Acta et Coomentationes, Exact and Natural Sciences, nr. 2(8), 2019 ISSN 2537-6284
p. 22-28 E-ISSN 2587-3644

CZU: 517.925 DOI: 10.36120/2587-3644.v8i2.22-28

CENTER CONDITIONS FOR A CUBIC DIFFERENTIAL SYSTEM WITH ONE INVARIANT STRAIGHT LINE

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Abstract. We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one invariant straight line. The presence of a center at $O(0,0)$ is proved by using the method of rational reversibility.

Keywords: Cubic differential system, center problem, invariant straight lines, rational reversibility.

CONDIȚII DE EXISTENȚĂ A CENTRULUI PENTRU UN SISTEM DIFFERENTIAL CUBIC CU O DREAPTĂ INVARIANTĂ

Rezumat. Se determină condițiile de existență a centrului pentru un sistem diferențial cubic cu punctul singular $O(0,0)$ de tip centru sau focar ce posedă o dreaptă invariantă. Prezența centrului se demonstrează aplicând metoda reversibilității raționale.

Cuvinte-cheie: Sistem diferențial cubic, problema centrului, drepte invariante, reversibilitate rațională.

1. Introduction

We consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{1.1}$$

in which variables $x = x(t), y = y(t)$ and coefficients are assumed to be real. Let the origin $O(0,0)$ be an isolated singularity of (1.1) with purely imaginary eigenvalues $(\lambda_{1,2} = \pm i,\ i^2 = -1)$. In this case the origin is either a focus or a center. The trajectories in some neighborhood of $O(0,0)$ can be spirals or closed trajectories.

It arises the problem of distinguishing between a center and a focus, i.e. of finding the conditions under which $O(0,0)$ is, for example, a center. We study the problem of the center for cubic system (1.1) assuming that the system has invariant straight lines.

Definition 1.1. An algebraic curve $\Phi(x,y) = 0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x,y]$ is said to be an invariant algebraic curve of system (1.1) if the following identity holds

$$P(x,y)\frac{\partial \Phi}{\partial x} + Q(x,y)\frac{\partial \Phi}{\partial y} = \Phi(x,y)K(x,y) \tag{1.2}$$

for some polynomial $K(x,y) \in \mathbb{C}[x,y]$ called the cofactor of the invariant algebraic curve $\Phi(x,y) = 0$.

Let the cubic system (1.2) have at least one invariant straight line

$$C + Ax + By = 0, \quad A, B, C \in \mathbb{C}, \ (A,B) \neq (0,0). \tag{1.3}$$

Then by Definition 1.1, a straight line (1.3) is an invariant straight for system (1.1) if and only if the following identity holds

$$AP(x,y)+BQ(x,y)=(C+Ax+By)K(x,y), \tag{1.4}$$

where $K(x,y)=c_{00}+c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2$ and A, B, C, c_{ij} are unknown complex coefficients.

Identifying the coefficients of $x^i y^j$ in (1.4), we find that

$$\begin{gathered}(m-c_{11})A-(c_{20}+q)B=0,\ \ (c-c_{01})A-(c_{10}+d)B-c_{11}C=0,\\ (p-c_{02})A-(c_{11}+n)B=0,\ \ fA-(b+c_{01})B-c_{02}C=0,\\ (a-c_{10})A-gB-c_{20}C=0,\ \ A-c_{00}B-c_{01}C=0,\ \ c_{00}C=0,\\ (k-c_{20})A-sB=0,\ \ rA-(c_{02}+l)B=0,\ c_{00}A+c_{10}C+B=0.\end{gathered} \tag{1.5}$$

The coefficients in (1.1) are real and the complex invariant straight lines occur in complex conjugated pairs $C+Ax+By=0$ and $\overline{C}+\overline{A}x+\overline{B}y=0.$ According to [1] the cubic system (1.1) cannot have more than four nonhomogeneous invariant straight lines, i.e. invariant straight lines of the form

$$1+Ax+By=0, \quad (A,B)\neq(0,0).$$

As homogeneous straight lines $Ax+By=0$, the system (1.1) can have only the lines

$$x\pm iy=0, i^2=-1.$$

The problem of the center was completely solved for cubic system (1.1) with four invariant straight lines [1], [2] and with three invariant straight lines [2], [3]. In [4] by using the method of Darboux integrability and the method of rational reversibility it was obtained the center conditions for cubic system (1.1) with two invariant straight lines.
The goal of this paper is to obtain center conditions for a cubic differential system (1.1) with one invariant straight line by using the method of rational reversibility.

2. Time-reversibility in cubic systems

In this Section for cubic system (1.1) we find conditions under which the system can be brought to a system with an axis of symmetry.

Definition 2.1. We say that system (1.1) is *time-reversible* if its phase portrait is invariant under reflection with respect to a line and reversion of time.

The classical condition is that the system is invariant under one or other of the transformations $(x,y,t)\to(-x,y,-t)$ or $(x,y,t)\to(x,-y,-t)$. The first corresponds to reflection in the y-axis and the second to reflection in the x-axis. If the system (1.1) is time-reversible, for example, $x=0$ is the axis of symmetry, then the origin is a center.
An algorithm for finding all time-reversible systems within a given family of 2-dim systems of ODE's whose right-hand sides are polynomials was presented by Romanovski in [5]. The relation between time-reversibility and the center-focus problem was discussed by Teixeira in [6].

Suppose system (1.1) is not time-reversible. It is clear that (1.1) has a center at $O(0,0)$ if there exists a diffeomorphism

$$\Psi : U \to V,\ \Psi = \{X = \varphi(x,y), Y = \psi(x,y)\},\ \Psi(0,0) = (0,0),$$

which brings system (1.1) to a system with the axis of symmetry. In particular, if $\varphi(x,y)$ and $\psi(x,y)$ are rational functions, then we say that (1.1) is rationally reversible.

In [7] Zoladek classified all rationally reversible cubic systems. An algorithm for checking whether it is possible to transform a given system to one which is time-reversible by means of a bilinear transformation was described by Lloyd in [8]. This algorithm was applied to find the center conditions for some families of cubic systems.

Consider the polynomial differential system

$$\begin{aligned} \dot{X} &= Y + M(X^2, Y), \\ \dot{Y} &= -X(1 + N(X^2, Y)). \end{aligned} \tag{2.1}$$

The critical point at the origin is clearly monodromic (locally the trajectories encircle the critical point). However a change of coordinates $(X,Y,t) \to (-X,Y,-t)$ leaves the system invariant. Clearly the Y- axis is a line of symmetry for the trajectories. Close to the origin the trajectories must therefore be closed.

In [4] by using the method of rational reversibility there were obtained the center conditions for cubic system (1.1) with two invariant straight lines. It was proved that the cubic system (1.1) by a rational transformation

$$x = \frac{a_1 X + b_1 Y}{a_3 X + b_3 Y - 1},\ y = \frac{a_2 X + b_2 Y}{a_3 X + b_3 Y - 1}, \tag{2.2}$$

where $a_1 b_2 - b_1 a_2 \neq 0$ and $a_j, b_j \in \mathrm{R},\ j = 1,2,3$ can by brouthgt to the form (2.1) if and only if the following equalitites are satisfied

$$\begin{gathered}
a_3[sb_1^4 + rb_2^4 + b_1 b_2((k+q)b_1^2 + (m+n)b_1 b_2 + (l+p)b_2^2)] = 0, \\
a_3[(2p - 3k - q)a_1 a_2 b_2^2 + da_2 b_1^2 a_3 + na_2^2 b_1^2 + ca_1 b_2^2 a_3 + (3l + p - 2q)a_2^2 b_1 b_2 + ma_1^2 b_2^2 + \\
+ (3(r+s) - 2(m+n))a_2^2 b_2^2 + (2b + c - 2g)a_2 b_1 b_2 a_3 + (2f - 2a - d)a_2 b_2^2 a_3 + a_3^2] = 0, \\
2aa_1^2 b_2 a_3 + [(m-s)a_1 + (p-q)a_2 + 2(c-g)a_3]a_1 a_2 b_2 + ka_1^3 b_2 + \\
+ a_2^3(lb_1 - nb_2 + rb_2 + 2a_2^2 a_3(bb_1 - db_2 + fb_2) = 0, \\
(qa_2 + 2ga_3)b_1^3 + [2(a+d)a_3 + (m + 2n - 3s)a_2]b_1^2 b_2 + \\
+ [(3l - 3k + 2p - 2q)a_2 + 2(b+c)a_3]b_1 b_2^2 + [2fa_3 + pa_1 - (2m + n - 3r)a_2]b_2^3 = 0, \\
(ka_2 - ga_3)b_1^3 + [(m-s)a_2 - (a+d)a_3]b_1^2 b_2 + \\
+ [(p-q)a_2 - (b+c)a_3]b_1 b_2^2 + [la_1 - (n-r)a_2 - fa_3]b_2^3 = 0, \\
qa_1^3 b_2 + (m + 2n - 3s)a_1^2 a_2 b_2 + (d-a)a_1^2 a_3 b_2 + (3l - 3k + 2p - 2q)a_1 a_2^2 b_2 + \\
+ [pb_1 + (3r - 2m - n)b_2]a_2^3 + (2b - g)a_1 a_2 a_3 b_2 + [(f - 2a)b_2 - (b - c)b_1]a_2^2 a_3 = 0, \\
aa_2 b_1^2 + (c - g)b_1 b_2 a_2 + (ba_1 - da_2 + fa_2)b_2^2 - a_3 = 0, \\
ga_1^3 + (a+d)a_1^2 a_2 + (b+c)a_1 a_2^2 + fa_2^3 + 2a_3 = 0, \qquad (2.3) \\
[db_1 + (2b + c - 2g)b_2]a_2 b_1 + [ca_1 + (2f - 2a - d)a_2]b_2^2 + 3a_3 = 0,
\end{gathered}$$

$$a_1^2 + a_2^2 - 1 = 0,\ b_1^2 + b_2^2 - 1 = 0,\ a_1 b_1 + a_2 b_2 = 0$$

in the coefficients of system (1.1) are the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ of the transformation.

The equations $a_1^2 + a_2^2 - 1 = 0$, $b_1^2 + b_2^2 - 1 = 0$ of (2.3) admit the parametrizations

$$a_1 = \frac{2u}{u^2+1},\ a_2 = \frac{u^2-1}{u^2+1} \quad \text{and} \quad b_1 = \frac{2v}{v^2+1},\ b_2 = \frac{v^2-1}{v^2+1}. \tag{2.4}$$

Taking into account (2.4) and the equation $a_1 b_1 + a_2 b_2 = 0$ of (2.3), two symmetric cases 1) $a_1 = b_2 = \dfrac{2u}{u^2+1},\ a_2 = -b_1 = \dfrac{u^2-1}{u^2+1}$;

2) $a_1 = -b_2 = \dfrac{2u}{u^2+1},\ a_2 = b_1 = \dfrac{u^2-1}{u^2+1}$

will be considered in studying the consistency of the algebraic system (2.3).

3. Reversible cubic systems with one invariant straight line

Assume that $a_1 = b_2, a_2 = -b_1$ and let $a_3 \neq 0.$ In this case from the equation

$$a a_2 b_1^2 + (c-g) b_1 b_2 a_2 + (b a_1 - d a_2 + f a_2) b_2^2 - a_3 = 0$$

of (2.3) we find that

$$a_3 = [a(u^2-1)^3 + u^2(8bu - 4(d-f)(u^2-1)) - 2(c-g)u(u^2-1)^2]/(u^2+1)^3.$$

We have the following theorem.

Theorem 3.1. The cubic differential system (1.1) is rationally reversible and has one invariant straight line if one of the following sets of conditions holds:

(i) $g = -2b,\ c = -3b,\ q = b(a-d),\ m = 2b^2,\ l = bf,\ r = 0,\ k = -2ab,\ p = -2bf;$

(ii) $f = -2a,\ d = -3a,\ p = a(b-c),\ n = 2a^2,\ k = ag,\ s = 0,\ l = -2ab,\ q = -2ag;$

(iii) $a = [(6b+2c)(10u^2 - 3u^4 - 3)u + 2d(u^6 - 11u^4 + 11u^2 - 1) +$
$+ 5f(7u^4 - u^6 - 7u^2 + 1)]/[4(u^4+1)(u^2-1)],$

$$g = [14bu(u^4 - 6u^2 + 1) + 2cu(5u^4 - 14u^2 + 5) + (3f - 2d)(u^6 - 15u^4 + 15u^2 - 1)]/[8u(u^4+1)],$$

$$\begin{aligned} k = {} & [(2df - 3f^2 + 4s)(u^{20}+1) + 4(bd - 5bf + cd - 4cf)(u^{19} - u) + 2(4s - 14b^2 - 24bc - 10c^2 + \\ & + 4d^2 - 24df + 31f^2)(u^{18} + u^2) + (256b^2 + 336bc + 112c^2 - 96d^2 + 362df - 391f^2 + 4s)(u^{16} + \\ & + u^4) + 8(157f^2 - 106b^2 - 140bc - 42c^2 + 20d^2 - 124df)(u^{14} + u^6) + 2(1536b^2 + 1752bc + \\ & + 456c^2 - 464d^2 + 1866df - 1851f^2 - 4s)(u^{12} + u^8) + 4(1389f^2 - 1994b^2 - 1592bc - 334c^2 + \\ & + 428d^2 - 1528df - 4s)u^{10} + 4(65bf - 31bd - 23cd + 50cf)(u^{17} - u^3) + 8(67bd - 152bf + \\ & + 39cd - 93cf)(u^{15} - u^5) + 8(287cf - 181bd + 476bf - 129cd)(u^{13} - u^7) + 8(760bd - 1385bf + \\ & + 332cd - 617cf)(u^{11} - u^9)]/[8u(u^4+1)^2(u^2+1)^4(u^2-1)], \end{aligned}$$

$$l = [(2df - 3f^2 - 8s)(u^{16} + 1) + 4(bd - 5bf + cd - 4cf + 4q)(u^{15} - u) + 4(32s - 7b^2 - 12bc - 5c^2 + 2d^2 - 17df + 21f^2)(u^{14} + u^2) + 4(123bf - 41bd - 33cd + 74cf - 28q)(u^{13} - u^3) + 4(158b^2 + 204bc + 54c^2 - 44d^2 + 196df - 195f^2 - 64s)(u^{12} + u^4) + 4(479bd - 913bf + 247cd - 442cf + 36q)(u^{11} - u^5) + 4(96s - 969b^2 - 868bc - 187c^2 + 254d^2 - 879df + 747f^2)(u^{10} + u^6) + 4(2543bf - 1527bd - 615cd + 1016cf - 60q)(u^9 - u^7) + 2(3656b^2 + 2832bc + 552c^2 - 848d^2 + 2798df - 2289f^2 - 248s)u^8]/[128u^3(u^4 + 1)^2(u^2 - 1)],$$

$$m = [f(2d - 3f)(u^{20} + 1) + 4(bd - 5bf + cd - 4cf + 2q)(u^{19} - u) + 2(5088b^2 + 5496bc + 1416c^2 - 1456d^2 + 5850df - 5787f^2)(u^{12} + u^8) + 2(4d^2 - 14b^2 - 24bc - 10c^2 - 32df + 39f^2)(u^{18} + u^2) + (320b^2 + 528bc + 240c^2 - 160d^2 + 586df - 711f^2)(u^{16} + u^4) + 8(453f^2 - 250b^2 - 396bc - 122c^2 + 52d^2 - 348df)(u^{14} + u^6) + 4(4293f^2 - 6026b^2 - 4920bc - 1038c^2 + 1324d^2 - 4720df)u^{10} + 8(2356bd - 4289bf + 1032cd - 1917cf + 2q)(u^{11} - u^9) + 4(6q - 39bd + 81bf - 31cd + 74cf)(u^{17} - u^3) + 8(119bd - 304bf + 91cd - 233cf + 4q)(u^{15} - u^5) + 8(4q - 577bd + 1508bf - 397cd + 883cf)(u^{13} - u^7)]/[16u^2(u^4 + 1)^2(u^2 + 1)^4],$$

$$n = [3(2df - 3f^2 - 8s)(u^{16} + 1) + 12(bd - 5bf + cd - 4cf + 4q)(u^{15} - u) + 4(306b^2 + 388bc + 106c^2 - 84d^2 + 380df - 397f^2 - 96s)(u^{12} + u^4) + 4(48s - 21b^2 - 36bc - 15c^2 + 6d^2 - 39df + 53f^2)(u^{14} + u^2) + 4(144s - 1467b^2 - 1356bc - 305c^2 + 378d^2 - 1369df + 1227f^2)(u^{10} + u^6) + 2(5112b^2 + 4144bc + 856c^2 - 1200d^2 + 4106df - 3523f^2 - 360s)u^8 + 4(265bf - 99bd - 75cd + 174cf - 52q)(u^{13} - u^3) + 4(789bd - 1579bf + 413cd - 782cf + 76q)(u^{11} - u^5) + 4(3773bf - 2181bd - 917cd + 1592cf - 116q)(u^9 - u^7)]/[64u^2(u^4 + 1)^2(u^2 - 1)^2],$$

$$p = [3(2df - 3f^2 - 8s)(u^{24} + 1) + 12(bd - 5bf + cd - 4cf + 4q)(u^{23} - u) + 2(444b^2 + 744bc + 348c^2 - 248d^2 + 978df - 1101f^2 + 120s)(u^{20} + u^4) + 4(24s - 21b^2 - 36bc - 15c^2 + 6d^2 - 49df + 60f^2)(u^{22} + u^2) + 4(24s - 1521b^2 - 2836bc - 1123c^2 + 462d^2 - 2589df + 3612f^2)(u^{18} + u^6) + 4(237bf - 119bd - 95cd + 222cf - 4q)(u^{21} - u^3) + 4(763bd - 1857bf + 611cd - 1598cf - 60q)(u^{19} - u^5) + 4(12747bf - 4535bd - 3735cd + 9252cf - 76q)(u^{17} - u^7) + (45984b^2 + 62912bc + 18592c^2 - 12608d^2 + 62394df - 71367f^2 + 24s)(u^{16} + u^8) + 8(13331bd - 29307bf + 7427cd - 15456cf - 44q)(u^{15} - u^9) + 8(5910d^2 - 23421b^2 - 23012bc - 5575c^2 - 23257df + 22740f^2 - 24s)(u^{14} + u^{10}) + 8(61569bf - 33111bd - 14927cd + 28154cf - 28q)(u^{13} - u^{11}) + 4(76404b^2 + 66616bc + 14932c^2 - 18024d^2 + 66126df - 61515f^2 - 120s)u^{12}]/[128u^3(u^4 + 1)^4(u^2 + 1)^4(u^2 - 1)],$$

$r=[(2df-3f^2-8s)(u^{26}-1)+4(bd-5bf+cd-4cf+4q)(u^{25}+u)+(104s-28b^2-48bc-$
$-20c^2+8d^2-62df+75f^2)(u^{24}-u^2)+2(146b^2+208bc+78c^2-76d^2+292df-267f^2+$
$+56s)(u^{22}-u^4)+2(1691f^2-634b^2-1200bc-582c^2+252d^2-1164df-56s)(u^{20}-u^6)+$
$+8(38bf-19bd-15cd+31cf-8q)(u^{23}+u^3)+8(115bd-213bf+75cd-178cf-20q)(u^{21}+$
$+u^5)+8(1486bf-471bd-483cd+1331cf-8q)(u^{19}+u^7)+(1082b^2+18992bc+6500c^2-$
$-3112d^2+18250df-2344f^2-88s)(u^{18}-u^8)+4(8223bd-20523bf+5471cd-12300cf-$
$-4q)(u^{17}+u^9)+(18768d^2-71256b^2-79392bc-20808c^2-80118df+83069f^2-$
$-200s)(u^{16}-u^{10})+16(8q-7947bd+1662bf-3847cd+7511cf)(u^{15}+u^{11})+4(44618b^2+$
$+40224bc+9222c^2-10748d^2+40200df-37909f^2-56s)(u^{14}-u^{12})+16(12157bd-$
$-22291bf+5381cd-10054cf+20q)u^{13}]/[256u^4(u^4+1)^2(u^2+1)^4].$

Proof. We study the compatibility of (2.3) in three possible cases: $u=0, u=1$ and $u(u-1)\neq 0$. If $u=0$, then from the equations of (2.3) we obtain the set of conditions (i) and the equations of (1.5) give the invariant straight line $1-2bx=0$.

If $u=1$, then from the equations of (2.3) we determine the set of conditions (ii) and the equations of (1.5) yield the invariant straight line $1-2ay=0$.

Assume that $u(u-1)\neq 0$. In this case from (2.3) we get the set of conditions (iii) and the equations of (1.5) imply for system (1.1) the existence of the invariant straight line

$$[(fu^2+2bu-f)(u^4-6u^2+1)+2u(u^2-1)(cu^2+2du-c)](2ux+u^2y-y)+$$
$$+(u^4+1)(u^2+1)^2=0.$$

Theorem 3.1 is proved.

Assume that $a_1=b_2, a_2=-b_1$ and let $a_3=0$. We have the following theorem

Theorem 3.2. The cubic differential system (1.1) is rationally reversible if one of the following sets of conditions holds:

(i) $a=d=f=k=l=p=q=0$;

(ii) $b=c=g=k=l=p=q=0$;

(iii) $a=[(d-f)(u^6-7u^4+7u^2-1)+b(20u^3-6u^5-6u)]/[2(u^2-1)^3]$,

$c=[f(1-u^6)+b(12u^3-2u^5-2u)+(4d-7f)(u^2-u^4)]/[2u(u^2-1)^2]$,

$g=[(d+f)(1-u^6)+(7d-f)(u^4-u^2)+b(2u^5-12u^3+2u)]/[4u(u^2-1)^2]$,

$l=[(m-s)(2u^7-14u^5+14u^3-2u)+k(20u^6+20u^2-u^8-54u^4-1)]/[2u^2(u^2-1)^2]$,

$q=[3k(u^4-6u^2+1)^2+8pu^2(u^2-1)^2-6u(m-s)(u^4-6u^2+1)(u^2-1)]/[8u^2(u^2-1)^2]$,

$n=[3k(u^4-6u^2+1)(u^8-20u^6+54u^4-20u^2+1)+2u(4pu^3-4pu+3su^4-18su^2+$
$+3u)(u^4-6u^2+1)(u^2-1)-mu(3u^2-1)(u^2-3)(u^4-14u^2+1)(u^2-1)]/[32u^3(u^2-1)^3]$,

$r=[k(u^4-6u^2+1)(u^8-4u^6+22u^4-4u^2+1)+2m(u^4-6u^2+1)^2(1-u^2)+8pu^2(u^8-$
$-8u^6+14u^4-8u^2+1)+2su(u^2+1)^4(u^2-1)]/[32u^3(u^2-1)^3]$.

Proof. We study the compatibility of (2.3) in three possible cases: $u=0$, $u=1$ and $u(u-1)\neq 0$. We obtain the center conditions (i), (ii) and (iii), respectively. When one of these three sets of conditions holds the cubic system (1.1) has no invariant straight lines. Theorem 3.2 is proved.

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