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ON METRIZABLE FAMILIES OF SUBSPACES OF LOCALLY CONVEX SPACES VIA SELECTIONS

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Summary. Using some methods from the works of E. Michael [11, 12, 13], T. Dobrowolski and J. van Mill [8] and of one of the authors [1, 2, 3, 5], a few new results about existence of the selections with conditions of continuity are proved.

Keywords: pseudometric, family of subspaces, set-valued mapping, selection, linear space.
MSC2010: 54C60, 54C65.

DESPRE FAMILIILE METRIZABILE DE SUBSPAŢII ALE SPAŢIILOR LOCAL CONVEXE REFERITOR LA SELECŢII

Rezumat. Utilizând unele metode din lucrările lui E. Michael [11, 12, 13], T. Dobrowolski şi J. van Mill [8] şi ale unuia dintre autorii [1, 2, 3, 5], au fost demonstrate câteva noi rezultate cu privire la existenţa selecţiilor cu condi[ţ]ii de continuitate.

Cuvinte cheie: pseudometric, familie de subspaţii, set-valued mapping, selecţie, spaţiu liniar.

1. Introduction

Any space is considered to be a Hausdorff space. Let X and Y be topological spaces. We say that $F : X \longrightarrow Y$ is a set-valued mapping if $F(x)$ is a non-empty subset of Y for any point $x \in X$.

A single-valued mapping $f : X \longrightarrow Y$ of a space X into a space Y is said to be a selection of a given set-valued mapping $F : X \longrightarrow Y$ if $f(x) \in F(x)$ for each $x \in X$. Note that by the Axiom of Choice selections always exist. In the category of topological spaces and continuous single-valued mappings the situation is more complex.

The following problem is important: Under what conditions there exist continuous selections? There exist many theorems on continuous selections.

The set-valued mapping $F : X \longrightarrow Y$ is called;

- lower semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \varnothing\}$ is an open subset of the space X for any open subset H of the space Y;

- upper semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \varnothing\}$ is a closed subset of the space X for any closed subset H of the space Y.

One of them is the famous Michael's selection theorem for convex-valued mappings from [11]. There are lower semicontinuous convex-valued mappings $F : X \longrightarrow Y$ without any continuous single-valued selections, even for $X = [0; 1]$ (see Example 6.2 from [11]). An important example is published in [9]. It was proved that every convex-valued lower semicontinuous mapping of a metrizable domain into a separable Banach space admits a continuous selection, provided that all values are finite-dimensional ([11], special case of Theorem 3.1). Distinct results of this kind were proved in [1, 2, 3, 5, 7, 8, 9, 12, 14, 16, 17].

The next localization principle is from the folklore field which do to H. Corson, J. Lindenstrauss, T. Dobrowolski, J. van Mill, V. Valov (see [7, 8, 14, 15]).

Proposition 1 (Localization Principle). *Suppose that a convex-valued mapping $F : X \longrightarrow Y$ of a paracompact space X into a topological vector space Y admits a single-valued continuous selection over each member of some open covering γ of the space X. Then F admits a global single-valued continuous selection.*

2. Pseudometrics on linear spaces

Let d be a pseudo-metric on a space E. For all $x \in E$ and $\epsilon > 0$ we put $V(x, d, \epsilon) = \{y \in E : d(x, y) < \epsilon\}$.

Let E be a topological linear space. A pseudo-metric d on a space E is called a continuous invariant pseudo-metric if $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in E$. A pseudo-metric d is called a uniform pseudo-metric if it is invariant and if:

- from $\{a_n \in \mathbb{R} : n \in\in \mathbb{N}\}$ and $lim_{n\to\infty} a_n = 0$ it follows $lim_{n\to\infty} d(0, a_n x) = 0$ for every $x \in E$;

- from $\{x_n \in E : n \in \mathbb{N}\}$ and $lim_{n\to\infty} d(0, x_n) = 0$ it follows $lim_{n\to\infty} d(0, tx_n) = 0$ for any $t \in \mathbb{R}$;

- the set $\{x \in E : d(0, x) < r\}$ is open in E for every $r \in \mathbb{R}$.

A pseudo-metric d on the linear space is called a convex pseudo-metric if it is uniform and the set $\{x \in E : d(0, x) < r\}$ is open and convex in E for every $r \in \mathbb{R}$.

Fix a pseudo-metric d on a space E. There exist a metric space $(E/d, \bar{d})$ and a mapping $p_d : E \longrightarrow E/d$ such that $d(x, y) = \bar{d}(p_d(x), p_d(y))$ for all $x, y \in E$. On X/d we consider only the topology generated by the metric $\bar{d}$. If the sets $V(x, d, \epsilon)$ are open in E for all $x \in E$ and $\epsilon > 0$, then we say that d is a continuous pseudo-metric on the space E. Obviously, the pseudo-metric d is continuous if and only if the mapping p_d is continuous. If the metric space $(E/d, \bar{d})$ is complete, then we say that d is a complete pseudo-metric. If $F : X \longrightarrow E$ is a set-valued mapping and the mapping $F_d : X \longrightarrow E$, where $F_d(x) = p_d(F(x))$ for each $x \in X$ has the property $\mathcal{P}$, then we say that F is a mapping with the d-property $\mathcal{P}$.

Assume that E is a topological linear space and d is a continuous invariant pseudo-metric. Then $H_d = \{x \in E : d(0, x) = 0\}$ is a closed subgroup of E, on E/d there exists a structure of group for which $p_d : E \longrightarrow E/d$ is a continuous homomorphism of the topological group E on the topological group E/d metrizable by the invariant metric $\bar{d}$. If the pseudo-metric d is uniform, then H_d is a closed liniar subspace of E, E/d admits a structure of a linear topological space metrizable by the uniform metric $\bar{d}$ on E/d. If the pseudo-metric d is convex, then $(E/d, \bar{d})$ is a locally convex linear space.

A family $\mathcal{A}$ of subsets of a space E is called metrizable by the pseudo-metric d if d is a continuous pseudo-metric on the space X and for any $L \in \mathcal{A}$, any point $x \in L$ and any open subset U of E with $x \in U$ there exist an open subset V of E and a number $\epsilon > 0$ such that $x \in V$ and $V(y, d, \epsilon) \cap M \subseteq U$ provided $M \in \mathcal{A}$ and $y \in M \cap V$. We observe that any set $L \in \mathcal{A}$ is metrizable by the metric d on L. If $L \in \mathcal{A}$ and the metric $\bar{d}$ is complete on the subspace $p_d(L)$, then we say that the set L is complete relatively to the pseudo-metric d. If any set $L \in \mathcal{A}$ is complete relatively to the pseudo-metric d, then we say that the family $\mathcal{A}$

is complete metrizable by the pseudo-metric d.

Proposition 2. *Let d be an invariant continuous pseudometric on a topological locally convex space E. Then on E there exists a convex pseudometric ρ such that:*

1. If $d(x, y) = 0$, then $\rho(x, y) = 0$ too.

2. If the family $\mathcal{A}$ of subsets of a space E is metrizable by the pseudo-metric d, then the family $\mathcal{A}$ of subsets of a space E is metrizable by the pseudo-metric ρ too.

3. If any set $L \subset E$ is metrizable relatively to the pseudo-metric d, then we say that the family $\mathcal{A}$ is metrizable by the pseudo-metric ρ too.

4. If any set $L \subset E$ is complete metrizable relatively to the pseudo-metric d, then we say that the family $\mathcal{A}$ is complete metrizable by the pseudo-metric ρ too.

5. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that from $x, y \in E$ and $d(x, y) < \delta$ it follows that $\rho(x, y) < \varepsilon$.

Proof. We fix a sequence of $\{V_n : n\mathbb{N}\}$ of open convex subsets of the space E such that:

- $V_{n+1} + V_{n+1} \subset V_n \supset V_n \subset \{x \in E : d(0, x) < 2^{-n}\}$ for each $n \in \mathbb{N}$;

- $cl_E V_{n+1} \subset V_n = -V_n$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ and $x \in E$ we consider the Minkowski functional $\nu_n(x) = infimum\{t \in \mathbb{R} : t \geq 0, t \cdot x \in V_n\}$ on E associated to V_n which has the next properties:

- $\{x \in E : \nu_n(x) < 1\} \subset V_n \subset \{x \in E : \nu_n(x) \leq 1\}$;

- $\nu_n(tx) = |t|\nu_n(x)$ for all $x \in E$ and $t \in \mathbb{R}$;

- $\nu_n(x + y) \leq \nu_n(x) + \nu_n(y)$ for all $x, y \in E$.

For any $n \in \mathbb{N}$ the linear space E/ν_n is a normed space and the projection $p_n : E \longrightarrow E/\nu_n$ is continuous and linear.

Now we put $\rho(x, y) = \Sigma\{2^{-n} \cdot min\{1, 2\nu_n(x - y)\} : n \in \mathbb{N}\}$ for all $x, y \in E$.

By construction, ρ is an invariant pseudo-metric on E and the space E/ρ is homeomorphic by a subspace of the topological product $\Pi\{E/\nu_n : n \in \mathbb{N}\}$. Moreover, we can assume that $p_\rho(x) = (\nu_n(x) : n \in \mathbb{N}) \in E/\rho \subset \Pi\{E/\nu_n : n \in \mathbb{N}\}$. Hence ρ is a continuous convex pseudo-metric on E. Since $V_{n+2} \subset \{x \in E : \rho(x, y) < 2^{-n-1}\} \subset V_n\}$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that from $x, y \in E$ and $d(x, y) < \delta$ it follows that $\rho(x, y) < \varepsilon$. Assertion 5 is proved. Assertions 1 - 3 follows from Assertion 1. Assume that $L \subset E$ and (L, d) is a complete metric space. Assume that $\{x_n \in L : n \in \mathbb{N}\}$ and $lim_{n,m\to\infty} d(x_n, x_m) = 0$. Fix $\varepsilon > 0$. There exists $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and $\rho(x_n, x_m) < 2^{-k-1}$ for $n, m \geq k$. Then $x_n - x_m \in V_k$ for $n, m \geq k$. Therefore $x_n - x_m\{x \in E : d(0, x) < 2^{-k}\}$ and $d(x_n, x_m) < \varepsilon$ for $n, m \geq k$. Hence, there exists a point $a \in L$ such that $lim_{n\to\infty} d(a, x_n) = 0$. Since d and ρ are equivalent metrics on E, we have $lim_{n\to\infty} \rho(a, x_n) = 0$ and ρ is complete on L. The proof is complete.

3. Local properties of families of subspaces and pseudometrics

Let d be a continuous pseudo-metric on a topological linear space E. A family $\mathcal{A}$ of subsets of a space E is called:

- weakly locally d-complete if for any element $L \in \mathcal{A}$ there exists an open subset U of E such that $L \cap U \neq \varnothing$ and the set $M \cap cl_E U$ is complete relatively to the pseudo-metric d

for any $M \in \mathcal{A}$;

- locally d-complete if for any element $L \in \mathcal{A}$ and any open subset V of E with $V \cap E \neq \varnothing$ there exists an open subset U of E such that $U \subset V$, $L \cap U \neq \varnothing$ and the set $M \cap cl_E U$ is complete relatively to the pseudo-metric d for any $M \in \mathcal{A}$;

- pseudo locally linear finite dimensional if for any element $L \in \mathcal{A}$ there exists an open subset U of E such that $L \cap U \neq \varnothing$ and $M \cap U$ for any $M \in \mathcal{A}$ is a subset of some finite dimensional linear subspace $Y(U, M)$ of E;

- almost locally linear finite dimensional if for any element $L \in \mathcal{A}$, each point $x \in L$ and any open subset V of X with $x \in V$ there exists an open subset U of E such that $U \subset V$, $U \cap L \neq \varnothing$ and $M \cap U$ is a subset of some linear subspace $Y(M, U)$ of E for any $M \in \mathcal{A}$;;

- weakly locally closed if for any element $L \in \mathcal{A}$ there exists an open subset U of E such that $L \cap U \neq \varnothing$ and $M \cap cl_E V$ is a closed subset of the subspace E for any $M \in \mathcal{A}$;

- locally closed if for any element $L \in \mathcal{A}$, each point $x \in L$ and any open subset V of E with $x \in V$ there exists an open subset U of E such that $U \subset V$, $U \cap L \neq \varnothing$ and $M \cap cl_E U$ is is a closed subset of the space E for any $M \in \mathcal{A}$.

The following assertion is well known and obvious

Proposition 3. *Let L be a convex subset of the topological linear space E, H be a linear subspace of E and U be an open subset of E. If $\varnothing \neq U \cap L \subset H$, then $L \subset H$.*

4. Main results

Theorem 4. *Let d be an invariant pseudo-metric on a topological locally convex space E, $\mathcal{A}$ be a family of non-empty convex subspaces of the space E metrizable by the pseudo-metric d and for any element $L \in \mathcal{A}$ there exists an open subset U of E such that $L \cap U \neq \varnothing$ and for any set $M \in \mathcal{A}$ the intersection $M \cap cl_E U$ is complete relatively to the pseudo-metric d. Assume that $F : X \longrightarrow E$ is a lower semicontinuous mapping of a paracompact space X into the space E and $F(x) \in \mathcal{A}$ for each $x \in X$. Then there exists a single-valued continuous mapping $f : X \longrightarrow E$ such that $f(x) \in F(x)$ for each point $x \in X$.*

Proof. By virtue of Proposition 2, we can assume that that d is a convex pseudo-metric on E. Consider the locally convex space E/d metrizable by the convex metric $\bar{d}$ and the continuous linear projection $p_d : E \longrightarrow E/d$.

We put $\mathcal{B} = \{L \cap \Phi : L \in \mathcal{A}, L \cap \Phi \neq \varnothing, \Phi \textit{ is a closed convex subset of } E\}$. Obviously, the family $\mathcal{B}$ is metrizable by the pseudo-metrics d.

Fix a point $a \in X$ and an open subset U of E such that $F(a) \cap U \neq \varnothing$ and for any set $M \in \mathcal{B}$ the intersection $M \cap cl_E U$ is complete relatively to the pseudo-metric d. Fix a point $b \in F(a) \cap U$, an open convex subset V of E and an open F_σ-subset W of X such that $b \in V \subset cl_E V \subset U$ and $a \in W \subset F^{-1}(V)$. We put $F_a(x) = F(x) \cap cl_E V$. Then $F_a : W \longrightarrow E$ is a lower semicontinuous mapping of a paracompact space W into the space E, $F_a(x) \in \mathcal{B}$ and the set $F_a(x)$ is convex and complete metrizable by the metric d for any point $x \in W$.

Now we put $\Psi_a(x) = p_d(F_a(x))$ for any $x \in W$. Then the mapping $\Psi_a : W \longrightarrow E/d$ is lower semicontinuous, convex-valued, closed-valued and for any $x \in W$ the image $\Psi_a(x)$ is a closed and convex subset of the metrizable by the convex metic d locally convex space E/d. Any set $\Psi_a(x)$ is complete by the convex metric $\bar{d}$.

Since W is a paracompact space, by virtue of Michael's Theorem from [11], there exists a single-valued continuous mapping $g_a : W \longrightarrow E/d$ such that $g_a(x) \in \Psi_a(x)$ for each $x \in W$.

By virtue of the reduction principles from [4, 5], there exists a single-valued continuous mapping $f_a : W \longrightarrow E$ such that $f_a(x) \in F_a(x) = F(x) \cap Y(F(a), F(x))$ and $g_a(x) = p_d(f_a(x))$ for each $x \in X_a$. Proposition 1 completes the proof.

Corollary 5. *Let d be an invariant pseudo-metrics on a topological locally convex space E, $\mathcal{A}$ be a family of non-empty convex subspaces of the space E metrizable by the pseudo-metric d. Assume that the family $\mathcal{A}$ is a weakly locally d-complete, $F : X \longrightarrow E$ is a lower semicontinuous mapping of a paracompact space X into the space E and $F(x) \in \mathcal{A}$ for each $x \in X$. Then there exists a single-valued continuous mapping $f : X \longrightarrow E$ such that $f(x) \in F(x)$ for each point $x \in X$.*

Proof. Fix an element $L \in \mathcal{A}$.

Since $\mathcal{A}$ is weakly locally closed there exist an open subset U_1 of the space E such that $L \cap U_1 \neq \varnothing$ and $M \cap cl_E U_1$ is a closed subset of E for any $M \in \mathcal{A}$.

Since $\mathcal{A}$ is a locally linear finite-dimensional family, there exist a number $n(L) \in \mathbb{N}$, an open subset U_2 of the space E and the linear subspaces $Y(L, M) \subset E$, $M \in \mathcal{A}$, of the dimension $\leq n(L)$ such that $U_2 \subset U_1$, $U_2 \cap L \neq \varnothing$ and $M \cap U_2 \subset Y(L, M)$ for each $M \in \mathcal{A}$.

Since $\mathcal{A}$ is a locally convex family, there exists an open subset U of the space E such that $U \subset U_2$, $U \cap L \neq \varnothing$ and $M \cap U \subset Y(L, M)$ is a convex subset of the linear subspace $Y(L, M)$ for each $M \in \mathcal{A}$. Then for any $x \in U$ the set $F(x) \cap cl_Y U$ is a closed convex subset of the linear subspace $Y(a, x)$ of dimension $\leq n(a)$. Theorem 3 completes the proof.

Corollary 6. *Let d be an invariant pseudo-metrics on a topological locally convex space E, $\mathcal{A}$ be a family of non-empty convex subspaces of the space E metrizable by the pseudo-metric d. Assume that the family $\mathcal{A}$ is pseudo locally linear finite-dimensional and weakly locally closed, and $F : X \longrightarrow E$ is a lower semicontinuous mapping of a paracompact space X into the space E such that $F(x) \in \mathcal{A}$ for each $x \in X$. Then there exists a single-valued continuous mapping $f : X \longrightarrow E$ such that $f(x) \in F(x)$ for each point $x \in X$.*

Proof. Since $\mathcal{A}$ is a weakly locally linear finite-dimensional family, then from Proposition 3 it follows that for any point $x \in X$ there exists a finite-dimensional linear subspace $Y(x)$ of E such that $F(x) \subset Y(x)$. We can assume that $\mathcal{A} = \{F(x) : x \in X\}$.

Fix $a \in X$ and put $L = F(a)$. Since $\mathcal{A}$ is weakly locally closed there exists an open subset U of the space E such that $L \cap U \neq \varnothing$ and $M \cap cl_E U$ is a closed subset of E for any $M \in \mathcal{A}$. We put $V = F^{-1}(U)$. Then $\{F(x) : x \in V = \{M \in \mathcal{A} : M \cap U \neq \varnothing\}$.

By virtue of Proposition 2, we can assume that that d is a convex pseudo-metric on E. Consider the topological linear space E/d metrizable by the convex metric $\bar{d}$ and the continuous projection $p_d : E \longrightarrow E/d$.

By virtue of the V.L. Klee theorem [10], the metric $\bar{d}$ is complete on any finite dimensional linear subspace L of E/d (see also [6]). In particular, any subspace $p_d(Y(x))$ is complete relatively to the metric $\bar{d}$. Hence, for any point $x \in V$, the set $F(x) \cap cl_E U$ is complete relatively to the pseudo-metric d. Theorem 4 completes the proof.

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