3 \ \Rightarrow A_7(x,y) = \delta(a_{10}b_1 + b_2)\cdot A_{11}(x,y) \equiv 0 \Rightarrow b_2 = -a_{10}b_1 \ \Rightarrow A_8(x,y) = 4\,\delta^2(x - a_{10}y)^3 \not\equiv 0.$

Thus, we have obtain $\mathbb{E}_\infty = Z^8(4\, x^3 - 12a_{10}\, x^2 y + 12\, a_{10}^2\, xy^2 - 4\, a_{10}^3\, y^3 + 3\, b_6\, x^2 Z - 6\, a_{10}b_6\, xyZ + 3\, a_{10}^2 b_6\, y^2 Z + 2\, b_3\, xZ^2 + 2\, a_{10}b_3\, yZ^2 + b_1 Z^3)\delta^2$ and the algebraic multiplicity $m_\infty = 9$.

The quartic system {(3), (4)} takes the form:

$$\dot{x} = a_{10}x^4 - 4\, a_{10}^2\, x^3 y + 6\, a_{10}^3\, x^2\, y^2 - 4a_{10}^4\, xy^3 + a_{10}^5\, y^4 + a_{10}b_6\, x^3 - 3\, a_{10}^2 b_6\cdot x^2 y \\ + 3\, a_{10}^3 b_6\, xy^2 - a_{10}^4 b_6 y^3 + a_{10}b_3\, x^2 - 2\, a_{10}^2 b_3\, xy + a_{10}^3 b_3\, y^2 + \\ +a_{10}b_1 x - a_{10}^2 b_1 y + a_{10}b_0 + \delta, \qquad (7)$$

$$\dot{y} = x^4 - 4a_{10}x^3 y + 6\, a_{10}^2\, x^2 y^2 - 4\, a_{10}^3\, x\, y^3 + a_{10}^4 y^4 + b_6 x^3 - 3\, a_{10}b_6 x^2\, y + \\ +3\, a_{10}^2\, b_6\, x\, y^2 - a_{10}^3\, b_6\, y^3 + b_3\, x^2 - 2\, a_{10}\, b_3\, x\, y + a_{10}^2\, b_3\, y^2 + b_1\, x - a_{10}b_1\, y + b_0.$$

In the conditions 1.1.2) we have $A_4(x,y) = A_{11}(x,y)\cdot A_{41}(x,y)$, where

$A_{41}(x,y) = -4a_0\, a_{10}\, x^3 + 4\, a_{10}^2\, b_0\, x^3 - a_0\, b_{11}\, x^3 + a_{10}\, b_0\, b_{11}\, x^3 + 12\, a_0\, a_{10}^2\, x^2\, y -$

$$-12\, a_{10}^3\, b_0\, x^2\, y + 3\, a_0 a_{10}\, b_{11}\, x^2\, y - 3\, a_{10}^2 b_0 b_{11} x^2\, y - 12\, a_0\, a_{10}^3\, x\, y^2 +$$
$$+12\, a_{10}^4\, b_0\, x y^2 - 3\, a_0\, a_{10}^2\, b_{11}\, x\, y^2 + 3\, a_{10}^3\, b_0 b_{11}\, x\, y^2 + 4\, a_0 a_{10}^4\, y^3 - 4\, a_{10}^5 b_0\, y^3 +$$
$$+a_0 a_{10}^3 b_{11}\, y^3 - a_{10}^4 b_0\, b_{11}\, y^3 + a_{10} b_6\, x^3\, \gamma + b_{11} b_6\, x^3\, \gamma - b_7\, x^3\, \gamma - 9\, a_{10}^2\, b_6\, x^2 y\, \gamma -$$
$$-3\, a_{10} b_{11} b_6\, x^2 y\, \gamma - a_{10}\, b_7\, x^2\, y\, \gamma - 2\, b_8\, x^2\, y\, \gamma - 6\, a_{10}^2\, b_7\, x\, y^2\, \gamma -$$
$$-2 a_{10} b_{11} b_7\, x\, y^2 \gamma - 3\, a_{10}\, b_8\, x\, y^2 \gamma - b_{11} b_8\, x y^2 \gamma - 3\, b_9\, x y^2\, \gamma - 3\, a_{10}^2 b_8 \cdot\, y^3 \gamma -$$
$$-a_{10} b_{11} b_8\, y^3 \gamma - 5\, a_{10} b_9\, y^3 \gamma - 2 b_{11} b_9\, y^3 \gamma\ .$$

The identity $A_{41}(x, y) \equiv 0$ yields

$$b_7 = (-4 a_0 a_{10} + 4 a_{10}^2 b_0 - a_0 b_{11} + a_{10} b_0 b_{11} + a_{10} b_6\, \gamma + b_{11} b_6\, \gamma)/\gamma,$$
$$b_8 = a_{10} (8\, a_0 a_{10} - 8 a_{10}^2 b_0 + 2 a_0 b_{11} - 2 a_{10} b_0 b_{11} - 5 a_{10} b_6 \gamma - 2 b_{11} \cdot b_6\, \gamma)/\gamma,$$
$$b_9 = a_{10}^2\, (-4\, a_0 a_{10} + 4\, a_{10}^2\, b_0 - a_0\, b_{11} + a_{10}\, b_0\, b_{11} + 3\, a_{10}\, b_6\, \gamma + b_{11}\, b_6\, \gamma)/\gamma.$$

In these conditions $A_5(x, y) = -A_{11}(x, y) \cdot A_{51}(x, y)/\gamma$, where

$$A_{51}(x, y) = 4\, a_0^2\, a_{10}\, x^2 - 8\, a_0 a_{10}^2 b_0\, x^2 + 4\, a_{10}^3\, b_0^2\, x^2 + a_0^2\, b_{11}\, x^2 -$$
$$-2\, a_0\, a_{10}\, b_0\, b_{11}\, x^2 + a_{10}^2\, b_0^2\, b_{11}\, x^2 - 8\, a_0^2 a_{10}^2\, x\, y + 16\, a_0\, a_{10}^3\, b_0\, x\, y -$$
$$-8\, a_{10}^4 b_0^2\, x\, y - 2 a_0^2\, a_{10} b_{11}\, x\, y + 4\, a_0 a_{10}^2 b_0 b_{11}\, x\, y - 2\, a_{10}^3 b_0^2 b_{11}\, x\, y +$$
$$+4 a_0^2 a_{10}^3\, y^2 - 8\, a_0 a_{10}^4 b_0\, y^2 + 4\, a_{10}^5 b_0^2\, y^2 + a_0^2 a_{10}^2 b_{11}\, y^2 - 2\, a_0 a_{10}^3 b_0 b_{11} y^2 +$$
$$+a_{10}^4 b_0^2 b_{11}\, y^2 - 4\, a_0 a_{10} b_6\, x^2 \gamma + 4\, a_{10}^2\, b_0 b_6\, x^2 \gamma - a_0 b_{11} b_6\, x^2\, \gamma + a_{10} b_0 b_{11} b_6\, x^2\, \gamma +$$
$$+8\, a_0\, a_{10}^2\, b_6 x\, y\, \gamma - 8\, a_{10}^3 b_0 b_6\, x\, y\, \gamma + 2\, a_0 a_{10} b_{11} b_6\, x\, y\, \gamma - 2\, a_{10}^2 b_0 b_{11} b_6\, x\, y\, \gamma -$$
$$-4\, a_0 a_{10}^3 b_6\, y^2\, \gamma + 4\, a_{10}^4 b_0 b_6 \cdot\, y^2\, \gamma - a_0 a_{10}^2\, b_{11}\, b_6\, y^2\, \gamma + a_{10}^3\, b_0\, b_{11}\, b_6\, y^2\, \gamma +$$
$$+ 2 a_{10}\, b_3\, x^2\, \gamma^2 + b_{11}\, b_3\, x^2\, \gamma^2 - b_4 x^2\, \gamma^2 - 6\, a_{10}^2\, b_3\, x\, y\, \gamma^2 - 2\, a_{10}\, b_{11}\, b_3\, x\, y\, \gamma^2 -$$
$$-2\, b_5\, x\, y\, \gamma^2 - 3\, a_{10}^2\, b_4\, y^2\, \gamma^2 - a_{10}\, b_{11}\, b_4\, y^2\, \gamma^2 - 2\, a_{10}\, b_5\, y^2\, \gamma^2 - b_{11}\, b_5\, y^2\, \gamma^2.$$

The identity $A_{51} \equiv 0 \Longrightarrow b_4 = (4\, a_0^2\, a_{10} - 8\, a_0\, a_{10}^2\, b_0 + 4\, a_{10}^3\, b_0^2 + a_0^2\, b_{11} -$

$$-2 a_0 \cdot a_{10} b_0 b_{11} + a_{10}^2 b_0^2\, b_{11} - 4 a_0 a_{10} b_6\, \gamma + 4 a_{10}^2 b_0\, b_6\, \gamma - a_0 b_{11} b_6\, \gamma$$
$$+ a_{10} b_0 b_{11} b_6\, \gamma + 2\, a_{10} \cdot\, b_3\, \gamma^2 + b_{11}\, b_3\, \gamma^2)/\gamma^2,$$
$$b_5 = -a_{10}\, (4\, a_0^2\, a_{10} - 8\, a_0\, a_{10}^2\, b_0 + 4\, a_{10}^3\, b_0^2 + a_0^2\, b_{11} - 2 a_0 a_{10} \cdot b_0 b_{11} + a_{10}^2 b_0^2 b_{11} -$$
$$-4 a_0 a_{10} b_6\, \gamma + 4\, a_{10}^2 b_0 b_6\, \gamma - a_0 b_{11} b_6\, \gamma + a_{10}\, b_0\, b_{11}\, b_6\, \gamma +$$
$$3\, a_{10}\, b_3 \cdot\, \gamma^2 + b_{11}\, b_3\, \gamma^2)/\gamma^2 \Rightarrow$$

$A_6(x, y) = -(x - a_{10}\, y)^2\, (x + 3\, a_{10}\, y + b_{11}\, y) \cdot A_{61}(x, y)/\gamma^2$, where

$$A_{61}(x, y) = -4\, a_0^3\, a_{10}\, x^2 + 12\, a_0^2\, a_{10}^2\, b_0\, x^2 - 12\, a_0\, a_{10}^3\, b_0^2\, x^2 + 4\, a_{10}^4\, b_0^3\, x^2 -$$
$$-a_0^3\, b_{11}\, x^2 + 3\, a_0^2\, a_{10}\, b_0\, b_{11}\, x^2 - 3\, a_0\, a_{10}^2\, b_0^2\, b_{11}\, x^2 + a_{10}^3\, b_0^3\, b_{11}\, x^2 +$$
$$+8\, a_0^3\, a_{10}^2\, xy - 24 a_0^2 \cdot a_{10}^3\, b_0\, x\, y + 24 a_0\, a_{10}^4\, b_0^2\, x\, y - 8\, a_{10}^5\, b_0^3\, x\, y +$$
$$+2\, a_0^3\, a_{10}\, b_{11}\, x\, y - 6\, a_0^2\, a_{10}^2\, b_0\, b_{11}\, xy + 6 a_0 \cdot a_{10}^3\, b_0^2\, b_{11}\, x\, y - 2 a_{10}^4\, b_0^3\, b_{11} x\, y -$$
$$-4 a_0^3\, a_{10}^3\, y^2 + 12\, a_0^2\, a_{10}^4\, b_0\, y^2 - 12\, a_0 a_{10}^5\, b_0^2\, y^2 + 4\, a_{10}^6 \cdot\, b_0^3 y^2 - a_0^3 a_{10}^2\, b_{11}\, y^2 +$$
$$+ 3\, a_0^2\, a_{10}^3\, b_0\, b_{11}\, y^2 - 3 a_0\, a_{10}^4\, b_0^2\, b_{11}\, y^2 + a_{10}^5\, b_0^3\, b_{11}\, y^2 + 4\, a_0^2\, a_{10} \cdot\, b_6\, x^2\, \gamma -$$
$$-8\, a_0\, a_{10}^2\, b_0\, b_6\, x^2\, \gamma + 4\, a_{10}^3\, b_0^2\, b_6\, x^2 \gamma + a_0^2\, b_{11} b_6\, x^2 \gamma - 2\, a_0\, a_{10}\, b_0\, b_{11}\, b_6\, x^2\, \gamma +$$
$$+a_{10}^2\, b_0^2\, b_{11} b_6\, x^2\, \gamma - 8\, a_0^2\, a_{10}^2\, b_6\, x\, y\, \gamma + 16\, a_0\, a_{10}^3\, b_0\, b_6\, x\, y\, \gamma -$$
$$-8\, a_{10}^4\, b_0^2\, b_6\, x\, y\, \gamma - 2\, a_0^2 \cdot a_{10}\, b_{11}\, b_6\, x\, y\, \gamma + 4\, a_0\, a_{10}^2\, b_0\, b_{11}\, b_6\, x\, y\, \gamma -$$
$$-2\, a_{10}^3\, b_0^2\, b_{11}\, b_6\, x\, y\, \gamma + 4\, a_0^2\, a_{10}^3\, b_6\, y^2\, \gamma - 8\, a_0\, a_{10}^4\, b_0\, b_6\, y^2\, \gamma +$$
$$+4\, a_{10}^5\, b_0^2\, b_6\, y^2\, \gamma + a_0^2\, a_{10}^2\, b_{11}\, b_6\, y^2\, \gamma - 2\, a_0\, a_{10}^3\, b_0\, b_{11}\, b_6\, y^2\, \gamma +$$

$$+a_{10}^4\, b_0^2\, b_{11}\, b_6\, y^2\, \gamma - 4\, a_0\, a_{10}\, b_3\, x^2\, \gamma^2 + \ 4\, a_{10}^2\, b_0\, b_3\, x^2\, \gamma^2 - a_0\, b_{11}\, b_3\, x^2\, \gamma^2 +$$
$$+a_{10}\, b_0\, b_{11} \cdot\, b_3\, x^2\gamma^2 + 8\, a_0\, a_{10}^2\, b_3\, x\, y\, \gamma^2 - 8\, a_{10}^3\, b_0\, b_3\, xy\, \gamma^2\ +$$
$$+\, 2\, a_0\, a_{10}\, b_{11}\, b_3\, xy\ \ \gamma^2 - \ a_{10}^2\, b_0\, b_{11} \cdot\, b_3\, xy\, \gamma^2 - 4\, a_0\, a_{10}^3\, b_3\, y^2\gamma^2 +$$
$$+4\, a_{10}^4\, b_0\, b_3\, y^2\ \ \gamma^2 - a_0\, a_{10}^2\, b_{11}\, b_3\, y^2\ \ \gamma^2 + a_{10}^3\, b_0\, b_{11}\, b_3 \cdot\, y^2\, \gamma^2 + 3\, a_{10}\, b_1\, x^2\, \gamma^3 +$$
$$+b_1\, b_{11}\, x^2\gamma^3 - b_2\, x^2\gamma^3 - 6\, a_{10}^2\, b_1\, x\, y\, \gamma^3 - 2\, a_{10}\, b_1\, b_{11}\, x\, y\, \gamma^3 + 2\, a_{10}\, b_2\, xy\, \gamma^3 +$$
$$+3\, a_{10}^3\, b_1\, y^2\gamma^3 + a_{10}^2\, b_1 b_{11}\, y^2\gamma^3 - a_{10}^2\, b_2 y^2\gamma^3 - 3\, x^2\gamma^4 - 18\, a_{10} xy\, \gamma^4 -$$
$$-6 b_{11}\, y\gamma^4 - 27\, a_{10}^2\, y^2\, \gamma^4 - 18\, a_{10}\, b_{11}\, y^2\, \gamma^4 - 3\, b_{11}^2\, y^2\gamma^4).$$

The identity $A_{61}(x,y) \equiv\ 0$ holds if $b_2 = -(a_{10}\, b_1 + 3\,\gamma),\ b_{11} = -4a_{10}$. In these conditions we have $A_7(x,y) = \gamma(x -\ a_{10}\, y)^4\, (4\, a_0 - 4\, a_{10}\, b_0 - b_6\, \gamma) \equiv 0\ \ \Rightarrow$ $a_0 = (4\, a_{10}\, b_0 + b_6\, \gamma)/4 \quad \Rightarrow \qquad A_8(x,y) = \gamma^2(8\, b_3 - 3\, b_6^2)(-x + a_{10}\, y)^3/4 \Rightarrow b_3 = 3b_6^2/8 \quad \Rightarrow \qquad A_9(x,y) = 3\gamma^2(-x + a_{10}\, y)\, (16\, b_1 x - b_6^3\, x - 16\, a_{10} b_1 y + a_{10} b_6^3\, y - 64\, y\, \gamma)/16 \not\equiv 0$.

So, $\mathbb{E}_\infty = -Z^9\, \gamma^2(-4\, x + 4\, a_{10}\, y - b_6\, Z)\, (-48\, b_1 x +\ 3\, b_6^3\, x + 48\, a_{10}\, b_1\, y - 3\, a_{10}\, b_6^3\, y - 64\, b_0\, Z + 4\, b_1\, b_6\, Z + 192\, y\, \gamma)/64$ and $m_\infty = 10$.

In this case the quartic system {(3), (4)} looks as:

$$\dot{x} = 8a_{10}\, x^4 - 32a_{10}^2\, x^3 y + 48a_{10}^3\, x^2y^2 - 32\, a_{10}^4\, xy^3 + 8\, a_{10}^5\, y^4 +$$
$$+8\, a_{10} b_6\, x^3 - 24a_{10}^2\, b_6\, x^2y + 24\, a_{10}^3\, b_6\, xy^2 - 8\, a_{10}^4\, b_6\, y^3 + 3\, a_{10} \cdot b_6^2 x^2 -$$
$$-6\, a_{10}^2 b_6^2\, xy + 3a_{10}^3 b_6^2 y^2 + 8\, a_{10} b_1 x - 8\, a_{10}^2\, b_1 y + 8\, a_{10} b_0 +$$
$$+\, 8\, x\, \gamma - 32\, a_{10} y\gamma + 2\, b_6\gamma)/8, \qquad (8)$$
$$\dot{y} = (8\, x^4 - 32\, a_{10}\, x^3 y + 48\, a_{10}^2\, x^2y^2 - 32\, a_{10}^3 xy^3\ \ + 8\, a_{10}^4\, y^4 +$$
$$+8\, b_6\, x^3 - 24\, a_{10}\, b_6\, x^2y + 24\, a_{10}^2 b_6\, xy^2\ -\ 8\, a_{10}^3\, b_6\, y^3 + 3\, b_6^2\, x^2 -$$
$$-\, 6\, a_{10} b_6^2\, xy + 3\, a_{10}^2\, b_6^2\, y^2 + 8\, b_1\, x - 8\, a_{10} b_1\, y\ \ + 8\, b_0 - 24\, y\, \gamma)/8.$$

The transformation of coordinates $X = b_6 + 4\, x - 4\, a_{10}\, y,\ Y = 4(64\, b_0 - 4\, b_1\, b_6 + (48\, b_1 - 3\, b_6^3)x - (48\, a_{10}\, b_1 - 3\, a_{10}\, b_6^3 + 192\, \gamma)y)/3$ and time rescaling $t = -\tau/\gamma$ reduce the system (8) to the system

$$\dot{X} = -X, \quad \dot{Y} = X^4 + 3\, Y. \qquad (9)$$

In the conditions 1.2) we have $A_3(x,y) = -A_{11}(x,y) \cdot\, A_{31}(x,y)$, where

$$A_{31}(x,y) = (-3\, a_1 a_{10} + a_2 + 3\, a_{10}^2 b_1 - a_1 b_{11} + a_{10} b_1 b_{11} - a_{10} b_2)x^4 -$$
$$-(4\, a_{10} a_2 + 2\, a_1 a_{10} b_{11} - 2a_{10}^2 b_1 b_{11} + 2a_1 b_{12} - 2a_{10} b_1 b_{12} - 4\, a_{10}^2 b_2)x^3 y +$$
$$+(12a_1 a_{10}^3 - 12a_{10}^4\, b_1 + 9a_1 a_{10}^2\, b_{11} - 3\, a_{10}\, a_2\, b_{11} - 9a_{10}^3\, b_1 b_{11} + 5a_1 a_{10} b_{12} -$$
$$a_2\, b_{12} - 5\, a_{10}^2 b_1 b_{12} + 3a_{10}^2 b_{11} b_2 + a_{10} b_{12} b_2)\, x^2y^2 -$$
$$-(12\, a_1 a_{10}^4 - 8a_{10}^3 a_2 - 12a_{10}^5 b_1 + 8a_1 a_{10}^3 b_{11} - 6a_{10}^2 a_2 b_{11} - 8\, a_{10}^4 b_1 b_{11} +$$
$$+4\, a_1 a_{10}^2 b_{12} - 2\, a_{10} a_2 b_{12} -\ 4\, a_{10}^3 b_1 b_{12} + 8a_{10}^4\, b_2 + 6a_{10}^3\, b_{11} b_2 + 2a_{10}^2 b_{12} b_2) \cdot\, xy^3 +$$
$$+(3a_1 a_{10}^5 - 5a_{10}^4\, a_2 - 3\, a_{10}^6 b_1 + 2a_1 a_{10}^4 b_{11} - 3a_{10}^3 a_2 b_{11} - 2a_{10}^5\, b_1\, b_{11} +$$
$$+a_1 a_{10}^3 b_{12} - a_{10}^2 a_2 b_{12} - a_{10}^4 b_1 b_{12} + 5\, a_{10}^5 b_2 + 3\, a_{10}^4 b_{11} b_2 + a_{10}^3 b_{12} b_2)y^4 +$$
$$+\, \beta(a_{10} b_6 + b_{11} b_6 - b_7)x^4 +$$
$$+\, \beta(2\, a_{10}^2 b_6 + 2a_{10} b_{11} b_6 + 2\, b_{12}\, b_6 - 2\, b_8)\, x^3 y - \beta(9\, a_{10}^3 b_6 + 6\, a_{10}^2 b_{11} b_6 +$$
$$+3\, a_{10}\, b_{12}\, b_6\ \ - a_{10}^2\, b_7 - a_{10}\, b_{11}\, b_7 - b_{12}\, b_7 + a_{10}\, b_8 + b_{11}\, b_8 + 3\, b_9)x^2y^2 -$$

$$-\beta\,(6\,a_{10}^3 b_7+4\,a_{10}^2 b_{11} b_7+2\,a_{10} b_{12} b_7+2a_{10} b_9+2b_{11} b_9)xy^3 -$$
$$-\beta(3\,a_{10}^3 b_8+2\,a_{10}^2\,b_{11}\,b_8+a_{10}\,b_{12}\,b_8+a_{10}^2\,b_9+a_{10}\,b_{11} b_9+b_{12}\,b_9)y^4\,.$$

As $A_{11}(x,y)\not\equiv 0$, we require that $A_{31}(x,y)\equiv\ 0$. The identity $A_{31}(x,y)\equiv 0$ holds if

$$a_2=3a_1 a_{10}-3a_{10}^2 b_1+a_1 b_{11}-a_{10} b_1 b_{11}+a_{10} b_2-a_{10} b_6 \beta-b_{11} b_6\,\beta+b_7\,\beta,$$
$$b_8=(-6a_1 a_{10}^2\ +6a_{10}^3 b_1-3\,a_1 a_{10} b_{11}+3\,a_{10}^2 b_1 b_{11}-a_1 b_{12}+a_{10} b_1 b_{12}+$$
$$+3a_{10}^2 b_6\,\beta+3a_{10} b_{11} b_6\,\beta+b_{12} b_6 \beta-2a_{10} b_7 \beta)/\beta,$$
$$b_9=-a_{10}\,(-6\,a_1 a_{10}^2+\ 6a_{10}^3 b_1-3a_1 a_{10} b_{11}+3\,a_{10}^2 b_1 b_{11}-a_1 b_{12}+a_{10} b_1 b_{12}+$$
$$+4a_{10}^2 b_6 \beta+3a_{10} b_{11} b_6 \beta+b_{12} b_6 \beta-a_{10} b_7 \beta)/\beta.$$

In these conditions $A_4(x,y)\equiv 0\ \Rightarrow$

$$b_4=-(4\,a_0 a_{10}-4a_{10}^2 b_0+a_0 b_{11}-a_{10} b_0 b_{11}-a_1 a_{10} b_6+a_{10}^2 b_1 b_6-a_1 b_{11} b_6+$$
$$+a_{10} b_1 b_{11} b_6+a_1 b_7-a_{10} b_1 b_7-2a_{10} b_3\,\beta-b_{11} b_3 \beta+a_{10} b_6^2 \beta+$$
$$+b_{11} b_6^2 \beta-b_6 b_7 \beta+2\beta^2)/\beta,$$
$$b_5=(4a_0 a_{10}^2-4a_{10}^3 b_0+a_0 a_{10} b_{11}-a_{10}^2 b_0 b_{11}-a_1 a_{10}^2 b_6+a_{10}^3 b_1 b_6-a_1 a_{10} b_{11} b_6+$$
$$+a_{10}^2 b_1 b_{11} b_6\ +a_1 a_{10} b_7-a_{10}^2 b_1 b_7-3a_{10}^2\cdot b_3 \beta-a_{10} b_{11} b_3 \beta+a_{10}^2 b_6^2 \beta+$$
$$+a_{10} b_{11} b_6^2 \beta-a_{10} b_6 b_7 \beta-6a_{10} \beta^2-2b_{11} \beta^2)\beta,$$

$b_{12}=-3(2a_{10}^2+a_{10} b_{11})\Rightarrow A_5(x,y)=-A_{11}(x,y)\cdot A_{51}(x,y)/\beta$, where

$$A_{51}=(-4a_0 a_1 a_{10}+\ 4a_1 a_{10}^2 b_0+4a_0 a_{10}^2 b_1-4a_{10}^3 b_0 b_1-a_0 a_1 b_{11}+a_1 a_{10} b_0 b_{11}+$$
$$+a_0 a_{10} b_1 b_{11}-a_{10}^2 b_0 b_1 b_{11}+a_1^2 a_{10} b_6-2a_1 a_{10}^2 b_1 b_6+a_{10}^3 b_1^2 b_6+a_1^2 b_{11} b_6-$$
$$-2a_1 a_{10} b_1 b_{11} b_6+a_{10}^2 b_1^2 b_{11} b_6-a_1^2 b_7+2a_1\cdot a_{10} b_1 b_7-a_{10}^2 b_1^2 b_7)x^2\ +\ (8\,a_0 a_1 a_{10}^2-$$
$$-8\,a_1 a_{10}^3 b_0-8\,a_0 a_{10}^3 b_1+8\,a_{10}^4 b_0 b_1+2\,a_0 a_1\,a_{10}\cdot b_{11}-2a_1 a_{10}^2 b_0 b_{11}-$$
$$-2a_0 a_{10}^2 b_1 b_{11}+2a_{10}^3 b_0 b_1 b_{11}-2a_1^2 a_{10}^2 b_6+4a_1 a_{10}^3 b_1 b_6-2a_{10}^4 b_1^2 b_6-2a_1^2 a_{10} b_{11} b_6+$$
$$+4a_1 a_{10}^2 b_1 b_{11} b_6-2a_{10}^3 b_1^2 b_{11} b_6+2a_1^2 a_{10} b_7-4a_1 a_{10}^2 b_1 b_7+2a_{10}^3 b_1^2 b_7)x\,y+$$
$$+(4a_1 a_{10}^4 b_0-4a_0 a_1 a_{10}^3+4a_0 a_{10}^4 b_1-4a_{10}^5 b_0 b_1-a_0 a_1 a_{10}^2 b_{11}+a_1 a_{10}^3 b_0 b_{11}+$$
$$+a_0 a_{10}^3 b_1 b_{11}-a_{10}^4 b_0 b_1 b_{11}+a_1^2\,a_{10}^3 b_6-2a_1 a_{10}^4 b_1 b_6+a_{10}^5 b_1^2 b_6+a_1^2 a_{10}^2 b_{11} b_6-$$
$$-2a_1 a_{10}^3 b_1 b_{11} b_6+a_{10}^4 b_1^2 b_{11}\cdot b_6-a_1^2 a_{10}^2 b_7+2a_1 a_{10}^3 b_1 b_7-a_{10}^4 b_1^2 b_7)\,y^2+$$
$$+\beta\,(3\,a_0 a_{10} b_6-3\,a_{10}^2 b_0 b_6-a_1 a_{10} b_6^2+a_{10}^2 b_1\cdot b_6^2\ -a_1 b_{11} b_6^2+a_{10} b_1 b_{11} b_6^2+$$
$$+a_0 b_7-a_{10} b_0 b_7+\ a_1 b_6 b_7-a_{10} b_1 b_6 b_7)x^2+\beta(6\,a_{10}^3 b_0 b_6-6a_0 a_{10}^2 b_6+2a_1 a_{10}^2 b_6^2-$$
$$-2a_{10}^3 b_1 b_6^2+2a_1 a_{10} b_{11} b_6^2-2a_{10}^2\,b_1 b_{11} b_6^2-2a_0 a_{10} b_7+2a_{10}^2 b_0 b_7-2a_1 a_{10} b_6 b_7+$$
$$+2a_{10}^2 b_1 b_6 b_7)\,x\,y\ +\beta(3\,a_0 a_{10}^3\,b_6-3\,a_{10}^4\,b_0\,b_6\ -\ a_1\,a_{10}^3\,b_6^2+a_{10}^4\,b_1\,b_6^2-$$
$$-a_1 a_{10}^2\,b_{11}\,b_6^2+a_{10}^3\,b_1\,b_{11}\,b_6^2+a_0\,a_{10}^2\,b_7-a_{10}^3\,b_0\,b_7+\ a_1\,a_{10}^2\,b_6\,b_7-$$
$$-a_{10}^3\,b_1\,b_6\,b_7)y^2+\beta^2\,(3a_1-6\,a_{10}\,b_1-b_1\,b_{11}+b_2+a_{10}\,b_3\,b_6+\ b_{11}\,b_3\,b_6-$$
$$-b_3\,b_7)x^2+\beta^2\,(18\,a_1 a_{10}-12\,a_{10}^2\,b_1+6\,a_1 b_{11}-4\,a_{10} b_1 b_{11}-2a_{10} b_2-$$
$$-2a_{10}^2\,b_3\,b_6\,-2a_{10}\,b_{11} b_3\,b_6+\ 2a_{10} b_3 b_7)\cdot x\,y+\beta^2\,(27\,a_1\,a_{10}^2-30\,a_{10}^3\,b_1\ +$$
$$+18\,a_1\,a_{10}\,b_{11}-19\,a_{10}^2\,b_1\,b_{11}+3\,a_1\,b_{11}^2-3\,a_{10}\,b_1 b_{11}^2+a_{10}^2\,b_2\ +\ a_{10}^3\,b_3\,b_6+$$
$$+a_{10}^2\,b_{11}\,b_3\,b_6-a_{10}^2\,b_3\,b_7\,)y^2-\ \beta^3 b_6\,x^2-\beta^3(10\,a_{10}\,b_6+\ 6\,b_{11}\,b_6-\ 4\,b_7)\,x\,y\ -$$
$$-\beta^3(\,13\,a_{10}^2 b_6+\ 12\,a_{10} b_{11} b_6+\ 3\,b_{11}^2 b_6-4\,a_{10} b_7-\ 2\,b_{11}\,b_7)y^2.$$

The identity $A_5(x,y)\equiv 0$ holds if

$$b_2 = (-12a_1^3a_{10} + 36a_1^2a_{10}^2b_1 - 36a_1a_{10}^3b_1^2 + 12a_{10}^4b_1^3 - 3a_1^3b_{11} + 9a_1^2a_{10}b_1b_{11} - 9a_1a_{10}^2b_1^2b_{11} + 3a_{10}^3b_1^3b_{11} + 20a_0a_1a_{10}\beta - 20a_1a_{10}^2b_0\beta - 20a_0a_{10}^2b_1\beta + +20a_{10}^3b_0b_1\beta + 5a_0a_1b_{11}\beta - 5a_1a_{10}b_0b_{11}\beta - 5a_0a_{10}b_1b_{11}\beta + 5a_{10}^2b_0b_1b_{11}\beta + +16a_1^2a_{10}b_6\beta - 32a_1a_{10}^2b_1b_6\beta + 16a_{10}^3b_1^2b_6\beta + 4\,a_1^2b_{11}b_6\beta - 8a_1a_{10}b_1\cdot b_{11}b_6\beta + +4a_{10}^2b_1^2b_{11}b_6\beta - 12a_1a_{10}b_3\beta^2 + 12a_{10}^2b_1b_3\beta^2 - 3a_1b_{11}b_3\beta^2 + 3a_{10}b_1b_{11}b_3\beta^2 - +12a_0a_{10}b_6\beta^2 + 12a_{10}^2b_0b_6\beta^2 - 3a_0b_{11}b_6\beta^2 + 3a_{10}b_0b_{11}b_6\,\beta^2 - 4a_1a_{10}b_6^2\beta^2 + +4a_{10}^2b_1b_6^2\cdot\beta^2 - a_1b_{11}b_6^2\beta^2 + a_{10}b_1b_{11}b_6^2\beta^2 - 6a_1\beta^3 + 12a_{10}b_1\beta^3 + 2b_1b_{11}\beta^3 + +4a_{10}b_3b_6\beta^3 + b_{11}b_3\cdot b_6\beta^3 + 2b_6\beta^4)/(2\beta^3) \text{ and}$$

$$b_7 = -3(4a_1a_{10} - 4a_{10}^2b_1 + a_1b_{11} - a_{10}b_1b_{11} - 2a_{10}b_6\beta - b_{11}b_6\beta)/(2\beta).$$

In these conditions $A_6(x,y) \not\equiv 0$, therefore $m_\infty = 7$.

In the conditions 2) the identity $A_2(x,y) \equiv 0$ leads us to the following conditions

$$a_4 = 2a_{10}a_3 + a_3b_{11} - 2a_{10}^2b_3 - a_{10}b_{11}b_3 + a_{10}b_4 + a_{10}b_6\alpha + b_{11}b_6\alpha - b_7\alpha + \alpha^2,$$

$$a_5 = 3a_{10}^2a_3 + 2a_{10}a_3b_{11} + a_3b_{12} - 3a_{10}^3b_3 - 2a_{10}^2b_{11}b_3 - a_{10}b_{12}b_3 + a_{10}b_5 + +2a_{10}^2b_6\alpha + 2a_{10}\cdot b_{11}b_6\alpha + b_{12}b_6\alpha - a_{10}b_7\alpha - b_8\alpha + 3a_{10}\alpha^2 + b_{11}\alpha^2,$$

$$b_9 = -a_{10}^3b_6 - a_{10}^2b_7 - a_{10}b_8 + 6a_{10}^2\alpha + 3a_{10}b_{11}\alpha + b_{12}\alpha,$$

$$b_{13} = -a_{10}(4a_{10}^2 + 3a_{10}b_{11} + 2b_{12}).$$

In the above conditions we have: $A_3(x,y) \equiv 0 \;\Rightarrow$

$$a_2 = 3a_1a_{10} - 3a_{10}^2b_1 + a_1b_{11} - a_{10}b_1b_{11} + a_{10}b_2 - a_{10}a_3b_6 - a_3b_{11}b_6 + a_{10}^2b_3b_6 + +a_{10}b_{11}b_3b_6 + a_3b_7 - a_{10}b_3b_7 - 3a_3\alpha + 5a_{10}b_3\alpha + b_{11}b_3\alpha - b_4\alpha - a_{10}b_6^2\alpha - -b_{11}b_6^2\alpha + b_6b_7\alpha - 2b_6\alpha^2,$$

$$b_5 = -4a_{10}a_3 - a_3b_{11} + 3a_{10}^2b_3 + a_{10}b_{11}b_3 - a_{10}b_4 - a_{10}b_6\alpha - b_{11}b_6\alpha + b_7\alpha - 2\alpha^2,$$

$$b_8 = -3a_{10}^2b_6 - 2a_{10}b_7 + 8a_{10}\alpha + 2b_{11}\alpha, \quad b_{12} = -3a_{10}(2a_{10} + b_{11}) \Rightarrow$$

$$\Rightarrow A_4(x,y) \equiv 0 \Rightarrow\; b_2 = -a_1,\; b_4 = -2a_3, \quad b_7 = 3(-a_{10}b_6 + \alpha),\; b_{11} = -4a_{10} \Rightarrow$$

$$\Rightarrow A_5(x,y) \not\equiv 0,\; m_\infty = 6.$$

Thus, the maximal algebraic multiplicity of the line at infinity is not greater than ten (see the case 1.1.2). In this way we have proved the Theorem.

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