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SOLUTION OF THE CENTER-FOCUS PROBLEM FOR A CUBIC DIFFERENTIAL SYSTEM WITH A REAL INVARIANT STRAIGHT LINE AND THE LINE AT INFINITY OF MULTIPLICITY TWO

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Abstract. In this paper, we show that for cubic differential system $\dot{x} = y(x-1)^2,\ \dot{y} = -(x + gx^2 + dxy + by^2 + qx^2y)$ the critical point (0,0) is a center if and only if the first four Lyapunov quantities vanish ($L_1 = L_2 = L_3 = L_4 = 0$) or, equivalently, if at least one of the following two sets of conditions: 1) $b = 0,\ q = dg$; 2) $d = q = 0$ holds.

Keywords: cubic differential system, invariant straight line, multiplicity, the center problem.

REZOLVAREA PROBLEMEI CENTRULUI PENTRU UN SISTEM DIFERENŢIAL CUBIC CE ARE O DREAPTĂ INVARIANTĂ REALĂ ŞI LINIA DE LA INFINIT DE MULTIPLICITATEA DOI

Rezumat. În lucrarea de faţă se arată că pentru sistemul diferenţial cubic $\dot{x} = y(x-1)^2,\ \dot{y} = -(x + gx^2 + dxy + by^2 + qx^2y)$, punctul critic (0,0) este de tip centru, dacă şi numai dacă se anulează primele patru mărimi Lyapunov ($L_1 = L_2 = L_3 = L_4 = 0$) sau, echivalent, dacă are loc cel puţin unul dintre seturile de condiţii: 1) $b = 0,\ q = dg$; 2) $d = q = 0$.

Cuvinte cheie: sistem diferențial cubic, dreaptă invariantă, multiplicitate, problema centrului.

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1. Introduction

We consider the polynomial differential system

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y), \quad \gcd(P,Q) = 1, \tag{1}$$

$P, Q \in \mathbb{R}[x,y]$, and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to systems (1). Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ ($n = 3$), then the system (1) is called *quadratic* (*cubic*).

An algebraic curve $f(x,y) = 0, f \in \mathbb{C}[x,y]$ (a function $f = exp\left(\frac{g}{h}\right); g, h \in \mathbb{C}[x,y], \deg(g) \leq \deg(h), \gcd(g,h) = 1$) is called an *invariant algebraic curve* (*exponential factor*) of the system (1) if there exists a polynomial $K_f \in \boldsymbol{C}[x,y], degK_f \leq n-1$ such that the identity

$$\frac{\partial f(x,y)}{\partial x}\cdot P(x,y) + \frac{\partial f(x,y)}{\partial y}\cdot Q(x,y) \equiv f(x,y)\cdot K_f(x,y) \tag{2}$$

holds. The polynomial $K_f(x,y)$ is called the *cofactor* of the invariant algebraic curve (exponential factor) f. In particular, a straight line $f(x,y) \equiv \alpha x + \beta y + \gamma = 0, (\alpha, \beta) \neq$

$(0,0)$ is invariant for (1) if the following identity $\alpha P(x,y)+\beta Q(x,y)\equiv(\alpha x+\beta y+\gamma)\cdot K_f(x,y)$ holds.

If $f=exp\left(\frac{g}{h}\right), \deg(h)>0$ is an exponential factor, then $h(x,y)=0$ is an invariant algebraic curve for (1) [1].

Let $f(x,y)=0$ be an invariant straight line and m is the greatest natural number such that f^m divide $P\cdot\mathbb{X}(Q)-Q\cdot\mathbb{X}(P)$. In this case we say that the line $f(x,y)=0$ has *multiplcity* m. By work [2], if the invariant straight line $h(x,y)=0$ has multiplicity $m, m>1$, then the system (1) has $m-1$ exponential factors of the form: $exp\left(\frac{g_1}{h}\right),\cdots,exp\left(\frac{g_{m-1}}{h^{m-1}}\right), \deg(g_l)\le\deg(h^l)$.

Assume that $P(x,y)=P_0+P_1(x,y)+\cdots+P_n(x,y), Q(x,y)=Q_0+Q_1(x,y)+\cdots+Q_n(x,y)$, where P_j, Q_j are omogeneous polynomials of degree j, and $\tilde{P}(x,y,z),\tilde{Q}(x,y,z)$, are homogenization of $P(x,y)$ and $Q(x,y)$, respectively, i.e.

$$\tilde{P}(x,y,z)=P_0\cdot z^n+P_1(x,y)\cdot z^{n-1}+\cdots+P_{n-1}(x,y)\cdot z+P_n(x,y),$$
$$\tilde{Q}(x,y,z)=Q_0\cdot z^n+Q_1(x,y)\cdot z^{n-1}+\cdots+Q_{n-1}(x,y)\cdot z+Q_n(x,y).$$

In this paper we suppose that $yP_n(x,y)-xQ_n(x,y)\not\equiv 0$. Denote $\tilde{\mathbb{X}}=\tilde{P}(x,y,z)\frac{\partial}{\partial x}+\tilde{Q}(x,y,z)\frac{\partial}{\partial y}$. We say that the natural number $m+1$ is *the multiplicity of the line at infinity* $z=0$ if m is the greatest number such that z^m divide $\tilde{P}\cdot\tilde{\mathbb{X}}(\tilde{Q})-\tilde{Q}\cdot\tilde{\mathbb{X}}(\tilde{P})$. According to [3], if the line at infinity has the multiplicity $m, m>1$, then (1) has $m-1$ exponential factors of the form $e^{g_1},\cdots,e^{g_{m-1}}$, where $g_l, l=1,\cdots,m-1$, are polynomial in x and y.

Up to now a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines (see, for example, [4]-[17]).

In this paper the cubic system

$$\dot{x}=y(x-1)^2,\quad \dot{y}=-(x+gx^2+dxy+by^2+qx^2y) \tag{3}$$

is examined. For this system the invariant straight line $x-1=0$, together with the line at infinity $z=0$, has the multiplicity two. Indeed, for (3) we have

$$\begin{aligned}P\cdot\mathbb{X}(Q)-Q\cdot\mathbb{X}(P)=-(x-1)^2(&x^2+2gx^3+g^2x^4+dx^2y+dgx^3y+qx^3y\\&+gqx^4y+y^2+2gxy^2-x^2y^2-2gx^2y^2+dy^3-bdxy^3+2qxy^3\\&-dx^2y^3-2qx^2y^3-bqx^2y^3+2by^4-b^2y^4-2bxy^4)\end{aligned}$$

and

$$\begin{aligned}\tilde{P}\cdot\tilde{\mathbb{X}}(\tilde{Q})-\tilde{Q}\cdot\tilde{\mathbb{X}}(\tilde{P})=-z(x-z)^2(&gqx^4y-dx^2y^3-2qx^2y^3-bqx^2y^3-2bxy^4\\&+g^2x^4z+dgx^3yz+qx^3yz-x^2y^2z-2gx^2y^2z-bdxy^3z+2qxy^3z+2by^4z\\&-b^2y^4z+2gx^3z^2+dx^2yz^2+2gxy^2z^2+dy^3z^2+x^2z^3+y^2z^3).\end{aligned}$$

Note that the system (3) is contained in the family of cubic systems

$$\begin{cases}\dot{x}=y+ax^2+cxy+fy^2+kx^3+mx^2y+pxy^2+ry^3,\\ \dot{y}=-(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3).\end{cases} \tag{4}$$

For (4) the origin (0,0) is a critical point of a focus or a center type. The problem of distinguishing between a center and a focus is called *the problem of the center* or *the center-focus problem.* At present, this problem has been solved only for some particular cases of systems (4). For a better initiation in the study of the center problem we recommend to readers the monographs [19], [20] and [21].

2. The problem of the center and integrability of polynomial differential systems

Let D be a domain in $\mathbb{R}^2$ and $F \in C^1(D, \mathbb{R})$ $\big(\mu \in C^1(D, \mathbb{R})\big)$. The function $F(x, y)$ $(\mu(x, y))$ is called a *first integral* (an *integrating factor*) of system (1) if the following identity

$$P(x,y)\frac{\partial F}{\partial x} + Q(x,y)\frac{\partial F}{\partial y} \equiv 0$$

$$\left(P(x,y)\frac{\partial \mu}{\partial x} + Q(x,y)\frac{\partial \mu}{\partial y} \equiv -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mu(x,y) \right)$$

occurs in D.

Let $f_1, \dots, f_\delta$ $\left(f_{\delta+1} = \exp\left(\frac{g_{\delta+1}}{h_{\delta+1}}\right), \dots, f_\sigma = \exp\left(\frac{g_\sigma}{h_\sigma}\right)\right)$ be invariant algebraic curves (exponential factors) of system (1). If this system has a first integral (an integrating factor) of the form

$$F(x,y) = \prod_{j=1}^{\sigma} f_j^{\alpha_j} \qquad \left(\mu(x,y) = \prod_{j=1}^{\sigma} f_j^{\alpha_j}\right), \tag{5}$$

where $\alpha_j \in \boldsymbol{C}, j = 1, \dots, \sigma,\ |\alpha_1| + \dots + |\alpha_\sigma| \neq 0,$ then we say that (1) is *Darboux integrable.*

It is easy to show that the function $F(x, y)$ $\big(\mu(x, y)\big)$ (5) is a first integral (an integrating factor) of (1) if and only if the following identity holds:

$$\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \dots + \alpha_\sigma K_{f_\sigma} \equiv 0 \tag{6}$$

$$\left(\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \dots + \alpha_\sigma K_{f_\sigma} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0\right), \tag{7}$$

where $K_{f_j}, j = 1,2, \dots, \sigma,$ are the cofactors of the invariant algebraic curves $f_j, j = 1, \dots, \delta,$ and the exponential factors $f_j, j = r + 1, \dots, \sigma,$ respectively.

According to [18] the system (4) has a center at the origin (0,0) if and only if it has in some neighborhood of (0,0) an analitic first integral $F(x, y)$, i.e. $F \in C^\omega$. Also, the system (4) has a center at (0,0) if and only if it has in some neighborhood of (0,0) an integrating factor of the form $\mu(x, y) = 1 + \sum \mu_j(x, y),\ \ \mu \in C^\omega$.

3. Calculation of the Lyapunov quantities

It is known there exists a formal power series $F(x, y) = \sum F_j(x, y)$ such that the rate of change of $F(x, y)$ along trajectories of (4) is a linear combination of the polynomials $\{(x^2 + y^2)^j\}_{j=2}^{\infty}$, i.e. $\mathbb{X}(F) \equiv \sum_{j=2}^{\infty} L_{j-1}(x^2 + y^2)^j$. Quantities $L_j,\ j = \overline{1, \infty}$ are polynomials in the coefficients of system (4) called to be *the Lyapunov quantities*. The origin (0,0) is a center for (4) if and only if $L_j = 0, j = \overline{1,\infty}$. In the case $L_j = 0, j = \overline{1,\infty}$ the function $F(x,y)$ is a first integral for (4).

With exactity of a non-constant factor, the Lyapunov quantities can be calculated by formula:

$$L_{j-1} = (2(j+1)Av_{j-2,j+1} + 2jCv_{j-1,j} + 2(j-1)Fv_{j,j-1} + (j+1)Kv_{j-3,j+1} + jMv_{j-2,j} + (j-1)Pv_{j-1,j-1} + (j-2)Rv_{j,j-2} - 2(j+1)Bv_{j+1,j-2} - 2jDv_{j,j-1} - 2(j-1)Gv_{j-1,j} - (j+1)Lv_{j+1,j-3} - jNv_{j,j-2} - (j-1)Qv_{j-1,j-1} - (j-2)Sv_{j-2,j})/(2i), \quad (8)$$

$j = 2,3,4,\ldots$, where $i^2 = -1$,
$A = \mathrm{g} - \mathrm{b} - \mathrm{c} + \mathrm{i}(\mathrm{a} + \mathrm{d} - \mathrm{f}), \mathrm{C} = 2(\mathrm{b} + \mathrm{g}) + 2\mathrm{i}(\mathrm{a} + \mathrm{f}), F = \mathrm{c} + \mathrm{g} - \mathrm{b} + \mathrm{i}(\mathrm{a} - \mathrm{d} - \mathrm{f})$,
$K = \mathrm{r} - \mathrm{m} - \mathrm{n} + \mathrm{s} + \mathrm{i}(\mathrm{k} + \mathrm{q} - \mathrm{p} - \mathrm{l}), \mathrm{M} = 3\mathrm{s} - \mathrm{m} + \mathrm{n} - 3\mathrm{r} + \mathrm{i}(3\mathrm{k} + \mathrm{q} + \mathrm{p} + 3\mathrm{l})$,
$P = 3\mathrm{s} + \mathrm{m} + \mathrm{n} + 3\mathrm{r} + \mathrm{i}(3\mathrm{k} - \mathrm{q} + \mathrm{p} - 3\mathrm{l}), \mathrm{R} = \mathrm{m} + \mathrm{s} - \mathrm{n} - \mathrm{r} + \mathrm{i}(\mathrm{k} - \mathrm{q} - \mathrm{p} + \mathrm{l})$,
$B = \bar{A}, D = \bar{C}, G = \bar{F}, L = \bar{K}, N = \bar{M}, Q = \bar{P}, S = \bar{R}$; $v_{0,0} = v_{1,0} = v_{0,1} = v_{2,0} = v_{1,1} - 1 = v_{0,2} = 0$, $v_{u,u} = 0$, if $u > 1$, $v_{u,j} = 0$, if $u < 0$ or $j < 0$, and

$$v_{u,j} = \frac{1}{8(u-j)}[2(j+1)Av_{u-2,j+1} + 2jCv_{u-1,j} + 2(j-1)Fv_{u,j-1} + (j+1)Kv_{u-3,j+1} + jMv_{u-2,j} + (j-1)Pv_{u-1,j-1} + (j-2)Rv_{u,j-2} - 2(u+1)Bv_{u+1,j-2} - 2uDv_{u,j-1} - 2(u-1)Gv_{u-1,j} - (u+1)Lv_{u+1,j-3} - uNv_{u,j-2} - (u-1)Qv_{u-1,j-1} - (u-2)Sv_{u-2,j}],$$

if $u + j = 3,4,\ldots$, $u \geq 0, j \geq 0, u \neq j$.

The first Lyapunov quantity of (4), calculated by formula (8), looks as

$$L_1 = bd - ac + 2bf - cf - 2ag + dg + 3k - 3l + p - q.$$

4. Axis of symmetry

Denote by $\mathcal{P}^-$ and $\mathcal{P}^+$ the semi-planes bounded by a straight line $\alpha x + \beta y = 0$, $\alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0,0)$. If the trajectories (trajectories segments) of the system (4) are symmetric with respect to $\alpha x + \beta y = 0$ and the directions of the trajectories (trajectories segments) on $\mathcal{P}^-$ and $\mathcal{P}^+$ are of the opposite directions, then $\alpha x + \beta y = 0$ is called an *axis of symmetry* for (4).

If the system (4) has an axis of symmetry, then the critical point (0,0) is of center type.

The necessary and sufficient conditions for (4) to have an axis of symmetry are obtained in the monograph [21]:

$$AD^3 - BC^3 = AF^3 - BG^3 = A^4L^3 - B^4K^3 = A^2N^3 - B^2M^3 = A^2R^3 - B^2S^3 = CF - DG = C^4L - D^4K = C^2N - D^2M = C^2R - D^2S = F^4K - G^4L = F^2M - G^2N = F^2S - G^2R = KN^2 - LM^2 = KR^2 - LS^2 = MR - NS = P - Q = 0. \quad (9)$$

The conditions (9) ensure the existence of an axis of symmetry and at the same time they are also sufficient conditions for system (4) to have a center at the origin.

Using the vector field, generated in the phase plane by system (1), it is easy to show that the coordinate axis Oy (respectively, Ox) is an axis of symmetry for (1), if the folowing two equalities are realized:

$$P(-x,y) = P(x,y), \qquad Q(-x,y) = -Q(x,y)$$

(respectively, $P(x,-y) = -P(x,y),\ Q(x,-y) = Q(x,y)$).

5. Solution of the problem of the center for system (3)

Lemma 1. *Let one of the following conditions be satissfied:*

$$1)\ b = 0,\ q = dg \qquad or \qquad 2)\ d = q = 0.$$

Then a critical point $(0,0)$ *of system* (3) *is a center, i.e. the conditions* 1) *and* 2) *are sufficient for the origin* $(0,0)$ *to be a center for* (3).

Proof. **1)** Assume that b$= 0, q = dg$. Then the system (3) looks as:

$$\dot{x} = y(x-1)^2,\ \dot{y} = -x(gx+1)(dy+1). \qquad (10)$$

Using the formula (2) it is easy to verify that the straight lines $f_1 = x - 1$ and $f_2 = dy + 1$ are invariant for (10) with cofactors: $K_{f_1}(x,y) = y(x-1)$ and $K_{f_2}(x,y) = -dx(gx+1)$, respectively. Similarly, using (2) we show that $f_3 = \exp\left(\frac{1}{x-1}\right)$ and $f_4 = \exp(dgx + y)$ are exponential factors with cofactors: $K_{f_3}(x,y) = -y$ and $K_{f_4}(x,y) = -x + dgy - gx^2 - dxy - 2dgxy$. Putting s = 4 and replacing the expressions of the cofactors $K_{f_j}(x,y), j = 1,2,3,4$ in (6), we obtain: $\alpha_1 = -d^2(1+2g)$, $\alpha_2 = 1,\ \alpha_3 = d^2(1+g),\ \alpha_4 = -d$. Thus,

$$F(x,y) = (1+dy)(x-1)^{-d^2(1+2g)} \cdot e^{\frac{d^2(1+g)}{x-1} - d(dgx+y)}$$

is a first integral for system (10) and according to [18], the critical point (0,0) is a center for the given system.

2) Assume that $d = q = 0$. Under these conditions, the system (3) has the form

$$\dot{x} = y(x-1)^2,\ \dot{y} = -(x + gx^2 + by^2). \qquad (11)$$

The last system (11) has the invariant straight line $f_1 = x - 1$ and the exponential factors $f_2 = \exp\left(\frac{1}{x-1}\right),\ \ f_3 = \exp(y)$, which have respectively the cofactors: $K_{f_1}(x,y) = y(x-1), K_{f_2}(x,y) = -y,\ \ K_{f_3}(x,y) = -x - gx^2 - by^2$. For these cofactors the identitaty {(6), $\sigma = 3$} holds only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and the identity {(7), $\sigma = 3$} holds only if $\alpha_1 = -2, \alpha_2 = -2b, \alpha_3 = 0$. Thus, the system (11) has an integrating factor

$$\mu(x,y) = (x-1)^{-2} \exp\left(\frac{2b}{1-x}\right) \qquad (12)$$

and, therefore, *(0,0)* is a center for (11). Lemma 1 is proved.

Note that the exponential factor $f_3 = \exp(y)$ does not appear in the expression of the integrating factor (12).

The existence of a center at the critical point (0,0) is also assured by the fact that the system (11) has an axis of symmetry. Indeed, it is easy to verify that for (11) the equalities (9) hold. Moreover, if we denote, $P(x,y) = y(x-1)^2$, $Q(x,y) = -(x+gx^2+by^2)$, then, obviously, $P(x,-y) = -P(x,y)$, $Q(x,-y) = Q(x,y)$, and therefore the axis coordinate Ox is an axis of symmetry for system (11).

Theorem 1. *The differential system* (3) *has a critical point of center type at origin* ($0,0$) *if and only if the first four Lyapunov quantities vanish* ($L_1 = L_2 = L_3 = L_4 = 0$).

Proof. The first Lyapunov quantity, calculated at the critical point ($0,0$) of system (3) looks as $L_1 = d(b+g) - q$. Putting $q = d(b+g)$ and using the formula (8) we calculate the following three Lyapunov quantities: L_2, L_3 and L_4. We obtain that $L_2 = -bdG_2(b,g)/3$, $L_3 = bdG_3(b,d,g)/288$ and $L_4 = -bdG_4(b,d,g)/34560$, where

$G_2(b,g) = 1 + (b+g)(6+3b+5g)$;

$G_3(b,d,g) = 289 + 2516b + 6391b^2 + 6618b^3 + 2064b^4 + 61d^2 + 366bd^2 + 183b^2d^2 + 2020g + 9498bg + 14614b^2g + 7046b^3g + 366d^2g + 488bd^2g + 3351g^2 + 9846bg^2 + 7642b^2g^2 + 305d^2g^2 + 1850g^3 + 2890bg^3 + 230g^4$;

$G_4(b,d,g) = 115249 + 1338802b + 5874795b^2 + 13391788b^3 + 16403746b^4 + 9903060b^5 + 2218422b^6 + 73634d^2 + 644968bd^2 + 1615430b^2d^2 + 1571316b^3d^2 + 471768b^4d^2 + 4699d^4 + 28194bd^4 + 14097b^2d^4 + 917402g + 7365308bg + 24061412b^2g + 39266708b^3g + 30862268b^4g + 9099296b^5g + 523952d^2g + 2470260bd^2g + 3666596b^2d^2g + 1665412b^3d^2g + 28194d^4g + 37592bd^4g + 2121257g^2 + 13024452bg^2 + 31714428b^2g^2 + 34362848b^3g^2 + 13462002b^4g^2 + 900198d^2g^2 + 2649420bd^2g^2 + 1929044b^2d^2g^2 + 23495d^4g^2 + 1932972g^3 + 9270116bg^3 + 15795456b^2g^3 + 8814760b^3g^3 + 554140d^2g^3 + 839660bd^2g^3 + 524938g^4 + 2230156bg^4 + 2263282b^2g^4 + 104260d^2g^4 - 161660g^5 - 43080bg^5 - 72730g^6$.

The case $bd = 0$ was investigated in Lemma 1. Suposse that $bd \neq 0$. Then, the Lyapunov quantities L_2, L_3 and L_4 vanish if and only if the polynomials $G_2(b,g)$, $G_3(b,d,g)$ and $G_4(b,d,g)$ are equal to zero. The resultant of $G_2(b,g)$ and $G_3(b,d,g)$ with respect to the variable b look as

$$Resultant[G_2, G_3, b] = 15552(1+g)^2(5g^2-8).$$

If $g = -1$, then $G_2(b,-1) = b(3b-2), G_3\left(\frac{2}{3}, d, -1\right) = \frac{16}{3} \neq 0$. If $g = \pm 2\sqrt{\frac{2}{5}}$, then $G_2\left(b, \pm 2\sqrt{\frac{2}{5}}\right) = (45 \pm 12\sqrt{10} + 30b \pm 16\sqrt{10}b + 15b^2)/5$ and

$$G_3\left(b,d,\pm 2\sqrt{\frac{2}{5}}\right)=(31197\pm 9960\sqrt{10}+91348b\pm 28244\sqrt{10}b+93091b^2\pm 29228\sqrt{10}b^2$$

$$+33090b^3\pm 14092\sqrt{10}b^3+10320b^4+305G_2\left(b,\pm 2\sqrt{\frac{2}{5}}\right)d\hat{}2)\ /\ 5.$$

The system $\begin{cases} G_2\left(b,\pm 2\sqrt{\frac{2}{5}}\right)=0, \\ G_3\left(b,d,\pm 2\sqrt{\frac{2}{5}}\right)=0 \end{cases}$ has a single solution $b=\frac{1}{3}\left(-1\mp\sqrt{10}\right)$, but

$$G_4\left(\frac{1}{3}\left(-1\mp\sqrt{10}\right),d,\pm 2\sqrt{\frac{2}{5}}\right)=\frac{64}{75}(68\pm 5\sqrt{10})\neq 0.$$

Theorem 1 is proved.

From the proof of Theorem 1 and the statement of Lemma 1 it follows the following assertion.

Theorem 2. *The cubic differential system (3) has a critical point of the center type at the origin (0,0) if and only if its coefficients verify at least one of the following two sets of conditions*:

1) $b=0,\ q=dg$; 2) $d=q=0.$

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