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CUBIC SYSTEMS WITH SEVEN INVARIANT STRAIGHT LINES ALONG ONE DIRECTION, WHEN SOME STRAIGHT LINES ARE COMPLEX CONJUGATE

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Abstract. Consider the generic cubic differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, where $P, Q \in \mathbb{R}[x,y]$, $\max\{\deg P, \deg Q\} = 3$, $GCD(P,Q) = 1$. If this system has enough invariant straight lines considered with their multiplicities, then, according to [1], we can construct a Darboux first integral. In this paper we obtain 8 canonical forms for cubic differential systems which possess invariant straight lines along one direction of total multiplicity seven including the straight line at the infinity and at least one planar invariant straight line is complex.

Keywords: cubic differential system, invariant straight line, Darboux integrability.

SISTEME CUBICE CU ŞAPTE DREPTE INVARIANTE DE-A LUNGUL UNEI DIRECŢII, CAND UNELE DREPTE SUNT COMPLEX CONJUGATE

Rezumat. Considerăm sistemul diferențial cubic general $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, unde, $P, Q \in \mathbb{R}[x,y]$, $\max\{\deg P, \deg Q\} = 3$, $GCD(P,Q) = 1$. Dacă acest sistem are suficiente linii drepte invariante considerate împreună cu multiplicitățile lor, atunci, conform [1], putem construi integrala primă Darboux. În această lucrare obținem 8 forme canonice pentru sistemele diferențiale cubice ce posedă drepte invariante de-a lungul unei direcții de multiplicitate totală șapte, inclusiv dreapta de la infinit și cel puțin o dreaptă invariantă planară este complexă.

Cuvinte cheie: sistem diferențial cubic, dreaptă invariantă, integrabilitate Darboux.

Introduction

We consider the real polynomial system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases}, \qquad GCD(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{2}$$

associated with system (1). Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 3$, then the system (1) is called cubic.

Definition 1. An algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$, is called invariant algebraic curve for the system (1), if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, such that the identity

$$\mathbb{X}(f) = f(x,y)K_f(x,y) \tag{3}$$

holds.

In this paper we are interested in invariant straight lines, i.e. invariant algebraic curves in the form $f(x,y) = \alpha x + \beta y + \gamma$. There are different types of multiplicities of invariant algebraic curves of differential polynomial systems, see for instance [2]. In this paper we will use the notion of algebraic multiplicity of an invariant straight line.

Definition 2: Let $\mathbb{C}_m[x]$ be the the $\mathbb{C}$ -vector space of polynomials in $\mathbb{C}[x]$ of degree at most *m*. Then it has dimension $R = C_{n+m}^{n}$. Let $v_1, v_2, \dots, v_R$ be a base of $\mathbb{C}_m[x]$. If k is the greatest positive integer such that the k-th power of $f(x,y)$ divides $\det M_R$, where

$$M_R = \begin{pmatrix} v_1 & v_2 & \dots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \dots & \mathbb{X}(v_R) \\ \dots & \dots & \dots & \dots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \dots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic cursve *f* of degree *m* of the vector field X has algebraic multiplicity *k*.

In the above definition, the expression $X^{R-1}(v_1)$ means that the operator X is applied *R-1* times on vector v_1, i.e. $X^{k+1}(v_i) = X(X^k(v_i))$.

There are a great number of articles dedicated to the investigation of polynomial differential systems with invariant straight lines. The problem of finding how many invariant straight lines a polynomial differential system can have is studied in [3]. In [4, 5, 6, 7] the authors study for the cubic differential system the problems of coexistence of invariant straight lines and limit cycles and coexistence of invariant straight lines and singular points of center type. In [8, 9] it was performed the classification of all cubic systems which have the maximum number of invariant straight lines including their multiplicities. In [10] were studied the cubic systems with exactly eight invariant straight lines. The cubic systems with six real invariant straight lines along two and three directions were studied in [11, 12].

In this paper we obtain all canonical forms of cubic differential systems with invariant straight lines along one direction from which at least one straight line is complex and with their total multiplicity equal to seven including the multiplicity of the line at the infinity.

Obtaining the system

Because we want that all the invariant straight lines to have a unique slope, it follows that the complex invariant straight lines don't pass through a real singular point, i.e. these complex invariant straight lines are of imaginary type. Using an affine transformation, we can make them parallel to *Oy* axe and we can bring them in the form $l_{1,2} \equiv x \pm i = 0$. Furthermore, if there exist a real invariant straight line, then it's equation is $l_3 \equiv x = 0$. Depending on the multiplicity of all invariant straight lines of the cubic differential system, we can distinguish the cases illustrated in Figure 1.

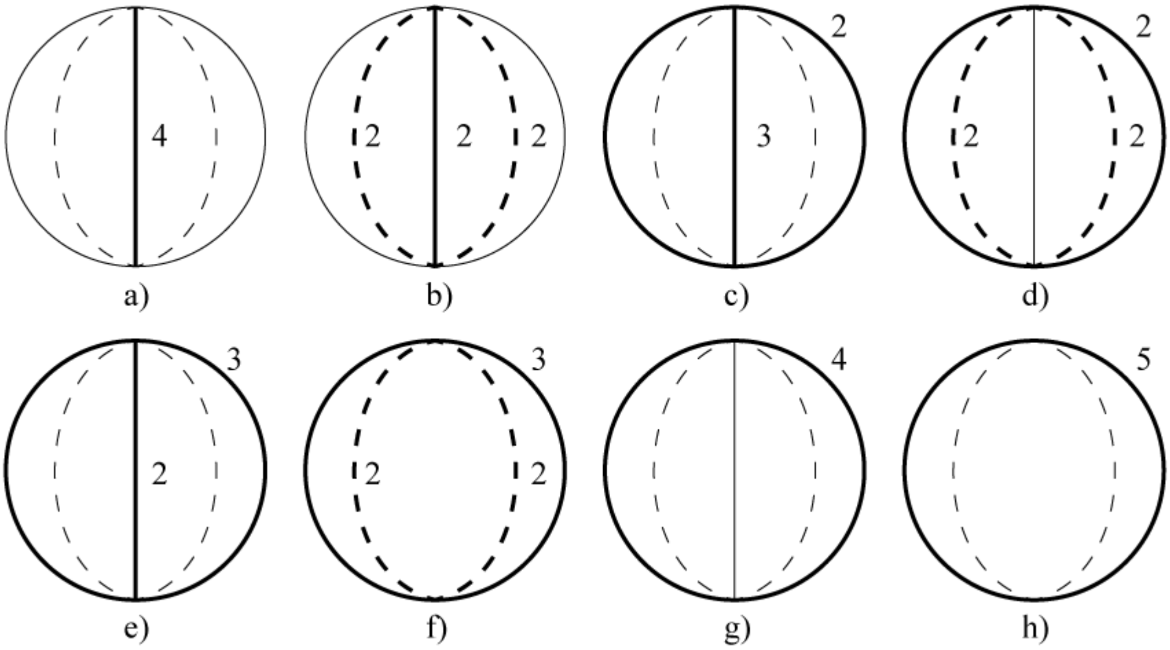

Figure 1

Therefore, we we'll seek that the generic cubic differential system

$$\begin{cases} \dot{x} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{cases} \tag{4}$$

where $a_{ij}\, b_{ij} \in \mathbb{R}$, to have invariant straight lines corresponding to Fig. 1.

a) The system (4) with the invariant straight lines $l_{1,2} \equiv x \pm i = 0$ and $l_{3,4,5,6} \equiv x = 0$ has the following form:

$$\begin{cases} \dot{x} = a_{10}x\left(x^2 + 1\right), \\ \dot{y} = b_{00} + a_{10}y + b_{00}x^2 + b_{30}x^3 + a_{10}x^2y. \end{cases}$$

By making the transformation $\bar{x} = x, \bar{y} = \frac{a_{10}}{b_{30}}y - \frac{b_{00}}{3b_{30}}$ and time rescaling $\bar{t} = a_{10}t$, we bring this system to the form

$$\begin{cases} \dot{x} = x\left(x^2 + 1\right), \\ \dot{y} = y + x^3 + x^2y. \end{cases} \tag{5}$$

b) The system (4) having all the invariant straight lines with the multiplicity 2 has the following form:

$$\begin{cases} \dot{x} = a_{10}x\left(x^2 + 1\right), \\ \dot{y} = b_{00} + b_{10}x + a_{10}y + b_{20}x^2 + b_{30}x^3 + 3a_{10}x^2y. \end{cases}$$

By making the transformation $\bar{x} = x, \bar{y} = \frac{3b_{30}}{6b_{00} - 2b_{20}}x + \frac{1}{b_{10}{}^2}y - \frac{b_{20}}{3b_{10}}$, rescaling the time by $\bar{t} = a_{10}t$ and using the notation $a = \frac{3b_{00} - b_{20}}{3b_{10}}$, we bring this system to the form

$$\begin{cases} \dot{x} = x\left(x^2+1\right), \\ \dot{y} = a + x + y + 3x^2 y. \end{cases} \quad (6)$$

c) In this case the infinity for the system (4) must have multiplicity equal to two. If it's invariant straight lines are $l_{1,2} \equiv x \pm i = 0$ and $l_{3,4,5} \equiv x = 0$, then it can be written

$$\begin{cases} \dot{x} = a_{10}x\left(x^2+1\right), \\ \dot{y} = b_{00} + a_{10}y + b_{20}x^2 + b_{30}x^3. \end{cases}$$

By making the transformation $\bar{x} = x, \bar{y} = \frac{a_{10}}{b_{30}}y - \frac{b_{00}}{b_{30}}$, rescaling the time by $\bar{t} = a_{10}t$ and using the notation $a = \frac{b_{20}}{b_{30}}$, we bring this system to the form

$$\begin{cases} \dot{x} = x\left(x^2+1\right), \\ \dot{y} = y + ax^2 + x^3. \end{cases} \quad (7)$$

d) The system (4) with invariant straight lines $l_{1,2,3,4} \equiv x \pm i = 0$, $l_5 \equiv x = 0$ and the multiplicity of the infinity equal to two can be brought to the form:

$$\begin{cases} \dot{x} = a_{10}x\left(x^2+1\right), \\ \dot{y} = b_{00} + b_{10}x - 2a_{10}y + b_{20}x^2 + b_{30}x^3. \end{cases}$$

By making the transformation $\bar{x} = x, \bar{y} = \frac{a_{10}}{b_{30}}y + \frac{b_{00}}{2b_{30}}$, rescaling the time by $\bar{t} = a_{10}t$ and using the notations $a = b_{10}, b = b_{20}$, we bring this system to the form

$$\begin{cases} \dot{x} = x\left(x^2+1\right), \\ \dot{y} = ax - 2y + bx^2 + x^3. \end{cases} \quad (8)$$

e) Let the system (4) have the infinity with the multiplicity equal to 3. In addition, if the straight lines $l_{1,2} \equiv x \pm i = 0$ and $l_{3,4} \equiv x = 0$ are invariant, then it has the form

$$\begin{cases} \dot{x} = a_{10}x\left(x^2+1\right), \\ \dot{y} = b_{00} + b_{10}x + a_{10}y + b_{30}x^3. \end{cases}$$

By making the transformation $\bar{x} = x, \bar{y} = \frac{a_{10}}{b_{30}}y - \frac{b_{00}}{b_{30}}$, rescaling the time by $\bar{t} = a_{10}t$ and using the notation $a = \frac{b_{10}}{b_{30}}$, we bring this system to the form

$$\begin{cases} \dot{x} = x\left(x^2+1\right), \\ \dot{y} = ax + y + x^3. \end{cases} \quad (9)$$

f) If the system (4) has the invariant straight lines $l_{1,2,3,4} \equiv x \pm i = 0$ and the infinity with the multiplicity equal to 3, then it can be brought to one of the following two forms:

$$\begin{cases} \dot{x} = a_{20}\left(x^2+1\right), \\ \dot{y} = b_{00} + b_{10}x + b_{20}x^2 + 2a_{20}xy + b_{30}x^3; \end{cases}$$

$$\begin{cases} \dot{x} = a_{10}\left(x^2+1\right), \\ \dot{y} = b_{00} + b_{10}x - 2a_{10}y + b_{30}x^3; \end{cases}$$

Let's note that in the second system appeared the invariant straight line $l_8 \equiv x = 0$. By making the transformation $\bar{x} = x, \bar{y} = \frac{b_{20}}{b_{30}}x + \frac{a_{20}}{b_{30}}y - \frac{b_{10}}{2b_{30}}$, rescaling the time by $\bar{t} = a_{20}t$ and using the notation $a = \frac{b_{00}+b_{20}}{b_{30}}$, we bring the first system to the form

$$\begin{cases} \dot{x} = x^2+1, \\ \dot{y} = a + 2xy + x^3. \end{cases} \tag{10}$$

We get the system

$$\begin{cases} \dot{x} = x\left(x^2+1\right), \\ \dot{y} = -2y + ax^3. \end{cases} \tag{11}$$

if in the second differential system we use the transformation $\bar{x} = x, \bar{y} = -\frac{b_{10}}{3b_{30}}x + \frac{a_{10}}{b_{30}}y + \frac{b_{00}}{2b_{30}}$, rescale the time $\bar{t} = a_{10}t$ and denote $a = 1 - \frac{b_{10}}{3b_{30}}$.

g) In this case the infinity for the system (4) must have multiplicity equal to 4. If its invariant straight lines are $l_{1,2} \equiv x \pm i = 0$ and $l_3 \equiv x = 0$, then it can be written

$$\begin{cases} \dot{x} = a_{10}x\left(x^2+1\right), \\ \dot{y} = b_{00} + b_{30}x + b_{30}x^3. \end{cases}$$

By making the transformation $\bar{x} = x, \bar{y} = -\frac{b_{30}}{b_{00}}x + \frac{a_{10}}{b_{00}}y$ and rescaling the time by $\bar{t} = a_{10}t$, we bring this system to the form

$$\begin{cases} \dot{x} = x\left(x^2+1\right), \\ \dot{y} = 1. \end{cases} \tag{12}$$

h) Asking that the system (4) to have invariant straight lines $l_{1,2} \equiv x \pm i = 0$ and the invariant straight line at the infinity to have multiplicity five, we obtain a system of algebraic equations which have no solution. Therefore, there is no differential cubic system whose invariant straight lines meet this case.

According to the above results, we proved the following theorem:

Theorem: Any cubic differential system with invariant straight lines along one direction with total algebraic multiplicity equal to seven including the line at the infinity, when at least one invariant straight line is complex, by an affine transformation and time rescaling can be brought to one of the systems (5) – (12).

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