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ON A CASE OF STABILITY OF UNPERTURBATED MOTION GOVERNED BY THE TERNARY DIFFERENTIAL CRITICAL SYSTEM WITH QUADRATIC NONLINEARITIES

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Abstract. The Lyapunov series and the conditions of stability of unperturbed motion has been determined by ternary differential critical system with quadratic nonlinearities in a non-singular case.

Keywords: Differential systems, stability of unperturbed motion, center-affine comitant and invariant, group.

ASUPRA UNUI CAZ DE STABILITATE A MIȘCĂRII NEPERTURBATE GUVERNATE DE SISTEMUL CRITIC DIFERENȚIAL TERNAR CU NELINIARITĂȚI PĂTRATICE

Rezumat. A fost determinată seria Lyapunov și condițiile de stabilitate a mișcării neperturbate guvernate de sistemul critic diferențial ternar cu neliniarități pătratice într-un caz nesingular.

Cuvinte-cheie: Sistem diferențial, stabilitatea mișcării neperturbate, comitanți și invatianți centro-afini, grup.

1. Introduction

We will consider the ternary differential system with quadratic nonlinearities $s^3(1,2)$ of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + cz + a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4xy + 2a_5xz + 2a_6yz, \\ \frac{dy}{dt} &= px + qy + rz + b_1 x^2 + b_2 y^2 + b_3 z^2 + 2b_4xy + 2b_5xz + 2b_6yz, \\ \frac{dz}{dt} &= sx + my + nz + c_1 x^2 + c_2 y^2 + c_3 z^2 + 2c_4xy + 2c_5xz + 2c_6yz, \end{aligned} \tag{1}$$

and the center-affine group $GL(3,\mathbb{R})$ given by transformations

$$\begin{aligned} \bar{x} &= \alpha_1 x + \alpha_2 y + \alpha_3 z, \\ \bar{y} &= \beta_1 x + \beta_2 y + \beta_3 z, \\ \bar{z} &= \gamma_1 x + \gamma_2 y + \gamma_3 z, \end{aligned} \quad \left(\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \right). \tag{2}$$

The coefficients and variables in (1) and (2) are given over the field of real numbers $\mathbb{R}$.

Let us write the system (1) in the tensorial form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j,\alpha,\beta = 1,2,3), \tag{3}$$

where $a^j_{\alpha\beta}$ is a symmetric tensors in lower indices in which the total convolution is done. Consider the non-singular invariant variety

$$\sigma_1 = a^{\alpha}_{\mu} a^{\beta}_{\delta} a^{\gamma}_{\alpha} x^{\delta} x^{\mu} x^{\nu} \varepsilon_{\beta\gamma\nu} \not\equiv 0 \tag{4}$$

$(\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1$ and $\varepsilon_{\beta\gamma\nu} = 0$ for the others $\beta\gamma\nu)$, where σ_1 is a center-affine comitant [1] for the system (3).

According to [1] and [2] under the condition (4) there exists a center-affine transformation (2) such that the system (3) can be brought to the form

$$\begin{aligned}
\frac{dx^1}{dt} &= x^2 + a^1_{\alpha\beta} x^{\alpha} x^{\beta},\\
\frac{dx^2}{dt} &= x^3 + a^2_{\alpha\beta} x^{\alpha} x^{\beta},\\
\frac{dx^3}{dt} &= a^3_1 x^1 + a^3_2 x^2 + a^3_3 x^3 + a^1_{\alpha\beta} x^{\alpha} x^{\beta},
\end{aligned} \tag{5}$$

where the nonlinear part of (5) contains five nonzero coefficients.

Passing from system (3) to system (1), which represents the same system with different notations of the coefficients and variables, we obtain that under the condition (4) the system (1) by a center-affine transformation (2) can be brought to the form

$$\begin{aligned}
\frac{dx}{dt} &= y + a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4 xy + 2a_5 xz + 2a_6 yz,\\
\frac{dy}{dt} &= z + b_1 x^2 + b_2 y^2 + b_3 z^2 + 2b_4 xy + 2b_5 xz + 2b_6 yz,\\
\frac{dz}{dt} &= sx + my + nz + c_1 x^2 + c_2 y^2 + c_3 z^2 + 2c_4 xy + 2c_5 xz + 2c_6 yz.
\end{aligned} \tag{6}$$

In the case of tensor terms $a^j_{\alpha\beta} x^{\alpha} x^{\beta}$ $(j, \alpha, \beta = 1,2,3)$ this system completely coincides by the form with the system (5).

2. The critical system and Lyapunov stability

Let us write for system (6) the characteristic equation

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ s & m & n-\lambda \end{vmatrix} = 0,$$

which is equivalent to the equation

$$\lambda^3 - n\lambda^2 - m\lambda - s = 0. \tag{7}$$

From (7) we obtain

Lemma 1. *The characteristic equation* (7) *of system* (6) *has one a zero root if and only if*

$$s = 0, m \neq 0. \tag{8}$$

Under conditions (8), the critical system (6) by a center-affine transformation

$$\bar{x} = -mx - ny + z, \qquad \bar{y} = y, \qquad \bar{z} = z \tag{9}$$

can be brought to form

$$\frac{dx}{dt} = a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4 xy + 2a_5 xz + 2a_6 yz,$$

$$\frac{dy}{dt} = z + b_1\, x^2 + b_2\, y^2 + b_3\, z^2 + 2b_4xy + 2b_5xz + 2b_6yz, \tag{10}$$
$$\frac{dz}{dt} = my + nz + c_1\, x^2 + c_2\, y^2 + c_3\, z^2 + 2c_4xy + 2c_5xz + 2c_6yz.$$

We will examine the case when

$$a_4 \neq 0. \tag{11}$$

According to [3] we analyze the non-critical equations

$$\begin{gathered} z + b_1\, x^2 + b_2\, y^2 + b_3\, z^2 + 2b_4xy + 2b_5xz + 2b_6yz = 0, \\ my + nz + c_1\, x^2 + c_2\, y^2 + c_3\, z^2 + 2c_4xy + 2c_5xz + 2c_6yz = 0. \end{gathered} \tag{12}$$

From the first equation of (12) we express z and from the second one y. We obtain that

$$\begin{gathered} y = -\frac{n}{m}z - \frac{c_1}{m}\, x^2 - \frac{c_2}{m}\, y^2 - \frac{c_3}{m}\, z^2 - 2\frac{c_4}{m}xy - 2\frac{c_5}{m}xz - 2\frac{c_6}{m}yz, \\ z = -b_1\, x^2 - b_2\, y^2 - b_3\, z^2 - 2b_4xy - 2b_5xz - 2b_6yz. \end{gathered} \tag{13}$$

We shall seek y and z as a holomorphic functions of x:

$$\begin{gathered} y(x) = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + \\ + A_9x^9 + A_{10}x^{10} + \cdots, \\ z(x) = B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + \\ + B_9x^9 + B_{10}x^{10} + \cdots. \end{gathered} \tag{14}$$

Substituting (14) into (13) we obtain

$$\begin{gathered} A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots = \\ = -\frac{n}{m}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \\ + B_{10}x^{10} + \cdots) - \frac{c_1}{m}\, x^2 - \frac{c_2}{m}\, (A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + \\ + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots)^2 - \frac{c_3}{m}\, (B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \\ + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots)^2 - 2\frac{c_4}{m}x(A_1x + A_2x^2 + A_3x^3 + \\ + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots) - 2\frac{c_5}{m}x(B_1x + \\ + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots) - \\ - 2\frac{c_6}{m}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + \\ + A_{10}x^{10} + \cdots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + \\ + B_9x^9 + B_{10}x^{10} + \cdots), \\ B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots = \\ = -b_1\, x^2 - b_2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + \\ + A_9x^9 + A_{10}x^{10} + \cdots)^2 - b_3\, (B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + \\ B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots)^2 - 2b_4x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \\ + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots) - 2b_5x(B_1x + B_2x^2 + \\ + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots) - \\ - 2b_6(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + \end{gathered}$$

$$+A_{10}x^{10}+\cdots)(B_1x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+$$
$$+B_9x^9+B_{10}x^{10}+\cdots).$$

From here we find

$$A_1=B_1=0,\quad A_2=-\frac{1}{m}(c_1+B_2n),\quad B_2=-b_1,$$
$$A_3=-\frac{1}{m}(2A_2c_4+2B_2c_5+B_3n),\quad B_3=-2(A_2b_4+B_2b_5),$$
$$A_4=-\frac{1}{m}(A_2^2c_2+B_2^2c_3+2A_3c_4+2B_3c_5+2A_2B_2c_6+B_4n),$$
$$B_4=-A_2^2b_2-B_2^2b_3-2A_3b_4-2B_3b_5-2A_2B_2b_6,$$
$$A_5=-\frac{1}{m}[2A_2A_3c_2+2B_2B_3c_3+2A_4c_4+2B_4c_5+2(A_3B_2+A_2B_3)c_6+B_5n],$$
$$B_5=-2[A_2A_3b_2+B_2B_3b_3+A_4b_4+B_4b_5+(A_3B_2+A_2B_3)b_6],$$
$$A_6=-\frac{1}{m}[(A_3^2+2A_2A_4)c_2+(B_3^2+2B_2B_4)c_3+2A_5c_4+2B_5c_5+$$
$$+2(A_4B_2+A_3B_3+A_2B_4)c_6+B_6n],$$
$$B_6=-(A_3^2+2A_2A_4)b_2-(B_3^2+2B_2B_4)b_3-2A_5b_4-2B_5b_5-$$
$$-2(A_4B_2+A_3B_3+A_2B_4)b_6,$$
$$A_7=-\frac{1}{m}[2(A_3A_4+A_2A_5)c_2+2(B_3B_4+B_2B_5)c_3+2A_6c_4+2B_6c_5+$$
$$+2(A_5B_2+A_4B_3+A_3B_4+A_2B_5)c_6+B_7n],$$
$$B_7=-2[(A_3A_4+A_2A_5)b_2+(B_3B_4+B_2B_5)b_3+A_6b_4+B_6b_5+$$
$$+(A_5B_2+A_4B_3+A_3B_4+A_2B_5)b_6],$$
$$A_8=-\frac{1}{m}[(A_4^2+2A_3A_5+2A_2A_6)c_2+(B_4^2+2B_3B_5+2B_2B_6)c_3+2A_7c_4+2B_7c_5+$$
$$+2(A_6B_2+A_5B_3+A_4B_4+A_3B_5+A_2B_6)c_6+B_8n],$$
$$B_8=-(A_4^2+2A_3A_5+2A_2A_6)b_2-(B_4^2+2B_3B_5+2B_2B_6)b_3-2A_7b_4-2B_7b_5-$$
$$-2(A_6B_2+A_5B_3+A_4B_4+A_3B_5+A_2B_6)b_6,$$
$$A_9=-\frac{1}{m}[2(A_4A_5+A_3A_6+A_2A_7)c_2+2(B_4B_5+B_3B_6+B_2B_7)c_3+2A_8c_4+$$
$$+2B_8c_5+2(A_7B_2+A_6B_3+A_5B_4+A_4B_5+A_3B_6+A_2B_7)c_6+B_9n],$$
$$B_9=-2[(A_4A_5+A_3A_6+A_2A_7)b_2+(B_4B_5+B_3B_6+B_2B_7)b_3+A_8b_4+$$
$$+B_8b_5+(A_7B_2+A_6B_3+A_5B_4+A_4B_5+A_3B_6+A_2B_7)b_6],$$
$$A_{10}=-\frac{1}{m}[(A_5^2+2A_4A_6+2A_3A_7+2A_2A_8)c_2+(B_5^2+2B_4B_6+2B_3B_7+$$
$$+2B_2B_8)c_3+2A_9c_4+2B_9c_5+2(A_8B_2+A_7B_3+A_6B_4+A_5B_5+$$
$$+A_4B_6+A_3B_7+A_2B_8)c_6+B_{10}n],$$
$$B_{10}=-(A_5^2+2A_4A_6+2A_3A_7+2A_2A_8)b_2-(B_5^2+2B_4B_6+2B_3B_7+$$
$$+2B_2B_8)b_3-2A_9b_4+2B_9b_5-2(A_8B_2+A_7B_3+A_6B_4+A_5B_5+$$
$$+A_4B_6+A_3B_7+A_2B_8)b_6,\ldots \quad (15)$$

Substituting (14) into the right-hand side of the critical differential equation (10) we get

$$a_1\, x^2 + a_2\, y^2 + a_3\, z^2 + 2a_4xy + 2a_5xz + 2a_6yz =$$
$$= C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7 + C_8x^8 + C_9x^9 + C_{10}x^{10} + \cdots,$$

or in the unfolded form

$$a_1\, x^2 + a_2\, (A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 +$$
$$+A_{10}x^{10} + \cdots)^2 + a_3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 +$$
$$+B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots)^2 + 2a_4x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 +$$
$$+A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots) + 2a_5x(B_1x + B_2x^2 + B_3x^3 +$$
$$+B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots) +$$
$$+2a_6(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 +$$
$$+A_9x^9 + A_{10}x^{10} + \cdots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 +$$
$$+B_8x^8 + B_9x^9 + B_{10}x^{10} + \cdots) =$$
$$= C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7 + C_8x^8 + C_9x^9 + C_{10}x^{10} + \cdots$$

From this, it follows that

$$C_2 = 0, \qquad C_2 = a_1, \qquad C_3 = 2(a_4A_2 + a_5B_2),$$
$$C_4 = 2a_4A_3 + 2a_5B_3 + a_2A_2^2 + 2a_6A_2B_2 + a_3B_2^2,$$
$$C_5 = 2[a_4A_4 + a_5B_4 + a_2A_2A_3 + a_6(A_2B_3 + A_3B_2) + a_3B_2B_3],$$
$$C_6 = 2a_4A_5 + 2a_5B_5 + a_2(2A_2A_4 + A_3^2) + 2a_6(A_2B_4 + A_3B_3 + A_4B_2) +$$
$$+a_3(2B_2B_4 + B_3^2),$$
$$C_7 = 2[a_4A_6 + a_5B_6 + a_2(A_2A_5 + A_3A_4) + a_6(A_2B_5 + A_3B_4 + A_4B_3 + A_5B_2) +$$
$$+a_3(B_2B_5 + B_3B_4)],$$
$$C_8 = 2a_4A_7 + 2a_5B_7 + a_2(2A_2A_6 + 2A_3A_5 + A_4^2) + 2a_6(A_2B_6 + A_3B_5 +$$
$$+A_4B_4 + A_5B_3 + A_6B_2) + a_3(2B_2B_6 + 2B_3B_5 + B_4^2),$$
$$C_9 = 2[a_4A_8 + a_5B_8 + a_2(A_2A_7 + A_3A_6 + A_4A_5) + a_6(A_2B_7 + A_3B_6 +$$
$$+A_4B_5 + A_5B_4 + A_6B_3 + A_7B_2) + a_3(B_2B_7 + B_3B_6 + B_4B_5)],$$
$$C_{10} = 2a_4A_9 + 2a_5B_9 + a_2(2A_2A_8 + 2A_3A_7 + 2A_4A_6 + A_5^2) + 2a_6(A_2B_8 + A_3B_7 +$$
$$+A_4B_6 + A_5B_5 + A_6B_4 + A_7B_3 + A_8B_2) +$$
$$+a_3(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2), \ldots \qquad (16)$$

We introduce the following notations:

$$A = a_4c_1 + a_5b_1m - a_4b_1n;$$
$$B = -4a_4^2a_5c_4 + 4a_4^3c_5 + a_3a_4^2b_1m + a_2a_5^2b_1m - 2a_4a_5a_6b_1m - 4a_4a_5^2b_4m +$$
$$+4a_4^2a_5b_5m + 4a_4^2a_5b_4n - 4a_4^3b_5n;$$
$$C = 2a_4^3a_5^2c_2 + 2a_4^5c_3 - 2a_3a_4^4c_4 - 2a_2a_4^2a_5^2c_4 + 4a_4^3a_5a_6c_4 - 4a_4^4a_5c_6 +$$
$$+a_2a_3a_4^2a_5b_1m + a_2^2a_5^3b_1m - a_3a_4^3a_6b_1m - 3a_2a_4a_5^2a_6b_1m + 2a_4^2a_5a_6^2b_1m +$$
$$+2a_4^2a_5^3b_2m + 2a_4^4a_5b_3m - 4a_3a_4^3a_5b_4m - 4a_2a_4a_5^3b_4m + 8a_4^2a_5^2a_6b_4m +$$
$$+2a_3a_4^4b_5m + 2a_2a_4^2a_5^2b_5m - 4a_4^3a_5a_6b_5m - 4a_4^3a_5^2b_6m - 2a_4^3a_5^2b_2n - 2a_4^5b_3n +$$
$$+2a_3a_4^4b_4n + 2a_2a_4^2a_5^2b_4n - 4a_4^3a_5a_6b_4n + 4a_4^4a_5b_6n;$$

$$D = a_3a_4^2 + a_2a_5^2 - 2a_4a_5a_6;$$
$$E = 8a_4^3a_5c_2 - 8a_2a_4^2a_5c_4 + 8a_4^3a_6c_4 - 8a_4^4c_6 + a_2a_3a_4^2b_1m + 5a_2^2a_5^2b_1m -$$
$$-10a_2a_4a_5a_6b_1m + 4a_4^2a_6^2b_1m + 12a_4^2a_5^2b_2m + 4a_4^4b_3m - 4a_3a_4^3b_4m -$$
$$-20a_2a_4a_5^2b_4m + 24a_4^2a_5a_6b_4m + 8a_2a_4^2a_5b_5m - 8a_4^3a_6b_5m - 16a_4^3a_5b_6m -$$
$$-8a_4^3a_5b_2n + 8a_2a_4^2a_5b_4n - 8a_4^3a_6b_4n + 8a_4^4b_6n;$$
$$F = a_4^3c_2 - a_2a_4^2c_4 + a_2^2a_5b_1m - a_2a_4a_6b_1m + 3a_4^2a_5b_2m - 4a_2a_4a_5b_4m +$$
$$+2a_4^2a_6b_4m + a_2a_4^2b_5m - 2a_4^3b_6m - a_4^3b_2n + a_2a_4^2b_4n;$$
$$G = a_2^2b_1 + 4a_4^2b_2 - 4a_2a_4b_4. \qquad (17)$$

Theorem 1. *The stability of unperturbed motion in the system of perturbed motion* (10), *when $a_4 \neq 0$ and $m, n < 0$, is described by one of the following possible cases:*

I. $a_1 \neq 0$, *then the unperturbed motion is unstable;*

II. $a_1 = 0$, $A > 0$, *then the unperturbed motion is unstable;*

III. $a_1 = 0, A < 0$, *then the unperturbed motion is stable;*

IV. $a_1 = A = 0$, $b_1B \neq 0$, *then the unperturbed motion is unstable;*

V. $a_1 = A = B = 0$, $b_1 \neq 0$, $C > 0$, *then the unperturbed motion is unstable;*

VI. $a_1 = A = B = 0$, $b_1 \neq 0$, $C < 0$, *then the unperturbed motion is stable;*

VII. $a_1 = A = B = C = 0$, $b_1DE \neq 0$, *then the unperturbed motion is unstable;*

VIII. $a_1 = A = B = C = E = 0$, $b_1D \neq 0$, $F > 0$, *then the unperturbed motion is unstable;*

IX. $a_1 = A = B = C = E = 0$, $b_1D \neq 0$, $F < 0$, *then the unperturbed motion is stable;*

X. $a_1 = A = B = C = E = F = 0$, $b_1DG \neq 0$, *then the unperturbed motion is unstable;*

XI. $a_1 = b_1 = A = 0$, *then the unperturbed motion is stable;*

XII. $a_1 = A = B = C = D = 0$, *then the unperturbed motion is stable;*

XIII. $a_1 = A = B = C = E = F = G = 0$, *then the unperturbed motion is stable.*

In the last three cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. The expressions for A, B, C, D, E, F, G are given in (17).

Proof. According to Lyapunov Theorem [3, §32], the coefficients of C_i series from (16) are analyzed.

If $C_2 \neq 0$, then we obtain the Case I of Theorem 1.

If $C_2 = 0$, then $a_1 = 0$ and $C_3 = -2m^{-1}A$. Then the stability or the instability of unperturbed motion is determined by the sign of expression C_3. By Lyapunov Theorem [3, §32] we get the Cases II and III.

Suppose that $a_1 = A = 0$. Then from (16) we have $C_4 = -a_4^{-2}b_1m^{-1}B$. If $b_1B \neq 0$, then we get the Cases IV.

Suppose that $a_1 = b_1 = A = 0$. Then $C_i = 0$ for all i. By Lyapunov Theorem [3, §32] we obtain the Case XI.

Assume that $a_1 = A = B = 0$ and let $b_1 \neq 0$. Then from (16) it follows that $C_5 = -a_4^{-2} b_1^2 m^{-1} C$. The stability or the instability of unperturbed motion is determined by the sign of expression C. According to Lyapunov Theorem [3, §32] we have the Cases V and VI.

Assume that $a_1 = A = B = C = 0$ and let $b_1 \neq 0$. Then (16) implies $C_6 = 4^{-1} a_4^{-6} b_1^3 m^{-1} DE$. If $DE \neq 0$, then we get the Cases VII.

When $a_1 = A = B = C = D = 0$ and $b_1 \neq 0$, then $C_i = 0$ for all i. By Lyapunov Theorem [3, §32] we obtain the Case XII.

Suppose that $a_1 = A = B = C = E = 0$ and let $b_1 D \neq 0$. Then from (16) it results that $C_7 = -2^{-1} a_4^{-8} b_1^4 m^{-1} D^2 F$. The stability or the instability of unperturbed motion is determined by the sign of expression F. According to Lyapunov Theorem [3, §32] we get the Cases VII and IX.

Assume that $a_1 = A = B = C = E = F = 0$ and let $b_1 D \neq 0$. Then from (16) we find that $C_8 = 16^{-1} a_4^{-10} b_1^5 D^3 G$. If $G \neq 0$, then we get the Cases X.

When $a_1 = A = B = C = E = F = 0$, then $C_i = 0$ for all i. By Lyapunov Theorem [3, §32] we have the Case XIII. Theorem is proved.

Conclusions

In this paper the Lyapunov series for the ternary differential critical system with quadratic nonlinearities was constructed. The invariant conditions for stability of unperturbed motion were obtained and included in thirteen cases.

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