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# ON THE REGULARIZATION OF SOME SINGULAR INTEGRAL OPERATORS. NOETHERIAN CRITERIA

**Vasile NEAGU**, prof. univ., dr. hab.
Tiraspol State University

**Abstract**. The work is centered on the study of algebra $\mathbf{A}$ generated by singular integral operators with shifts with continuous coefficients. Necessary and sufficient conditions are established for operators $\mathbf{A}$ to be Noetherian and to admit equivalent regularization in the space $L_p(\Gamma,\rho)$. There are constructed regularizators for Noetherian operators. The study is done in the space $L_p(\Gamma,\rho)$ with weight $\rho(t) = \prod_{k=1}^{n}|t-t_k|^{\beta_k}$ and is based on the theory of Ghelfand [1] concerning Banach algebras.

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**Keywords**: singular integral operator, compact operator, regularization.
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# ASUPRA REGULARIZĂRII OPERATORILOR INTEGRALI SINGULARI. CRITERII NOETHERIENE

**Rezumat**. Lucrarea este consacrată studiului algebrei (închise) generată de operatorii integrali singulari cu coeficienți continui pe conturul $\Gamma$. Sunt stabilite condiţiile necesare şi suficiente în care operatorii din algebra $\mathbf{A}$ sunt noetherieni în spaţiul $L_p(\Gamma,\rho)$; sunt construiţi regularizatorii operatorilor noetherieni. Studiul este efectuat în spaţiul $L_p(\Gamma,\rho)$ cu ponderea $\rho(t) = \prod_{k=1}^{n}|t-t_k|^{\beta_k}$ şi se bazează pe teoria lui I. Ghelfand cu privire la algebrele Banach comutative (a se vedea [1]).

Rezultatele principale ale acestei lucrări au fost prezentate la Congresul al 9-lea al Matematicienilor Români, care s-a desfășurat în orașul Galați, 2019.

**Cuvinte cheie**: operator integral singular, operator compact, regularizare.

## 1. Introduction

The results presented in this work are generalizations of the paper [2]. Thus, if, in particular, there are considered integral singular characteristic operators without translations, then the results of given work, coincide with these of [2].

We remind that an operator $A \in L(\mathbf{B})$ admits a regularization, if there exist an operator $M \in L(\mathbf{B})$ such that $AM = I + T_1$, $MA = I + T_2$, where $T_1$ and $T_2$ are compact operators in the space $\mathbf{B}$. The class of operators which admit a regularization is of special interest due to the fact that operators of this class have the fallowing properties (F.Noether theorems):

*1) The equation $Ax = y$ is solvable if and only if the right-hand side is orthogonal to all solutions of the equation $A^*\varphi = 0$. This condition is equivalent to the condition that the set of values of operator $A$ is a subspace, or such that the relation*

$$ImA = \bigcap_{f \in Ker A^*} Ker f$$

*holds.*

*2) The equations $Ax = 0$ and $A^*\varphi = 0$ have a finite number of linear independent solutions.*

Operators with properties 1) and 2) are called *Noetherian* and are essential generalizations of the class of operators of the form $I+T$, with $T$ compact, for which theorems similar to that of Fredholm are true.

In the case when conditions 1) and 2) are met, then the number $\mathrm{dimKerA} - \mathrm{dimKerA}^*$ is called *the index of noetherian operator* A and is denoted by IndA:

$$IndA = dimKerA - dimKerA^*.$$

Let $\Gamma$ be a closed Lyapunov type contour, $S$ be a singular integral operator with Cauchy kernel and $V$ be an operator of shifting, $(V\varphi)(t) = \varphi(\alpha(t))$, where function $\omega : \Gamma \to \Gamma$ satisfies the following conditions:

*a*) $\omega(\omega(t)) \equiv \omega(t)$ , $(\omega(t)) \neq t)$;

*b*) there exists derivative $\omega'(t) \neq 0$;

*c*) the function $\omega'(t)$ satisfies the Hölder condition on contour $\Gamma$.

Consider the complete singular integral equation

$$A\varphi = a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_{\Gamma} \frac{\varphi(\tau)}{\tau - t}\, d\tau + \int_{\Gamma} k(\tau, t)\varphi(\tau) d\tau = f(t) \quad (k \in C(\Gamma \times \Gamma)). \tag{1.1}$$

To Mihlin (see [3]) the following theorems belong

***Theorem 1.1.*** *The operator $M$ is an equivalent regularizer for singular equation $A\varphi = f$ for every right hand part $f$ if and only if $KerM = \{0\}$.*

A similar theorem can be formulated for the case of right-hand regularization. In this case, the operators $A$ and $M$ are interchanged. Operator $M$ is an equivalent regularizer for $A$ if it is a regularizer for $A$ and is invertible to the left.

***Theorem 1.2.*** *The integral singular equation $A\varphi = f$ admits an equivalent regularization for every right-hand part $f$, if and only if $IndA \geq 0$.*

It is well known [2] that a singular integral operator[1] $A = aI + bS + T$ admits a regularization if and only if $a^2(t) - b^2(t) \neq 0$ for all $t \in \Gamma$. For example, as a regularizator one can take the operator

$$R = \frac{a}{a^2 - b^2} I - \frac{b}{a^2 - b^2} S .$$

Under these conditions operator $A^*$, obviously, also admits a regularization and thus for $A$ and $A^*$ Noether theorems hold.

The main result of this work is given by

***Theorem 1.3.*** *Operator*

$$A = aI + bS + (cI + dS)V + T,\ a, b, c, d \in C_\alpha(\Gamma), \tag{1.2}$$

[1)] By $T$ *with indices we denote compact operators.*

*admits a regularization in* $L_p(\Gamma,\rho)$ *if and only if*

$$(a(t)+b(t))^2-(c(t)+d(t))^2\neq 0, \qquad (a(t)-b(t))^2-(c(t)-d(t))^2\neq 0 \tag{1.3}$$

*for every* $t\in\Gamma$. *Under conditions* (1.3) *operator*

$$R=\frac{\alpha}{\alpha^2-\delta^2}P+\frac{\beta}{\beta^2-\gamma^2}Q-(\frac{\delta}{\alpha^2-\delta^2}P+\frac{\gamma}{\beta^2-\gamma^2}Q)V, \tag{1.4}$$

*where* $\alpha=a+b$, $\beta=a-b$, $\delta=c+d$, $\gamma=c-d$, $P=\frac{1}{2}(I+S)$, $Q=\frac{1}{2}(I-S)$, *is a regularizator for* $A$.

This theorem is proved in §5.

## 2. Properties of operators S and V

It is known the fact that the operator

$$(S\varphi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau \ , t\in\Gamma, \tag{2.1}$$

where the integral is understood in the sense of main value, is defined on the set of rational functions on contour $\Gamma$. If $\beta_k>-1$, then this set is dense in the space $L_p(\Gamma,\rho)$, where

$$\rho(t)=\prod_{k=1}^n|t-t_k|^{\beta_k}.$$

Hvedelidze proved that if

$$-1<\beta_k<p-1,\ k=1,2,\dots,n, \tag{2.2}$$

then the operator $S$ is bounded in the space $L_p(\Gamma,\rho)$. Conditions (2.2) are also [4] necessary in which $S$ is bounded in $L_p(\Gamma,\rho)$. In what follows we suppose that conditions (2.2) are verified.

We shall also mention some properties of operator S, necessary in what follows:

$1^0$. $S^2=I$, *where* $I$ *is the identity operator on* $L_p(\Gamma,\rho)$.

$2^0$. *Operator* $S^*$, *which acts in the space* $(L_p(\Gamma,\rho))^*=L_q(\Gamma,\rho^{1-q})$, $p^{-1}+q^{-1}=1$, *differs from operator* $S$ *in this space by compact:* $S^*=S+T$.

$3^0$. *For every function* $a(t)$ *continuous on* $\Gamma$ *operator* $aS-SaI$ *is compact.*

$4^0$. *The relation* $HS^*H=-S+T$, *where* $(H\varphi)(t)=\overline{\varphi(t)}$ *is realized.*

The proof of properties $1^0-3^0$ is contained in monograph [4], and property $4^0$ can be easily proved, if respective results from [4-5] will be applied (see also [6-7])

From the properties of the function $\omega$ it results that the operator

$$(V\varphi)(t)=\varphi(\omega(t)) \tag{2.3}$$

is involutive, $V^2=I$. We shall establish its continuity in the space $L_p(\Gamma,\rho)$. Let $\varphi\in L_p(\Gamma,\rho)$, then

$$\|V\varphi\|^p = \int_\Gamma |\varphi(\omega(t))|^p \prod_1^n |t-t_k|^{\beta_k} |dt| = \int_\Gamma |\varphi(t)|^p \prod_1^n |\omega(t)-t_k|^{\beta_k} |\omega'(t)||dt|. \quad (2.4)$$

Since the function $\omega$ has derivative and $\omega'(t) \neq 0$, there exist constants $c_k > 0$ and $\tilde{c}_k > 0, k = 1,2,\ldots,n$, such that

$$\tilde{c}_k < \left| \frac{\omega(t)-t_k}{t-t_k} \right|^{\beta_k} < c_k. \quad (2.5)$$

Taking into account (2.4) and (2.5), we get

$$\|V\varphi\|^p < \prod_1^n c_k \cdot \int_\Gamma |\varphi(t)|^p \prod_1^n |t-t_k|^{\beta_k} |\omega'(t)||dt| \le c\|\varphi\|^p$$

and continuity of operator $V$ is proved.

We mention also the property of $V$ contained in

$5^0$. $VS = SV + T$.

Indeed

$$(VS-SV)\varphi = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-\omega(t)}\, d\tau - \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\omega(\tau))}{\tau - t}\, d\tau = \frac{1}{\pi i}\int_\Gamma \left(\frac{\omega'(\tau)}{\omega(\tau)-\omega(t)} - \frac{1}{\tau-t}\right)\varphi(\omega(\tau))\, d\tau.$$

From properties of function $\omega$ it follows [4] that the kernel of operator $VS - SV$,

$$k(t,\tau) = \frac{\omega'(\tau)}{\omega(\tau)-\omega(t)} - \frac{1}{\tau - t},$$

has weak singularities on contour $\Gamma$ and, hence, this operator is compact in the space $L_p(\Gamma,\rho)$.

Denote $P = \frac{1}{2}(I+S)$ and $Q = \frac{1}{2}(I-S)$. Then, using the above properties, the following relations are easily proved:

$6^0$. $P^2 = P$ , $Q^2 = Q$ , $PQ = QP = 0$ , $P + Q = I$ , $P - Q = S$, $VP = PV + T_1$, $VQ = QV + T_2$ , $HPH = Q + T_3$ , $HQH = P + T_4$ , *where* $(H\varphi)(t) = \overline{\varphi(t)}$.

## 3. Algebras A and Â

Denote by $C_\omega(\Gamma)$ $(\subset C(\Gamma))$ the set of functions $a(t)$ continuous on $\Gamma$ and satisfying condition $a(\omega(t)) = a(t)$. Evidently, this set forms a commutative algebra with identity and norm $\|a\|_{C_\omega(\Gamma)} = \|a\|_{C(\Gamma)}$. It is also obvious that every function of the form $a(t) = b(t)\cdot b(\omega(t))$, where $b \in C(\Gamma)$, is contained in $C_\omega(\Gamma)$. The converse to this statement is also true: every function $a \in C_\omega(\Gamma)$ may be represented in the form $a(t) = b(t)\cdot b(\omega(t))$, where $b \in C(\Gamma)$. We can join these remarks in assertion that algebra $C_\omega(\Gamma)$ is characterized by relation

$$C_\alpha(\Gamma) = \{b(t)\cdot b(\omega(t)) \,|\, b \in C(\Gamma)\}. \quad (3.1)$$

Representation of functions from $C_\omega(\Gamma)$ in the form $a(t) = b(t) \cdot b(\omega(t))$ is unique up to some constant factors $c_1$ and $c_2$, where $c_1 \cdot c_2 = 1$. Later on we shall assume that $c_1 = c_2 = 1$. Thus, for example, if $\Gamma$ is the unit circle and $\omega(t) = -t$, then functions $a_1(t) = -t^2$, $a_2(t) = t^2$ belong to $C_\omega(\Gamma)$ and they can be represented as $a_1(t) = t \cdot (-t)$ and, respectively $a_2(t) = it \cdot (-it)$.

Denote by $\mathbf{A}$ the algebra generated by $S, V$ and the set of operators $aI$ of multiplication by functions $a(t), a \in C_\omega(\Gamma)$. $\mathbf{A}$ is a subalgebra of algebra $L\left(L_p(\Gamma, \rho)\right)$ formed by set of linear and bounded operators, acting in the space $L_p(\Gamma, \rho)$.

***Theorem 3.1.*** $\mathbf{A}$ *is a closed algebra.*

In the proof of this theorem we use the properties of operators $S$ and $V$ and characterization of algebra $C_\omega(\Gamma)$. Preliminary is also necessary

***Lemma 3.1.*** *If operator $(M\varphi) = a(t)\varphi(t)$ of multiplication by function $a(t)$, continuous on $\Gamma$, can be represented in the form $M = B + T$, where $B$ is invertible and $T$ is compact operator in $L_p(\Gamma, \rho)$, then [2] the function $a(t)$ is not vanished on $\Gamma$.*

***Proof.*** Suppose, by absurd that the function $a(t)$ vanishes on a set $\sigma \in \Gamma$ of nonzero measure. Then equation $a(t)\varphi(t) = 0$ in $L_p(\Gamma, \rho)$ has an infinite set of linear independent solutions.

So the equation $(B + T)\varphi = 0$, which is equivalent to the equation $(I + B^{-1}T)\varphi = 0$, has an infinite number of linear independent solutions, which is absurd, since the operator $B^{-1}T$ is compact in $L_p(\Gamma, \rho)$. The function a(t) cannot be vanished even on a set of zero measure. Really, otherwise, the equation $a(t)\varphi(t) = 0$ would have only the trivial solution and, by virtue of Fredholm theorem, operator $M$ should be invertible. Since the operator $(M_1\varphi)(t) = \frac{1}{a(t)}\varphi(t)$ is unbounded in the space $L_p(\Gamma, \rho)$. It results that the operator $M$ is not invertible. Thus, function $a(t)$ does not vanished on $\Gamma$, and lemma is proved.

From this lemma it follows directly the following corollary.

***Corollary 3.1.*** *Operator$(M\varphi) = a(t)\varphi(t)$ is compact if and only if $a(t) \equiv 0$.*

***Proof of Theorem 3.1.*** Let the sequence $(A_n)$, where

$$A_n = a_n I + b_n S + (c_n I + d_n S)V + T_n,\ a_n, b_n, c_n, d_n \in C_\omega(\Gamma),$$

be fundamental. Then the sequence $(HA_n^*H), HA_n^*H = a_n I - b_n S + (c_n I - d_n S)V + \tilde{T}_n$, is fundamental. In consequence, the sequence

$$R_n = a_n I + c_n V + T_n', \ (\mathrm{R_n} = \frac{\mathrm{A_n + HA_n^* H}}{2}), \tag{3.2}$$

is also fundamental. Define the following operator $(N\varphi)(t) = (\omega(t) - t)\varphi(t)$, which linear and bounded in $L_p(\Gamma, \rho)$. Since function ω preserves orientation on $\Gamma$, it has no (see [5]) fixed points on $\Gamma$. Hence, there exists operator $N^{-1}$,

$$(N^{-1}\varphi)(t) = \frac{1}{\omega(t)-t}\varphi(t).$$

Obviously, $NaN^{-1} = aI$ and $NVN^{-1} = -V$.Together with sequence (3.2) it will be also fundamental the sequence $(N^{-1}R_nN)$, which by the above has the form

$$N^{-1}R_nN = a_nI - c_nV + T_n''. \tag{3.3}$$

From the fact that sequence strings (3.2) and (3.3) are fundamental, it is deduced that the half-sum of these strings is also fundamental. In other words, for every $\varepsilon > 0$ there exists a natural number $n_0$, such that for every $n > n_0$ and for every $m > n_0$ the inequality

$$\left\|(a_n - a_m)I + \tilde{\tilde{T}}_n - \tilde{\tilde{T}}_m\right\| < \varepsilon, \ \left(\tilde{\tilde{T}}_n = \frac{T_n' + T_n''}{2}\right) \tag{3.4}$$

holds.

From relation (3.4) it results that operator $(a_n - a_m)I$ can be represented in the form

$$(a_n - a_m)I = \tilde{\tilde{T}}_m - \tilde{\tilde{T}}_n + B_{n,m},$$

where $\|B_{n,m}\| < \varepsilon$. For every complex number $\lambda$, $|\lambda| > \varepsilon$ and every $n, m > n_0$ operator $\lambda I - B_{n,m}$ is invertible. Thus, for these values of $\lambda$ and $n, m$ the operator $(\lambda - (a_n - a_m))I$ can be represented as a sum of two operators, $(\lambda - a_n + a_m)\varphi = ((\lambda I - B_{n,m}) - (\tilde{\tilde{T}}_m - \tilde{\tilde{T}}_n))\varphi$, from which one is invertible and other is compact. Applying to operator $(\lambda - (a_n - a_m))I$ Corollary 2.1, we obtain that the values of function $a_n(t) - a_m(t)$ are in the disk with centre in zero of radius $\varepsilon$. That is

$$|a_n(t) - a_m(t)| < \varepsilon, \ \forall n, m > n_0, \text{ and } \forall t \in \Gamma.$$

Thus, the sequence of continuous function $(a_n)$ converges uniformly. Similarly, considering the half-difference of operators (3.2) and (3.3), we obtain that the sequence of continuous functions $(c_n)$ converges uniformly. Then, essentially repeating the reasoning that led us to the convergence of sequences $(a_n)$ and $(c_n)$, we shall obtain that the sequences $(b_n)$ and $(d_n)$ are uniformly convergent too. Let $a, b, c, d$ and $T$ be the limits of the sequences $(a_n)$, $(b_n)$, $(c_n)$, $(d_n)$ and $(T_n)$. Since

$$\|A_n - A\| \le \max|a_n(t) - a(t)| + \max|b_n(t) - b(t)|\|S\| +$$
$$\max|c_n(t) - c(t)|\|V\| + \max|d_n(t) - d(t)|\|S\|\|V\| + \|T_n - T\|,$$

it results that the sequence $(A_n)$, $A_n = a_nI + b_nS + (c_nI + d_nS)V + T_n$, converges to the operator

$$A = aI + bS + (cI + dS)V + T.$$

Theorem 3.1 is proved.

***Remark 3.1.*** *The norm of algebra* $\mathbf{A}$, *defined as operator norm,* is *topologically equivalent to the norm*

$$\|A\|_1 = \max|a(t)| + \max|b(t)| + \max|c(t)| + \max|d(t)| + \|T\|. \tag{3.5}$$

The set $\mathbf{T} = \mathbf{T}(L(L_p(\Gamma,\rho)))$ of compact operators in the space $L_p(\Gamma,\rho)$ is included in $\mathbf{A}$ and form a two-sided closed ideal. Consider the quotient algebra $\hat{\mathbf{A}} = \mathbf{A}/\mathbf{T}$, which is also a Banach algebra. Four continuous function $a(t), b(t), c(t))$ and $d(t)$ uniquely define an cosets from $\hat{\mathbf{A}}$ and, conversely, every element belonging to a coset adjacent class of $\hat{\mathbf{A}}$ is of form $aI + bS + (cI + dS)V + T$, where $T$ is a compact operator. Indeed, if elements $aI + bS + (cI + dS)V + T$ and $a_1 I + b_1 S + (c_1 I + d_1 S)V + T_1$ are cosets, then their difference $(a - a_1)I + (b - b_1)S + ((c - c_1)I + (d - d_1)S)V + T - T_1$ must be a compact operator. Under these conditions from Theorem 3.1 one can deduce that operators $(a - a_1)I$, $(b - b_1)I$, $(c - c_1)I$, $(d - d_1)I$ are compact, but from Lemma 3.1 this is possible if and only if $a(t) \equiv a_1(t)$, $b(t) \equiv b_1(t)$, $c(t) \equiv c_1(t)$, $d(t) \equiv d_1(t)$.

Let us return to algebra $\hat{\mathbf{A}}$. The element of $\hat{\mathbf{A}}$, determined by functions $a(t), b(t), c(t)$ and $d(t)$, is denoted by $\{aI + bS + (cI + dS)V\}$. From properties of operators $S$ and $V$ and by direct calculations we get

***Theorem 3.2.*** *Algebra $\hat{\mathbf{A}}$ is commutative and, besides, the equality*

$$\begin{gathered}\{aI + bS + (cI + dS)V\} \cdot \{a_1 I + b_1 S + (c_1 I + d_1 S)V\} = \\ \{(aa_1 + bb_1 + cc_1 + dd_1)I + (ab_1 + a_1 b + cd_1 + c_1 d)S + \\ ((ac_1 + a_1 c + bd_1 + b_1 d)I + (ad_1 + a_1 d + bc_1 + b_1 c)S)V\}\end{gathered} \tag{3.6}$$

*is true.*

The norm in $\hat{\mathbf{A}}$ is defined by the equality

$$|\{aI + bS + (cI + dS)V\}| = \inf_{T \in \mathbf{T}} \|aI + bS + (cI + dS)V\| \tag{3.7}$$

and it is topologically equivalent to the norm

$$|\{aI + bS + (cI + dS)V\}|_1 = \max|a(t)| + \max|b(t)| + \max|c(t)| + \max|d(t)|. \tag{3.8}$$

## IV. The structure of maximal ideals of algebra $\hat{\mathbf{A}}$

Further, elements of algebra $\hat{\mathbf{A}}$ will be expressed in the form

$$\{aP + bQ + (cP + dQ)V\},\ a,b,c,d \in C_\omega(\Gamma), \tag{4.1}$$

where $P = \frac{1}{2}(I + S)$ and $Q = \frac{1}{2}(I - S)$.

We shall describe all maximal ideals of $\hat{\mathbf{A}}$. This result will enable us to establish necessary and sufficient condition under which element of $\hat{\mathbf{A}}$ are invertible. Using this result we shall also construct regularizations for Noetherian operators.

***Theorem 4.1.*** *The set of elements $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathbf{A}}$ forms a maximal ideal of $\hat{\mathbf{A}}$ if the function $a(t) + c(t)$ vanishes at some point $t_0 \in \Gamma$. The set of elements $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathbf{A}}$ for which one of the functions $a(t) - c(t)$, $b(t) + d(t)$ or*

$b(t)-d(t)$ *vanishes at some point (every function at his own point) also form a maximal ideal. There are no other maximal ideals.*

***Proof.*** Denote by $M_{t_0}$ the set of elements $\{aP+bQ+(cP+dQ)V\}$ for which $a(t_0)+c(t_0)=0$ and let $\{a_1P+b_1Q+(c_1P+d_1Q)V\}$ be any element of algebra $\hat{\mathbf{A}}$. Then by relations (3.6), it is easily shown that

$$\{aP+bQ+(cP+dQ)V\}\cdot\{a_1P+b_1Q+(c_1P+d_1Q)V\} = \\ \{(aa_1+cc_1)P+(bb_1+dd_1)Q+((ac_1+a_1c)P+(bd_1+b_1dQ)V\} \qquad (4.2)$$

and

$$\big(a(t_0)a_1(t_0)+c(t_0)c_1(t_0)\big)+\big(a(t_0)c_1(t_0)+a_1(t_0)c(t_0)\big)= \\ (a(t_0)+c(t_0))(a_1(t_0)+c_1(t_0))=0.$$

Thus, the set $M_{t_0}$ forms an ideal. We shall prove that this ideal is maximal, that is it does not exist such an ideal M which contain $M_{t_0}$ and an element

$$\{a_2P+b_2Q+(c_2P+d_2Q)V\}\notin M_{t_0}\ (a_2(t_0)+c(t_0)).$$

Let us admit that such an ideal $\mathbf{M}$ exists. Then any element $\{aP+bQ+(cP+dQ)V\}\subset\hat{\mathbf{A}}$ can be expressed in the form

$$\{aP+bQ+(cP+dQ)V\}=\sigma(\{a_2P+b_2Q+(c_2P+d_2Q)V\})+ \\ \{(a-\sigma\,a_2)P+(b-\sigma\,b_2)Q+((c-\sigma\,c_2)P+(d-\sigma\,d_2)Q)V\}, \qquad (4.3)$$

where $\sigma$ is a complex number. We determine σ from condition that the second term from the right-hand side of (4.3) is an element of $M_{t_0}$. For this the number $\sigma$ must verify the equation

$$(a(t_0)-\sigma a_2(t_0))+(c(t_0)-\sigma c_2(t_0))=0\Rightarrow\exists\sigma=\frac{a(t_0)+c(t_0)}{a_2(t_0)+c_2(t_0)}. \qquad (4.4)$$

From relation (4.3), in which the number σ is determined from equality (4.4), it follows that the ideal $\mathbf{M}$ coincides with algebra $\hat{\mathbf{A}}$. Therefore, the ideal $M_{t_0}$is maximal. Similarly, cases, in which $a(t_0)-c(t_0)=0$, $b(t_0)+d(t_0)=0$ and $\mathrm{b}(\mathrm{t}_0)-\mathrm{d}(\mathrm{t}_0)=0$, are examined. It remains to prove that algebra $\hat{\mathbf{A}}$ does not contain other maximal ideals. Indeed, let $\mathbf{M}$ be a maximal ideal. We shall prove that for every element $\{aP+bQ+(cP+dQ)V\}\subset M$ one of the function $a(t)+c(t)$, $a(t)-c(t)$, $b(t)+d(t)$ or $b(t)-d(t)$ is vanished in a point $t_0\in\Gamma$. Assume contrary, that is for every $\mathrm{t}\in\Gamma$ in ideal $\mathbf{M}$ there exist elements $\{a_tP+\alpha Q+(c_tP+\beta Q)V\}$, $\{\tilde{a}_tP+\tilde{\alpha}Q+(\tilde{c}_tP+\tilde{\beta}Q)V\}$, $\{\gamma P+b_tQ+(\delta P+d_tQ)V\}$ and $\{\tilde{\gamma}P+\tilde{b}_tQ+(\tilde{\delta}P+\tilde{d}_tQ)V\}$, such that $a_t(t)-c_t(t)\neq 0$, $\tilde{a}_t(t)+\tilde{c}_t(t)\neq 0$, $b_t(t)-d_t(t)\neq 0$ and respectively $\tilde{b}_t(t)+\tilde{d}_t(t)\neq 0$. Hence there exist a neighbourhood U(t) of point $t$ such that

$$|a_t(\tau)-c_t(\tau)|>\lambda_t>0,\ |\tilde{a}_t(\tau)+\tilde{c}_t(\tau)|>\lambda_t>0,\ |b_t(\tau)-d_t(\tau)|>\lambda_t>0,\ |\tilde{b}_t(\tau)+\tilde{d}_t(\tau)|>\lambda_t>0$$

for every $\tau \in U(t)$. By Borel-Lebesgue theorem from the cover $\bigcup U(t)$ extract a finite one. Let $t_1, t_2, ..., t_n$ be respective point of obtained cover. Elements

$$a_{t_k} P + c_{t_k} PV,\ c_{t_k} P + a_{t_k} PV,\ \tilde{a}_{t_k} P + \tilde{c}_{t_k} PV,\ \tilde{c}_{t_k} P + \tilde{a}_{t_k} PV,$$

$$b_{t_k} Q + d_{t_k} QV,\ d_{t_k} Q + b_{t_k} QV,\ \tilde{b}_{t_k} Q + \tilde{d}_{t_k} QV,\ \tilde{d}_{t_k} Q + \tilde{b}_{t_k} QV,$$

$k = 1, 2, ... n$, belong to ideal M. Together with these elements the ideal M contains

$$\left\{a_1 P + c_1 PV = \sum_{k=1}^{n} (a_{t_k} \bar{a}_{t_k} P + c_{t_k} \bar{a}_{t_k} PV)\right\},\quad \left\{a_2 P + c_2 PV = \sum_{k=1}^{n} (c_{t_k} \bar{c}_{t_k} P + a_{t_k} \bar{c}_{t_k} PV)\right\},$$

$$\left\{a_3 P + c_3 PV = \sum_{k=1}^{n} (\tilde{a}_{t_k} \bar{\tilde{a}}_{t_k} P + \tilde{c}_{t_k} \bar{\tilde{a}}_{t_k} PV)\right\},\quad \left\{a_4 P + c_4 PV = \sum_{k=1}^{n} (\tilde{c}_{t_k} \bar{\tilde{c}}_{t_k} P + \tilde{a}_{t_k} \bar{\tilde{c}}_{t_k} PV)\right\},$$

$$\left\{b_1 Q + d_1 QV = \sum_{k=1}^{n} (b_{t_k} \bar{b}_{t_k} Q + \bar{b}_{t_k} d_{t_k} QV)\right\},\quad \left\{b_2 Q + d_2 QV = \sum_{k=1}^{n} (d_{t_k} \bar{d}_{t_k} Q + b_{t_k} \bar{d}_{t_k} QV)\right\},$$

$$\left\{b_3 Q + d_3 QV = \sum_{k=1}^{n} (\tilde{b}_{t_k} \bar{\tilde{b}}_{t_k} Q + \tilde{d}_{t_k} \bar{\tilde{b}}_{t_k} QV)\right\},\quad \left\{b_4 Q + d_4 QV = \sum_{k=1}^{n} (\tilde{d}_{t_k} \bar{\tilde{d}}_{t_k} Q + \tilde{b}_{t_k} \bar{\tilde{d}}_{t_k} QV)\right\},$$

as well as sum of these elements

$$\{a_0 P + b_0 Q + (c_0 P + d_0 Q)V\} =$$
$$\{(a_1 + a_2 + a_3 + a_4)P + (b_1 + b_2 + b_3 + b_4)Q + ((c_1 + c_2 + c_3 + c_4)P + (d_1 + d_2 + d_3 + d_4)Q)V\}.$$

On the other hand, the element $\{a_0 P + b_0 Q + (c_0 P + d_0 Q)V\}$ is invertible in $\hat{\mathbf{A}}$. Really, we have

$$a_0(\tau) \pm c_0(\tau) = \sum_{k=1}^{n} \left| a_{t_k}(\tau) \pm c_{t_k}(\tau) \right|^2 + \sum_{k=1}^{n} \left| \tilde{a}_{t_k}(\tau) \pm \tilde{c}_{t_k}(\tau) \right|^2 > 2n\lambda^2 > 0 \text{ and}$$

$$b_0(\tau) \pm d_0(\tau) = \sum_{k=1}^{n} \left| b_{t_k}(\tau) \pm d_{t_k}(\tau) \right|^2 + \sum_{k=1}^{n} \left| \tilde{b}(\tau)_{t_k} \pm \tilde{d}_{t_k}(\tau) \right|^2 > 2n\lambda^2 > 0,$$

where $\lambda = \min(\lambda_{t_1}, \lambda_{t_2}, ..., \lambda_{t_n})$. Hence, the element

$$\left\{ \frac{a_0}{a_0^2 - c_0^2} P + \frac{b_0}{b_0^2 - d_0^2} Q - (\frac{c_0}{a_0^2 - c_0^2} P + \frac{d_0}{b_0^2 - d_0^2} Q)V \right\}$$

belong to ideal M. Using relation (4.2), it is directly verified that

$$\{a_0 P + b_0 Q + (c_0 P + d_0 Q)V\} \cdot \left\{ \frac{a_0}{a_0^2 - c_0^2} P + \frac{b_0}{b_0^2 - d_0^2} Q - (\frac{c_0}{a_0^2 - c_0^2} P + \frac{d_0}{b_0^2 - d_0^2} Q)V \right\} = \{I\}.$$

From the last relation it results that M=$\hat{\mathbf{A}}$. The obtained contradictions prove the theorem.

By virtue of Ghelfand [1] results, according to which an element of some Banach algebra is invertible if and only if it does not belong to any maximal ideal, we obtain the following

***Theorem 4.2.*** *An element* $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathbf{A}}$ *is invertible in* $\hat{\mathbf{A}}$ *if and only if* functions $a(t) \pm c(t)$ *and* $b(t) \pm d(t)$ *are not vanished on contour* $\Gamma$.

We shall establish some other properties of algebra $\hat{\mathbf{A}}$. Observe that the intersection of all maximal ideals of $\hat{\mathbf{A}}$ coincides to the null ideal. In fact, by Theorem 4, if $\{aP+bQ+(cP+dQ)V\}\in\bigcap M_t$, then $a(t)+c(t)\equiv 0$, $a(t)-c(t)\equiv 0$, $b(t)+d(t)\equiv 0$ and $b(t)-d(t)\equiv 0$, that is $\{aP+bQ+(cP+dQ)V\}=\{0\}$. Consequently,

$1^0$. *Algebra* $\hat{\mathbf{A}}$ *has no radical.*

$2^0$. $\hat{\mathbf{A}}$ *is an involutive algebra.*

Define the involution by

$$\{aP+bQ+(cP+dQ)V\}^* = \{\bar{a}P+\bar{b}Q+(\bar{c}P+\bar{d}Q)V\}.$$

All properties of involution are evident. We shall show that only for every element $\{aP+bQ+(cP+dQ)V\}\in\hat{\mathbf{A}}$ there exists in $\hat{\mathbf{A}}$ the element

$$\left[I+\{(aP+bQ+(cP+dQ)V)\cdot(\bar{a}P+\bar{b}Q+(\bar{c}P+\bar{d}Q)V)\}\right]^{-1}.$$

Compute

$$\left[I+\{(aP+bQ+(cP+dQ)V)\cdot(\bar{a}P+\bar{b}Q+(\bar{c}P+\bar{d}Q)V)\}\right]=$$
$$\{(1+|a|^2+|c|^2)P+(1+|b|^2+|d|^2)Q+((a\bar{c}+\bar{a}c)P+(b\bar{d}+\bar{b}d)Q)V\},$$

$$1+|a(t)|^2+|c(t)|^2\pm(a(t)\bar{c}(t)+\bar{a}(t)c(t))=1+|a(t)\pm c(t)|^2>0,$$

$$1+|b(t)|^2+|d(t)|^2\pm(b(t)\bar{d}(t)+\bar{b}(t)d(t))=1+|b(t)\pm d(t)|^2>0.$$

Hence, there exists

$$\left[I+\{(aP+bQ+(cP+dQ)V)\cdot(\bar{a}P+\bar{b}Q+(\bar{c}P+\bar{d}Q)V)\}\right]^{-1}=$$
$$\left\{\begin{array}{l}\dfrac{1+|a|^2+|c|^2}{(1+|a-c|^2)(1+|a+c|^2)}P+\dfrac{1+|b|^2+|d|^2}{(1+|b-d|^2)(1+|b+d|^2)}Q-\\ -(\dfrac{a\bar{c}+\bar{a}c}{(1+|a-c|^2)(1+|a+c|^2)}P+\dfrac{b\bar{d}+\bar{b}d}{(1+|b-d|^2)(1+|b+d|^2)}Q)V\end{array}\right\}$$

and this element belongs to $\hat{\mathbf{A}}$. Property $2^0$ is proved.

Denote by $\mathbf{M}$ the bicompact of maximal ideals of $\hat{\mathbf{A}}$.

$3^0$. $\mathbf{M}$ *is isomorphic to the topological product* $(\Gamma\times j)\times(\Gamma\times k)$: $\mathbf{M}=(\Gamma\times j)\times(\Gamma\times k)$, *where* $j=\pm1$ *and* $k=\pm1$.

It is know [1] that every commutative Banach algebra without radical is isomorphically mapped into an algebra of functions, defined on bicompact of maximal ideals. In our case it is easy to observe that to element $A=\{aP+bQ+(cP+dQ)V\}\in\hat{\mathbf{A}}$ corresponds the function $A(M)=(a(t)+jc(t))(b(t)+kd(t))$.

$4^0$. *Algebra* $\hat{\mathbf{A}}$ *is a symmetric algebra without radical.*

In commutative and symmetric algebra $\mathbf{R}$ every element $x$ is invertible or is a generalized zero divisor (see [1]), that is, there exists a sequence $(y_n)$, $y_n\in\mathbf{R}, |y_n|=1$ and $\lim_{n\to\infty}\|y_n x\|=0$. Thus, every element $A=\{aP+bQ+(cP+dQ)V\}$, for which one of functions

$a(t)+c(t)$, $a(t)-c(t)$, $b(t)+d(t)$ or $b(t)-d(t)$ is vanished on $\Gamma$, is a generalized zero divisor.

Obvious, $L(L_p(\Gamma,\rho))\setminus\mathbf{T}$ is a (no commutative) Banach algebra including $\hat{\mathbf{A}}$.

$5^0$. *An element $A\in\hat{\mathbf{A}}$ is invertible in $L(L_p(\Gamma,\rho))\setminus\mathbf{T}$ if and only if it is invertible in $\hat{\mathbf{A}}$.*

In fact, let $A$ be invertible in $L(L_p(\Gamma,\rho))\setminus\mathbf{T}$ and suppose that it is not invertible in $\hat{\mathbf{A}}$, $A^{-1}\notin\hat{\mathbf{A}}$. Then, by virtue of $4^0$, $A$ is a generalized zero divisor. But this is impossible, since in this case the invertible operator $A$ should be a generalized zero divisor in $L(L_p(\Gamma,\rho))\setminus\mathbf{T}$.

## V. Regularization of operators of the form $A = aI + bS + (cI + dS)V + T$

Let us approach the problem of regularization of singular integral operators with shift $\omega$, $A=aI+bS+(cI+dS)V+T$. It is easy to observe that operator $A$ admits a regularization in algebra $L(L_p(\Gamma,\rho))$ if and only if element $\{aI+bS+(cI+dS)V\}\in\hat{\mathbf{A}}$ is invertibile in $L(L_p(\Gamma,\rho))\setminus\mathbf{T}$. In order to apply assertions of Theorem 4.2 and property $5^0$ we use operators

$$P=\tfrac{1}{2}(I+S),\ Q=\tfrac{1}{2}(I-S),\ I=P+Q \text{ and } S=P-Q.$$

Then operators $A$ is transcribed as $A=\alpha P+\beta Q+(\delta P+\gamma Q)V+T$, where $\alpha=a+b$, $\beta=a-b$, $\delta=c+d$, $\gamma=c-d$. From Theorem 4.2 and property $5^0$ it results that $\{\alpha P+\beta Q+(\delta P+\gamma Q)V\}$ is invertible in $L(L_p(\Gamma,\rho))\setminus\mathbf{T}$ if and only if functions $\alpha^2(t)-\delta^2(t)$ and $\beta^2(t)-\gamma^2(t)$ do not vanish on $\Gamma$. In other words, a singular integral operator $A$ with shift, $A=aI+bS+(cI+dS)V+T$, admits a regularization in $L(L_p(\Gamma,\rho))$ if and only if

$$\alpha^2(t)-\delta^2(t)=(a(t)+b(t))^2-(c(t)+d(t))^2\neq 0,$$
$$\beta^2(t)-\gamma^2(t)=(c(t)+d(t))^2-(c(t)+d(t))^2\neq 0.$$

Thus, condition (1.3) of Theorem 1.3 is satisfied. With the help of judgments used in the proof of Theorem 4.2 it is supplementary obtained that $AR=I+T_1$ and $RA=I+T_2$, where $R$ is defined by relation (1.4) and $T_1$, $T_2$ are compact operators.

***Theorem 5.1.*** *Operator $A=\alpha P+\beta Q+(\delta P+\gamma Q)V+T$ admits an equivalent regularization if and only if the following conditions*

$$\alpha^2(t)-\delta^2(t)\neq 0,\ \beta^2(t)-\gamma^2(t)\neq 0,\ ind\frac{\alpha^2(t)-\delta^2(t)}{\beta^2(t)-\gamma^2(t)}\leq 0$$

*are verified. Under these conditions*

$$IndA = -\frac{1}{2} ind \frac{\alpha^2(t) - \delta^2(t)}{\beta^2(t) - \gamma^2(t)}.$$

*For* $IndA < 0$ *all solutions of equation* $Ax = y$ *are obtained from din relation* $x = Rz$, *where* $z$ *runs all solutions to equation* $RAz = y$ *and* $R$ *is defined by (1.4).*

Cases when the function of shifting $\omega$, changes the orientation of contour $\Gamma$ and systems of singular integral equation with shift will be approached, possibly, in other works of author.

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