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THE CAUCHY PROBLEM FOR A PARABOLIC SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS WITH AN OPERATOR OF VOLTERRA-FREDHOLM TYPE

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Abstract. The Cauchy problem for a parabolic system of integro-differential equations with an operator of Volterra-Fredholm type is considered. A fundamental matrix of solutions of the problem in classical Hölder spaces is constructed, the estimates for the matrix and its derivatives are established. This makes it possible to prove the correctness theorem.

Keywords: parabolic system of integro-differential equations, Volterra-Fredholm operator, fundamental matrix of solutions, method of reduction to a system of integral equations, kernels of the integral operator, resolvent, conditions for correct solvability.

PROBLEMA CAUCHY PENTRU UN SISTEM PARABOLIC DE ECUAŢII INTEGRO DIFERENŢIALE CU UN OPERATOR DE TIP VOLTERRA-FREDHOLM

Rezumat. Se consideră problema Cauchy pentru un sistem parabolic de ecuaţii integro diferenţiale cu un operator de tip Volterra-Fredholm. Se construieşte o matrice fundamentală a soluţiilor problemei în spaţiile clasice Hölder, sunt stabilite estimările pentru matrice şi derivatele ei. Acest lucru face posibilă demonstrarea teoremei corectitudinii.

Cuvinte cheie: sistem parabolic de ecuaţii integro diferenţiale, operator Volterra-Fredholm, matricea fundamentală a soluţiilor, metoda de reducere la un sistem de ecuaţii integrale, nuclee ale operatorului integral, rezolvent, condiţii pentru solvabilitate corectă.

Introduction

The theory of the correct solvability of the Cauchy problem and the boundary value problems is almost completely developed for general Petrovsky parabolic systems [1]–[5]. The considerable theoretical interest is the further study of problems for parabolic systems, including those that contain integro-differential operators (IDOs) [6]–[8]. Such operators arise in mathematical modeling of processes in control theory, problems of financial theory, thermomechanics, acoustics, viscoelasticity, mathematical biology, where it is assumed that there is a global mechanism that affects the process itself. Parabolic IDOs also arise in problems associated with stochastic Markov-Feller processes with jumps [9].

In recent years, IDO problems for different classes of differential and integro-differential equations (IDEs) were studied in [10]–[18] and others. Moreover, those classes of problems, that were studied by imposing additional conditions or restrictions to the equation, boundary conditions, or the region in which the problem has been considered, have to be correctly posed.

The article investigates the fundamental matrix of solutions (FMS) of the Cauchy problem for a parabolic system of IDEs with an operator of Volterra-Fredholm type. Using the method of reduction to a system of integral equations with the kernel that is expressed through the kernel of the IDO of the system and the FMS of the corresponding parabolic system, the FMS of the Cauchy problem in classical Hölder spaces $C^{m+\alpha}(\Pi)$ is constructed. This is the class of functions $u(t,x)$, that have in $\Pi = (0,T) \times \mathbb{R}^n$ continuous derivatives $D_x^k u$, $D_x^k u \equiv D_{x_1}^{k_1} \cdots D_{x_n}^{k_n} u$, $k = (k_1, ..., k_n)$, $|k| = k_1 + ... + k_n$, up to and including m, elder derivatives are Hölder with exponent α, $0 < \alpha < 1$, and the norm

$$|u|_{m+\alpha} = \sum_{|k|\le m} \sup_{(t,x)\in\Pi} |D_x^k u(t,x)| + \sum_{|k|=m} \sup_{(t,x),(t,x+\Delta x)\in\Pi} \frac{|\Delta_x D_x^k u(t,x)|}{|\Delta x|^\alpha},$$

$$|\Delta x| = (\Delta x_1^2 + ... + \Delta x_n^2)^{1/2}$$

is finite.

It is established that the smoothness of the solution of the Cauchy problem depends on not only the smoothness of the initial function, but also on the differential properties of the kernels of the IDOs of the initial system. And here the question arises of what kind and under what conditions the IDO is included in the system itself for the possibility of establishing the correctness of the Cauchy problem in the framework of the classical theory.

1. Problem statement

In the domain $\Pi = (0,T) \times \mathbb{R}^n$ we consider the Cauchy problem for the uniformly parabolic system of N IDEs

$$\begin{gathered} L(t,x,D,B)u \equiv \frac{\partial u}{\partial t} - \sum_{|k|\le 2b} A_k(t,x) D_x^k u - \\ - \int_0^t d\tau \int_{\mathbb{R}^n} \sum_{|s|\le p} B_s(t,\tau,x,\xi) D_\xi^s u(\tau,\xi)\, d\xi = f(t,x), \end{gathered} \tag{1}$$

$$u\big|_{t=0} = \varphi(x), \qquad x \in \mathbb{R}^n, \qquad 0 \le p \le 2b-1. \tag{2}$$

The definition of uniform parabolicity of the system (1), only without Volterra-Fredholm-type IDOs, is given in [1].

Let's further construct the FMS of the Cauchy problem (1) – (2), and also establish conditions under which it can be built, and find estimates for the FMS and its derivatives.

2. Construction of the FMS of the Cauchy problem, obtaining estimates of the FMS and its derivatives

Let's introduce the substitution

$$\frac{\partial u}{\partial t} - \sum_{|k|\le 2b} A_k(t,x) D_x^k u = y(t,x), \tag{3}$$

where $y(t,x)$ – is an unknown N-vector-function. Then the problem (3), (2) is the Cauchy problem for a parabolic system with heterogeneity $y(t,x)$. For this Cauchy problem under the assumption that $y \in C_x^\alpha(\Pi)$ the solution is written through the FMS Z

$$Z(t,\tau,x,\xi) = G_0(t,\tau,x-\xi,\xi) + \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} G_0(t,\beta,x-y,y) R_0(\beta,\tau,y,\xi) dy \equiv G_0 + W_0$$

in the form [1, c. 269]

$$u(t,x) = \int\limits_{\mathbb{R}^n} Z(t,0,x,\xi)\varphi(\xi)\, d\xi + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} Z(t,\tau,x,\xi) y(\tau,\xi) d\xi \tag{4}$$

and for the derivatives Z the next estimates

$$\begin{gathered} \left| D_x^k G_0(t,\tau,x,\xi) \right| \le C(t-\tau)^{-\frac{n+|k|}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \\ \left| D_x^k W_0(t,\tau,x,\xi) \right| \le C(t-\tau)^{-\frac{n+|k|-\alpha}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \\ |k| \le 2b, \quad t>\tau, \quad x,\xi \in \mathbb{R}^n \end{gathered} \tag{5}$$

are correct.

Hereinafter, the letter C with and without indices will denote positive limited finite constants.

The FMS Z is the main component in constructing the FMS of the Cauchy problem for the considered parabolic system of IDEs. We substitute the expression for u (4) to the initial system (1):

$$\begin{aligned} y(t,x) = f(t,x) + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} \sum_{|s|\le p} B_s(t,\tau,x,\xi) \int\limits_{\mathbb{R}^n} D_\xi^s Z(\tau,0,\xi,z)\varphi(z) dz d\xi + \\ + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} \sum_{|s|\le p} B_s(t,\tau,x,\xi) \int\limits_0^\tau d\beta \int\limits_{\mathbb{R}^n} D_\xi^s Z(\tau,\beta,\xi,z) y(\beta,z) dz d\xi. \end{aligned} \tag{6}$$

Thus we have the Volterra-Fredholm integral equation of the second kind with respect to the unknown vector function y. In the last integral by the formula Dirichlet we change the order of integration and immediately redefine the integration variables

$$\int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} \sum_{|s|\le p} B_s(t,\tau,x,\xi) \int\limits_0^\tau d\beta \int\limits_{\mathbb{R}^n} D_\xi^s Z(\tau,\beta,\xi,z) y(\beta,z) dz d\xi =$$

$$= \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} \left[\int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} \sum_{|s|\le p} B_s(t,\beta,x,z) D_z^s Z(\beta,\tau,z,\xi)dz \right] y(\tau,\xi)d\xi$$

to highlight the kernel, which we denote by

$$H(t,\tau,x,\xi) \equiv \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} \sum_{|s|\le p} B_s(t,\beta,x,z) D_z^s Z(\beta,\tau,z,\xi)dz.$$

If in the second term of the integral equation (6) we change the order of integration over spatial variables, it will hightligt the kernel H and then the equation can be written in the form

$$y(t,x) = F(t,x) + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} H(t,\tau,x,\xi) y(\tau,\xi)d\xi, \tag{7}$$

where

$$F(t,x) \equiv f(t,x) + \int\limits_{\mathbb{R}^n} H(t,0,x,\xi)\varphi(\xi)d\xi.$$

According to the theory of integral equations with regular or with a quasiregular kernel, the kernel $H(t,\tau,x,\xi)$ corresponds to a resolvent

$$R(t,\tau,x,\xi) = \sum_{\nu=1}^{\infty} H_\nu(t,\tau,x,\xi),$$

where

$$H_\nu(t,\tau,x,\xi) = \int\limits_\tau^t d\beta \int\limits_{\mathbb{R}^n} H_1(t,\beta,x,z) H_{\nu-1}(\beta,\tau,z,\xi)dz, \quad \nu = 2,3,..., \quad H_1 = H.$$

The quasiregularity of the kernel H is established below. So we get a solution of the equation (7) in the form

$$y(t,x) = F(t,x) + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} R(t,\tau,x,\xi) F(\tau,\xi)d\xi$$

or

$$y(t,x) = f(t,x) + \int\limits_{\mathbb{R}^n} H(t,0,x,\xi)\varphi(\xi)d\xi + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} R(t,\tau,x,\xi) f(\tau,\xi)d\xi +$$

$$+ \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} R(t,\tau,x,\xi) \int\limits_{\mathbb{R}^n} H(\tau,0,\xi,z)\varphi(z)dzd\xi. \tag{8}$$

Thus, if we substitute the found solution $y(t,x)$ in the representation for u (4), then the solution of the initial Cauchy problem (1) – (2) will take the form

$$u(t,x) = \int\limits_{\mathbb{R}^n} Z(t,0,x,\xi)\varphi(\xi)d\xi + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} Z(t,\tau,x,\xi) f(\tau,\xi)d\xi +$$

$$+\int_0^t d\tau \int_{\mathbb{R}^n} Z(t,\tau,x,\xi) \int_0^\tau d\beta \int_{\mathbb{R}^n} R(\tau,\beta,\xi,z) f(\beta,z) dz d\xi +$$

$$+\int_0^t d\tau \int_{\mathbb{R}^n} \Big\{ Z(t,\tau,x,\xi) \int_{\mathbb{R}^n} \Big(H(\tau,0,\xi,\zeta) +$$

$$+\int_0^\tau d\beta \int_{\mathbb{R}^n} R(\tau,\beta,\xi,z) H(\beta,0,z,\zeta) dz \Big) \varphi(\zeta) d\zeta \Big\} d\xi. \qquad (9)$$

Let's write the solution (9) through the operator of symbolic convolution

$$u = (Z + Z**R)**f + Z*\varphi + Z**[(H + R**H)*\varphi]$$

and consider the last two terms

$$Z*\varphi + Z**[(H+R**H)*\varphi] = \{Z + Z**H + [Z**(R**H)]\}*\varphi =$$

$$= \{Z + Z**[H + R**H]\}*\varphi = \{Z + Z**R\}*\varphi,$$

because of $R = \sum\limits_{\nu=1}^{\infty} H_\nu$ and $H + R**H = R$. As a result, for the solution we get an image

$$u = \{Z + Z**R\}*\varphi + \{Z + Z**R\}**f.$$

So, the kernel of the inverse operator of the Cauchy problem (1) – (2) is allocated

$$\mathbf{\Gamma}(t,\tau,x,\xi) \equiv Z(t,\tau,x,\xi) + \int_\tau^t d\beta \int_{\mathbb{R}^n} Z(t,\beta,x,z) R(\beta,\tau,z,\xi) dz, \qquad (10)$$

by which the decision is written in the form

$$u(t,x) = \int_{\mathbb{R}^n} \mathbf{\Gamma}(t,0,x,\xi)\varphi(\xi)\, d\xi + \int_0^t d\tau \int_{\mathbb{R}^n} \mathbf{\Gamma}(t,\tau,x,\xi) f(\tau,\xi)\, d\xi.$$

Let's determine the conditions, under which the resolvent R can be constructed for the equation (7). Let's pretend that

$$|D_x^m B_s(t,\tau,x,\xi)| \le C_{ms}(t-\tau)^{-\frac{n+2b+|m|-\alpha}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \qquad (11)$$

$$t > \tau, \;\; x,\xi \in \mathbb{R}^n, \;\; |m| = 0,1, \;\; |s| \le p.$$

For further estimates we will use the statement.

Lemma 1. (On the estimation of an improper volume integral). *For volume integral*

$$\mathcal{I}(t,\tau,x,\xi) = \int_{\mathbb{R}^n} \frac{e^{-c\rho(t,\beta,x,y)}}{(t-\beta)^{\frac{n}{2b}}} \frac{e^{-c\rho(\beta,\tau,y,\xi)}}{(\beta-\tau)^{\frac{n}{2b}}}\, dy$$

the next inequality is correct [1, c. 39]

$$\mathcal{I}(t,\tau,x,\xi) \le C_\varepsilon (t-\tau)^{-\frac{n}{2b}} e^{-(c-\varepsilon)\rho(t,\tau,x,\xi)}, \quad 0 < \varepsilon < c.$$

We estimate the kernel H using estimates for derivatives of the FMS Z (5) and kernels B_s (11):

$$|H(t,\tau,x,\xi)| \le \int_\tau^t d\beta \int_{\mathbb{R}^n} \sum_{|s|\le p} |B_s(t,\beta,x,z)||D_z^s Z(\beta,\tau,z,\xi)|dz \le$$

$$\le C_0 \int_\tau^t d\beta \int_{\mathbb{R}^n} \frac{e^{-c\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n+2b-\alpha}{2b}}} \frac{e^{-c\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n+p}{2b}}} dz \le$$

$$\le C_0 \int_\tau^t \frac{d\beta}{(t-\beta)^{\frac{2b-\alpha}{2b}}(\beta-\tau)^{\frac{p}{2b}}} \int_{\mathbb{R}^n} \frac{e^{-c\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n}{2b}}} \frac{e^{-c\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n}{2b}}} dz \le$$

$$\le C_1 \mathrm{B}\left(\tfrac{2b-p}{2b}, \tfrac{\alpha}{2b}\right) \frac{e^{-c(1-\varepsilon)\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n+p-\alpha}{2b}}}, \tag{12}$$

$$C_1 = C_0 \cdot C_\varepsilon, \quad 0<\varepsilon<1, \quad t>\tau, \quad x,\xi \in \mathbb{R}^n.$$

In obtaining the estimate we use the lemma on estimating improper volume integral, and the integral over β is counted using the B - function. Next we will evaluate the repeated kernels. To estimate H_2, we again use the lemma, and in the integral over β we pass to the B - function:

$$|H_2(t,\tau,x,\xi)| \le C_1^2 \mathrm{B}^2\left(\tfrac{2b-p}{2b}, \tfrac{\alpha}{2b}\right) \int_\tau^t d\beta \int_{\mathbb{R}^n} \frac{e^{-c(1-\varepsilon)\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n+p-\alpha}{2b}}} \frac{e^{-c(1-\varepsilon)\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n+p-\alpha}{2b}}} dz \le$$

$$\le C_1^2 C_\varepsilon \cdot \mathrm{B}^2\left(\tfrac{2b-p}{2b}, \tfrac{\alpha}{2b}\right) \cdot \mathrm{B}\left(\tfrac{2b-p+\alpha}{2b}, \tfrac{2b-p+\alpha}{2b}\right) \frac{e^{-c(1-2\varepsilon)\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n-2b+2(p-\alpha)}{2b}}}, \quad t>\tau, \ \ x,\xi\in\mathbb{R}^n.$$

By induction, we have

$$|H_\nu(t,\tau,x,\xi)| \le C_1^\nu C_\varepsilon^{\nu-1} \cdot \mathrm{B}^\nu\left(\tfrac{2b-p}{2b}, \tfrac{\alpha}{2b}\right) \times$$

$$\times \mathrm{B}\left(\tfrac{2b-p+\alpha}{2b}, \tfrac{2b-p+\alpha}{2b}\right) \cdot \ldots \cdot \mathrm{B}\left(\tfrac{2b-p+\alpha}{2b}, \tfrac{(\nu-1)(2b-p+\alpha)}{2b}\right) \times$$

$$\times (t-\tau)^{-\frac{n-(\nu-1)2b+\nu(p-\alpha)}{2b}} e^{-c(1-\nu\varepsilon)\rho(t,\tau,x,\xi)}, \quad t>\tau, \ \ x,\xi\in\mathbb{R}^n.$$

As seen from the last estimate, starting with $\nu \ge \nu_0 = \left[\frac{n+2b}{2b-p+\alpha}\right]+1$, kernels do not have singularities and for H_{ν_0} the inequality

$$|H_{\nu_0}(t,\tau,x,\xi)| \le C_{\nu_0} e^{-c_{\nu_0}\rho(t,\tau,x,\xi)}, \quad t>\tau, \ \ x,\xi\in\mathbb{R}^n,$$

$$c_{\nu_0} \equiv c(1-\nu_0\varepsilon),$$

$$C_{\nu_0} \equiv C_1^{\nu_0} C_\varepsilon^{\nu_0-1} \cdot \mathrm{B}^{\nu_0}\Big(\tfrac{2b-p}{2b}, \tfrac{\alpha}{2b}\Big) \cdot \mathrm{B}\Big(\tfrac{2b-p+\alpha}{2b}, \tfrac{2b-p+\alpha}{2b}\Big) \cdot \ldots \cdot \mathrm{B}\Big(\tfrac{2b-p+\alpha}{2b}, \tfrac{(\nu_0-1)(2b-p+\alpha)}{2b}\Big)$$

is correct. For all the next kernels we obtain estimates

$$|H_{\nu_0+m}(t,\tau,x,\xi)| \le C C_1^m C_{\nu_0} \cdot \mathrm{B}^m\Big(\tfrac{2b-p}{2b}, \tfrac{\alpha}{2b}\Big) \times$$

$$\times\mathrm{B}\Big(\tfrac{2b-p+\alpha}{2b},1\Big)\cdot\ldots\cdot\mathrm{B}\Big(\tfrac{2b-p+\alpha}{2b},1+\tfrac{(m-1)(2b-p+\alpha)}{2b}\Big)\times$$
$$\times(t-\tau)^{\frac{(2b-p+\alpha)\cdot m}{2b}}e^{-c_{\nu_0}\rho(t,\tau,x,\xi)},\quad t>\tau,\ \ x,\xi\in\mathbb{R}^n,\quad m=1,2,3,...,$$

where in the power of exponent we split from $c(1-\varepsilon)$ the value c_{ν_0} and use the inequality $\rho(t,\beta,x,z)+\rho(\beta,\tau,z,\xi)\geq\rho(t,\tau,x,\xi)$. In the result, we obtain an integral of the Poisson type, and in integral over the time variable we get the B-function.

If we consider the residual series $\sum\limits_{\nu=\nu_0+1}^{\infty}H_\nu$ and the constants obtained in the estimates of the repeated kernels

$$A_m=\mathrm{B}^m\left(\tfrac{2b-p}{2b},\tfrac{\alpha}{2b}\right)\cdot\mathrm{B}\left(\tfrac{2b-p+\alpha}{2b},1\right)\cdot\ldots\cdot\mathrm{B}\left(\tfrac{2b-p+\alpha}{2b},1+\tfrac{(m-1)(2b-p+\alpha)}{2b}\right)=$$
$$=\frac{\Gamma^m\left(\frac{2b-p}{2b}\right)\cdot\Gamma^m\left(\frac{\alpha}{2b}\right)}{\Gamma\left(1+\frac{m(2b-p+\alpha)}{2b}\right)},$$

in which the transition from B- to Γ-functions is carried out, then the majorant number series $\sum\limits_{m=1}^{\infty}A_m$ converges according to the d'Alembert criterion, which means that the functional series converges uniformly and absolutely according to the Weierstrass criterion. This allows us to construct a resolvent for which from the inequalities for the repeated kernels we can obtain the following estimate

$$|R(t,\tau,x,\xi)|\leq C(t-\tau)^{-\frac{n+p-\alpha}{2b}}e^{-c_{\nu_0}\rho(t,\tau,x,\xi)},\quad t>\tau,\ \ x,\xi\in\mathbb{R}^n.\tag{13}$$

Now we return to the solution (4) of the Cauchy problem (3), (2), to find which it has been assumed that the function y is a Hölder function with respect to the spatial variable. We show that the function y is a Hölder function with the same exponent α as the coefficients and kernels of the IDOs of the initial system. To do this, we firstly establish estimates for the kernel increments B_s, H and the resolvent. For the kernels B_s, the estimates (11) are correct, from which we obtain estimates for the growth of the kernels for $|\triangle x|\geq\sqrt[2b]{t-\tau}$:

$$\begin{gathered}\big|\triangle_x B_s(t,\tau,x,\xi)\big|\leq|B_s(t,\tau,x+\triangle x,\xi)|+|B_s(t,\tau,x,\xi)|\leq\\ \leq C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+2b}{2b}}\Big(e^{-c\rho(t,\tau,x+\triangle x,\xi)}+e^{-c\rho(t,\tau,x,\xi)}\Big),\\ t>\tau,\ \ x,\xi\in\mathbb{R}^n,\quad |s|\leq p.\end{gathered}\tag{14}$$

If $|\triangle x|<\sqrt[2b]{t-\tau}$, then according to the Lagrange theorem

$$|\triangle_x B_s(t,\tau,x,\xi)|=|D_xB_s(t,\tau,x+\theta\triangle x,\xi)|\cdot|\triangle x|\leq C\cdot|\triangle x|\frac{e^{-c\rho(t,\tau,x+\theta\triangle x,\xi)}}{(t-\tau)^{\frac{n+2b+1-\alpha}{2b}}}.$$

Consider $e^{-c\rho(t,\tau,x+\theta\triangle x,\xi)}$, $|\theta|\leq 1$. Using the lemma from [19, p. 144] there exists such a constant $c_1>0$, that

$$e^{-c\left(\frac{|x+\theta\triangle x-\xi|}{(t-\tau)^{1/2b}}\right)^q}\leq e^{-c_1\left(\frac{|x-\xi|}{(t-\tau)^{1/2b}}\right)^q}e^{c'\left(|\theta|\frac{|\triangle x|}{(t-\tau)^{1/2b}}\right)^q}\leq C\cdot e^{-c_1\left(\frac{|x-\xi|}{(t-\tau)^{1/2b}}\right)^q},$$

as long as $\frac{|\triangle x|}{(t-\tau)^{1/2b}} < 1$. Thus

$$\left|\triangle_x B_s(t,\tau,x,\xi)\right| \le C\left(\frac{|\triangle x|}{(t-\tau)^{1/2b}}\right)^{1-\alpha}\frac{|\triangle x|^{\alpha}}{(t-\tau)^{\frac{n+2b}{2b}}}\, e^{-c_1\rho(t,\tau,x,\xi)} \le$$

$$\le C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+2b}{2b}} e^{-c_1\rho(t,\tau,x,\xi)}, \quad t>\tau,\ \ x,\xi\in\mathbb{R}^n, \quad |s|\le p.$$

So, the inequalities (14) are correct for the growth of B_s kernels. This means that the kernels satisfy the non-uniform Hölder condition with exponent α.

We estimate the kernel increment H. Let $|\triangle x| \ge \frac{1}{2}\sqrt[2b]{t-\tau}$. Then the estimate of the kernel increment H follows from the estimate for H (12):

$$\left|\triangle_x H(t,\tau,x,\xi)\right| \le \left|H(t,\tau,x+\triangle x,\xi)\right| + \left|H(t,\tau,x,\xi)\right| \le$$

$$\le C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+p}{2b}}\left(e^{-c\rho(t,\tau,x+\triangle x,\xi)} + e^{-c\rho(t,\tau,x,\xi)}\right), \quad t>\tau,\ \ x,\xi\in\mathbb{R}^n. \qquad (15)$$

Let's estimate the kernel increment H at $|\triangle x| < \frac{1}{2}(t-\tau)^{1/2b}$:

$$\triangle_x H(t,\tau,x,\xi) = \int\limits_{\tau}^{t} d\beta \int\limits_{\mathbb{R}^n} \sum_{|s|\le p} \triangle_x B_s(t,\beta,x,z) D_z^s Z(\beta,\tau,z,\xi)dz =$$

$$= \int\limits_{\tau}^{t_1} \ldots + \int\limits_{t_1}^{t-|\triangle x|^{2b}} \ldots + \int\limits_{t-|\triangle x|^{2b}}^{t} \ldots = I_1 + I_2 + I_3,$$

where $t_1 = \tau + \frac{t-\tau}{2}$. Using the estimate (14) and the fact that $t-\beta \ge \frac{t-\tau}{2}$, we have

$$|I_1| \le C\frac{|\triangle x|^{\alpha}}{t-\tau}\int\limits_{\tau}^{t_1} d\beta \int\limits_{\mathbb{R}^n} \frac{e^{-c\rho(t,\beta,x+\triangle x,z)} + e^{-c\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n}{2b}}}\frac{e^{-c\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n+p}{2b}}}dz \le$$

$$\le C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+p}{2b}}\left(e^{-c\rho(t,\tau,x+\triangle x,\xi)} + e^{-c\rho(t,\tau,x,\xi)}\right).$$

And now let us also estimate the second term. We use the Lagrange's mean value theorem:

$$|I_2| \le C\int\limits_{t_1}^{t-|\triangle x|^{2b}} d\beta \int\limits_{\mathbb{R}^n} |\triangle x| \frac{e^{-c\rho(t,\beta,x+\theta\triangle x,z)}}{(t-\beta)^{\frac{n+2b+1-\alpha}{2b}}}\frac{e^{-c\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n+p}{2b}}}dz \le$$

$$\le C|\triangle x|\frac{e^{-c\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n}{2b}}}\int\limits_{t_1}^{t-|\triangle x|^{2b}} \frac{d\beta}{(t-\beta)^{\frac{2b+1-\alpha}{2b}}(\beta-\tau)^{\frac{p}{2b}}} \le C|\triangle x|^{\alpha}\frac{e^{-c\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n+p}{2b}}}.$$

Similar to kernel evaluation B_s, $e^{-c\rho(t,\beta,x+\theta\triangle x,z)} \le C\cdot e^{-c_1\rho(t,\beta,x,z)}$, as $|\triangle x| < (t-\beta)^{1/2b}$ and $\beta-\tau \ge \frac{t-\tau}{2}$.

To estimate I_3, we use the estimate (11) for kernels B_s and the fact that $\beta-\tau > \frac{t-\tau}{2}$:

$$|I_3| \le \sum_{|s|\le p}\left[\int\limits_{t-|\triangle x|^{2b}}^{t} d\beta \int\limits_{\mathbb{R}^n} \left|B_s(t,\beta,x+\triangle x,z)\cdot D_z^s Z(\beta,\tau,z,\xi)\right| dz + \right.$$

$$+\int\limits_{t-|\triangle x|^{2b}}^{t} d\beta \int\limits_{\mathbb{R}^n} |B_s(t,\beta,x,z)\cdot D_z^s Z(\beta,\tau,z,\xi)|\,dz\Bigg] \le$$

$$\le C\int\limits_{t-|\triangle x|^{2b}}^{t} \frac{d\beta}{(t-\beta)^{\frac{2b-\alpha}{2b}}(\beta-\tau)^{\frac{p}{2b}}}\int\limits_{\mathbb{R}^n}\frac{e^{-c\rho(t,\beta,x+\triangle x,z)}+e^{-c\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n}{2b}}}\frac{e^{-c\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n}{2b}}}dz \le$$

$$\le C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+p}{2b}}\left(e^{-c\rho(t,\tau,x+\triangle x,\xi)}+e^{-c\rho(t,\tau,x,\xi)}\right).$$

Using estimates for I_1, I_2, I_3 and (15) finally we get

$$\big|\triangle_x H(t,\tau,x,\xi)\big| \le C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+p}{2b}}\left(e^{-c\rho(t,\tau,x+\triangle x,\xi)}+e^{-c\rho(t,\tau,x,\xi)}\right), \quad (16)$$

$$t>\tau, \quad x,\xi\in\mathbb{R}^n.$$

For the resolvent increment we obtain a similar estimate

$$\big|\triangle_x R(t,\tau,x,\xi)\big| \le C|\triangle x|^{\alpha}(t-\tau)^{-\frac{n+p}{2b}}\left(e^{-c\rho(t,\tau,x+\triangle x,\xi)}+e^{-c\rho(t,\tau,x,\xi)}\right), \quad (17)$$

$$t>\tau, \quad x,\xi\in\mathbb{R}^n.$$

Now let us show that the function y, which is represented as (8) and which can also be written as a sum

$$y\equiv f+J_1+J_2+J_3,$$

is Hölder function in the spatial variable x. f is a Hölder function by assumption. Using the inequalities (16) and (17), we estimate the growth of the following terms for $t>0$, $x\in\mathbb{R}^n$:

$$|\triangle_x J_1(t,x)| \le \int\limits_{\mathbb{R}^n}|\triangle_x H(t,0,x,\xi)||\varphi(\xi)|d\xi \le$$

$$\le C|\triangle x|^{\alpha}\cdot|\varphi|_C\int\limits_{\mathbb{R}^n}\frac{e^{-c\rho(t,x+\triangle x,\xi)}+e^{-c\rho(t,x,\xi)}}{t^{\frac{n+p}{2b}}}d\xi \le C|\varphi|_C\cdot|\triangle x|^{\alpha}\cdot t^{-\frac{p}{2b}}; \quad (18)$$

$$|\triangle_x J_2(t,x)| \le$$

$$\le C|f|_{\alpha}|\triangle x|^{\alpha}\int\limits_0^t d\tau\int\limits_{\mathbb{R}^n}\frac{e^{-c\rho(t,\tau,x+\triangle x,\xi)}+e^{-c\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n+p}{2b}}}d\xi \le C|f|_{\alpha}\cdot|\triangle x|^{\alpha}\cdot t^{\frac{2b-p}{2b}}; \quad (19)$$

$$|\triangle_x J_3(t,x)| \le$$

$$\le C|\varphi|_C|\triangle x|^{\alpha}\int\limits_0^t d\tau\int\limits_{\mathbb{R}^n}\frac{e^{-c\rho(t,\tau,x+\triangle x,\xi)}+e^{-c\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n+p}{2b}}}\int\limits_{\mathbb{R}^n}\frac{e^{-c(1-\varepsilon)\rho(\tau,0,\xi,z)}}{\tau^{\frac{n+p-\alpha}{2b}}}dzd\xi \le$$

$$\le C|\varphi|_C|\triangle x|^{\alpha}\int\limits_0^t\frac{d\tau}{(t-\tau)^{\frac{p}{2b}}\tau^{\frac{p-\alpha}{2b}}} \le CB\left(\tfrac{2b-p+\alpha}{2b},\tfrac{2b-p}{2b}\right)|\varphi|_C\cdot|\triangle x|^{\alpha}\cdot t^{\frac{2b-2p+\alpha}{2b}}. \quad (20)$$

From the inequalities (18) – (20) we get that y satisfies the non-uniform Hölder condition with exponent α. So the estimate

$$|\triangle_x y(t,x)| \le |\triangle x|^{\alpha}\left(C_1 + C_2|f|_{\alpha} + C_3|\varphi|_C \cdot t^{-\frac{p}{2b}}\right), \quad t>0, \;\; x\in\mathbb{R}^n$$

is correct.

Now we establish estimates for the FMS $\mathbf{\Gamma} \equiv Z + W$ (10) and its derivatives. For $Z = G_0 + W_0$ we have the estimates (5). Let's estimate the derivatives of the volume potential W. According to the volume potential differentiation lemma [1, p.68] the lower derivatives are found by direct differentiation under the integral sign, and the higher derivatives are calculated by special formulas

$$D_x^{2b}W(t,\tau,x,\xi) = \int\limits_{\tau}^{t_1} d\beta \int\limits_{\mathbb{R}^n} D_x^{2b}Z(t,\beta,x,z)R(\beta,\tau,z,\xi)dz +$$

$$+\int\limits_{t_1}^{t} d\beta \int\limits_{\mathbb{R}^n} \bar{D}_x^{2b}G_o(t,\beta,x-z,x)\left[R(\beta,\tau,z,\xi) - R(\beta,\tau,x,\xi)\right]dz +$$

$$+\int\limits_{t_1}^{t} d\beta \int\limits_{\mathbb{R}^n} \left[D_x^{2b}G_o(t,\beta,x-z,z) - \bar{D}_x^{2b}G_o(t,\beta,x-z,x)\right] R(\beta,\tau,z,\xi)dz +$$

$$+\int\limits_{t_1}^{t} R(\beta,\tau,x,\xi)d\beta \int\limits_{\mathbb{R}^n} \bar{D}_x^{2b}G_o(t,\beta,x-z,x)dz +$$

$$+\int\limits_{t_1}^{t} d\beta \int\limits_{\mathbb{R}^n} D_x^{2b}W_o(t,\beta,x,z)R(\beta,\tau,z,\xi)dz \equiv \sum_{i=1}^{5} A_i, \tag{21}$$

where $\bar{D}$ means differentiation only in the third argument.

The estimate A_1 is obtained from the estimates (5) and (13):

$$|A_1(t,\tau,x,\xi)| \le C(t-\tau)^{-\frac{n+p-\alpha}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \quad t>\tau, \;\; x,\xi\in\mathbb{R}^n.$$

We estimate A_2 due to the non-uniform Hölder condition (17) of the resolvent R by the third argument

$$|A_2(t,\tau,x,\xi)| \le C\int\limits_{t_1}^{t} d\beta \int\limits_{\mathbb{R}^n} \frac{e^{-c\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n+2b}{2b}}} \frac{|x-z|^{\alpha}}{(\beta-\tau)^{\frac{n+p}{2b}}}\left(e^{-c\rho(\beta,\tau,z,\xi)} + e^{-c\rho(\beta,\tau,x,\xi)}\right)dz \le$$

$$\le C(\varepsilon)\int\limits_{t_1}^{t} \frac{d\beta}{(t-\beta)^{\frac{2b-\alpha}{2b}}(\beta-\tau)^{\frac{p}{2b}}}\left[\int\limits_{\mathbb{R}^n} \frac{e^{-(c-\varepsilon)\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n}{2b}}} \frac{e^{-c\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n}{2b}}}dz + \right.$$

$$\left. + \int\limits_{\mathbb{R}^n} \frac{e^{-(c-\varepsilon)\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n}{2b}}} \frac{e^{-c\rho(\beta,\tau,x,\xi)}}{(\beta-\tau)^{\frac{n}{2b}}}dz\right],$$

where we have taken advantage of the obvious inequality $\left(\frac{|x-z|}{(t-\beta)^{1/2b}}\right)^{\alpha} e^{-\varepsilon\rho(t,\beta,x,z)} \leq C(\varepsilon)$, $\varepsilon > 0$. Further, applying the lemma on estimating the volume integral to the first volume integral, and in the second, using the fact that $\beta - \tau \geq \frac{t-\tau}{2}$ and n of Poisson integrals are bounded quantities, we finally obtain

$$|A_2(t,\tau,x,\xi)| \leq C(t-\tau)^{-\frac{n+p-\alpha}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \quad t > \tau, \ x,\xi \in \mathbb{R}^n.$$

In A_3 we use the Hölder property of the Green matrix G_0 in the fourth argument

$$|A_3(t,\tau,x,\xi)| \leq C \int\limits_{t_1}^{t} d\beta \int\limits_{\mathbb{R}^n} |x-z|^{\alpha} \frac{e^{-c\rho(t,\beta,x,z)}}{(t-\beta)^{\frac{n+2b}{2b}}} \frac{e^{-c_{\nu_0}\rho(\beta,\tau,z,\xi)}}{(\beta-\tau)^{\frac{n+p-\alpha}{2b}}} dz \leq$$

$$\leq C \frac{e^{-c\rho(t,\tau,x,\xi)}}{(t-\tau)^{\frac{n}{2b}}} \int\limits_{t_1}^{t} \frac{d\beta}{(t-\beta)^{\frac{2b-\alpha}{2b}} (\beta-\tau)^{\frac{p-\alpha}{2b}}} \leq C(t-\tau)^{-\frac{n+p-2\alpha}{2b}} e^{-c\rho(t,\tau,x,\xi)},$$

$$t > \tau, \ x,\xi \in \mathbb{R}^n.$$

$A_4 = 0$ according to the property of the Green matrix G_0, and the derivatives W_0 in A_5 have a lower order of singularity according to the FMS construction Z. Therefore, from the estimates for A_1, A_2, A_3, which have been obtained for $|k| = 2b$, the estimate of the derivatives of the volume potential for $|k| \leq 2b$, $0 \leq p \leq 2b-1$ is finally followed:

$$|D_x^k W(t,\tau,x,\xi)| \leq C(t-\tau)^{-\frac{n+|k|+p-2b-\alpha}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \quad t > \tau, \ x,\xi \in \mathbb{R}^n. \tag{22}$$

3. Theorems about the FMS and the correctness of the Cauchy problem

The following statements about the FMS and the correctness of the Cauchy problem are correct.

Theorem 1. (About FMS). *Suppose that the system* (1) *is uniformly parabolic, the coefficients of the system $A_k(t,x)$ are defined in the domain Π, continuous in t and uniformly in x for $|k| = 2b$, $A_k \in C_x^{\alpha}(\Pi)$. The kernels of the IDOs of the system $B_s = \left(B_{ij}^s\right)_{i,j=1}^N$, $|s| \leq p$, are continuous for $t > \tau$, $x,\xi \in \mathbb{R}^n$ and satisfy the inequalities* (11)*. Then there is the FMS of the system* (1)

$$\mathbf{\Gamma}(t,\tau,x,\xi) = Z(t,\tau,x,\xi) + \int\limits_{\tau}^{t} d\beta \int\limits_{\mathbb{R}^n} Z(t,\beta,x,z) R(\beta,\tau,z,\xi)\, dz \equiv Z + W,$$

which for $t > \tau$ satisfies a homogeneous system. Here R is the resolvent of the corresponding system of Volterra-Fredholm integral equations of the second kind, the repeated kernels of which are expressed through the kernels B_s of the IDOs of the system (1) *and the FMS Z of the corresponding parabolic system* (3)*. For the derivatives of the volume potential W, the estimates* (22) *are correct.*

Theorem 2. (On the correctness of the Cauchy problem). *Let the conditions of the theorem 1 on the existence of the FMS* $\mathbf{\Gamma}$ *are satisfied and* $f \in C_x^{\alpha}(\Pi)$. *Then*

1) if the initial function $\varphi \in C(\mathbb{R}^n)$, *then the solution of the Cauchy problem* (1) – (2) *is determined by the sum of the potentials*

$$u(t,x) = \int\limits_{\mathbb{R}^n} \mathbf{\Gamma}(t,0,x,\xi)\varphi(\xi)\,d\xi + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} \mathbf{\Gamma}(t,\tau,x,\xi) f(\tau,\xi)\,d\xi$$

and for the derivatives of the solution the estimates

$$\left|D_x^k u\right| \le C\left(t^{-\frac{|k|}{2b}}|\varphi|_C + |f|_\alpha\right), \quad |k| \le 2b$$

are correct.

2) If $\varphi \in C^{2b+\alpha}(\mathbb{R}^n)$ *and the kernels of the IDOs of the system* B_s *satisfy the inequality*

$$|D_x^m B_s(t,\tau,x,\xi)| \le C_{ms}(t-\tau)^{-\frac{n+2b+|m|-\gamma}{2b}} e^{-c\rho(t,\tau,x,\xi)}, \tag{23}$$

$$\gamma > \alpha, \quad t > \tau, \quad x,\xi \in \mathbb{R}^n, \quad |m| = 0,1, \quad |s| \le p, \quad 0 \le p \le 2b-1,$$

then the solution of the Cauchy problem belongs to the class $C^{2b+\alpha}(\Pi)$ *and*

$$|u|_{2b+\alpha} \le C(|\varphi|_{2b+\alpha} + |f|_\alpha). \tag{24}$$

Proof. In the case of only continuous and bounded initial function the estimates of the derivatives of the solution are obtained directly from the derivatives estimates of the FMS $\mathbf{\Gamma}$.

In the second case, the Cauchy problem can be reduced to a problem with zero initial conditions

$$L(t,x,D,B)v = f + \sum_{|k|\le 2b} A_k(t,x)D_x^k\varphi + \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n}\sum_{|s|\le p} B_s(t,\tau,x,\xi)D_\xi^s\varphi(\xi)\,d\xi \equiv f_1(t,x),$$

$$v|_{t=0} = 0,$$

where $v(t,x) = u(t,x) - \varphi(x)$. Under the conditions (23) on B_s kernels, their increments can be estimated in this way

$$|\Delta_x B_s(t,\tau,x,\xi)| \le C\,|\Delta x|^\alpha (t-\tau)^{-\frac{n+2b-\varepsilon}{2b}}\left(e^{-c\rho(t,\tau,x+\Delta x,\xi)} + e^{-c\rho(t,\tau,x,\xi)}\right),$$

$$\varepsilon = \gamma - \alpha > 0, \quad t > \tau, \quad x,\xi \in \mathbb{R}^n, \quad |s| \le p.$$

This estimate guarantees the Hölder heterogeneity of f_1 with respect to the argument x, which in turn ensures the inequality for the norm (24).

Concluding remarks

Thus, in this work, for the parabolic system of IDEs with Volterra-Fredholm type IDO, the estimates of the FMS $\mathbf{\Gamma}$ for the Cauchy problem are constructed and obtained.

The construction of the FMS $\mathbf{\Gamma}$ is preserved similar to the construction the FMS Z of a uniformly parabolic system. The main term is the FMS Z of the uniformly parabolic system. The additional term is an integral with the kernel Z and density, which is the resolvent of the system of Volterra-Fredholm integral equations of the second kind, the repeated kernels of which are expressed through the kernels of the IDOs B_s of the parabolic system of IDEs and FMS Z of the corresponding parabolic system. At the same time, the smoothness of the solution of the Cauchy problem depends not only on the smoothness of the initial function, but also on the differential properties of the kernels of the IDOs of the parabolic system of IDEs.

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