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A LYAPUNOV TYPE INVARIANCE THEOREM FOR THE BOOLEAN ASYNCHRONOUS DYNAMICAL SYSTEMS

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Abstract. The asynchronous circuits from digital electronics are modeled by Boolean asynchronous dynamical systems, consisting in functions $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$ that iterate their coordinates $\Phi_1, ..., \Phi_n$ independently on each other. The purpose of this work is to give a Lyapunov-Lagrange type theorem that characterizes the invariance of the sets $A \subset \{0,1\}^n$ in the case of these systems.
Keywords: asynchronous circuits from digital electronics, Boolean asynchronous dynamical systems, Lyapunov-Lagrange type theorem, invariance of the sets.

O TEOREMĂ A INVARIANŢEI DE TIP LYAPUNOV PENTRU SISTEMELE DINAMICE ASINCRONE BOOLEENE

Rezumat. Circuitele asincrone ale dispozitivelor digitale sunt modelate de sisteme dinamice asincrone booleane, constând în funcţii $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$ care iterează coordonatele lor $\Phi_1, ..., \Phi_n$ independent unul pe celalalt. Scopul acestei lucrari este de a formula o teoremă de tip Lyapunov-Lagrange care caracterizează invarianţa seturilor $A \subset \{0,1\}^n$ în cazul acestor sisteme.
Keywords: circuite asincrone ale dispozitivelor digitale, sisteme dinamice asincrone booleane, teorema de tip Lyapunov-Lagrange, invarianţa mulţimilor.

1 Introduction

When modeling the circuits from digital electronics, the coordinates $\Phi_1, ..., \Phi_n$ of $\Phi : \{0,1\}^n \rightarrow \{0,1\}^n$ are not computed at the same time, synchronously, but independently on each other, asynchronously.

Table 1: Example

(μ_1, μ_2, μ_3)	$\Phi(\mu_1, \mu_2, \mu_3)$
$(0, 0, 0)$	$(0, 1, 1)$
$(0, 0, 1)$	$(0, 1, 1)$
$(0, 1, 0)$	$(0, 1, 0)$
$(0, 1, 1)$	$(1, 1, 1)$
$(1, 0, 0)$	$(0, 0, 0)$
$(1, 0, 1)$	$(1, 0, 1)$
$(1, 1, 0)$	$(1, 0, 0)$
$(1, 1, 1)$	$(1, 1, 0)$

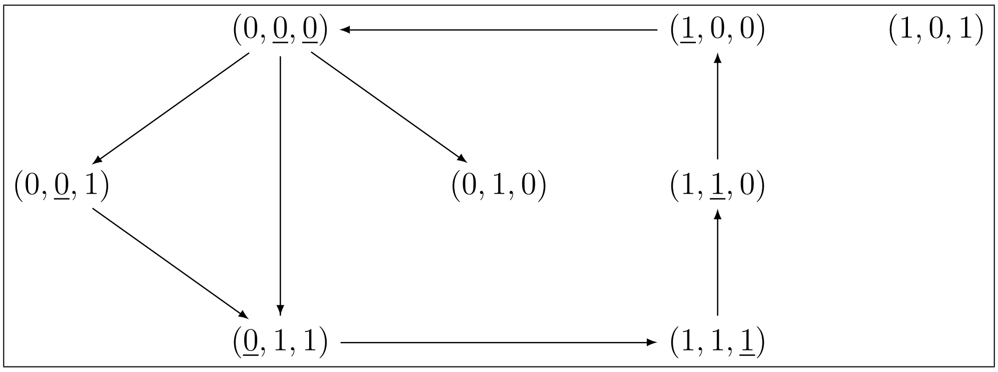

Figure 1. Dependence on the order in which Φ_1,Φ_2,$\Phi_3$are computed

The Boolean asynchronous dynamical systems were first introduced by the author in 2007. The dynamics of these systems is understood [1] by studying the function Φ from Table 1, whose state portrait was drawn in Figure 1. We analyze Figure 1, where the arrows show the increase of time. In the state $(\mu_1, \mu_2, \mu_3) \in \{0,1\}^3$ we have underlined $(0, \underline{0}, \underline{0})$ the coordinates called unstable, or excited, or enabled that fulfill $\mu_i \neq \Phi_i(\mu), i \in \{1,2,3\}$; these are the coordinates ready to switch, but the time instant and the order of the switches are not known. We must consider all the possibilities, as a value of the state has in general several possible successors.

$(1,0,1), (0,1,0)$ are fixed points (or equilibrium points, or rest positions, or final states), where the system stays indefinitely long; they have no underlined coordinates. The transition $(0,1,1) \longrightarrow (1,1,1)$ consists in the computation of $\Phi_1(0,1,1)$; even if we do not know when it happens, we know that it happens and the system, if it is in $(0,1,1)$, surely gets to $(1,1,1)$ sometime. And the transitions $(1,1,1) \longrightarrow (1,1,0), (1,1,0) \longrightarrow (1,0,0), (1,0,0) \longrightarrow (0,0,0)$ are similar. The interesting behavior is in $(0,0,0)$; since if $\Phi_3(0,0,0)$ is computed first, or if $\Phi_2(0,0,0)$, $\Phi_3(0,0,0)$ are computed at the same time, the system gets to $(0,1,1)$ sometime; but if $\Phi_2(0,0,0)$ is computed first, then $(0,1,0)$ is reached and the system rests there indefinitely long.

In the previous discussion:

a) a system is identified with a function $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$, with a state portrait and also with the flow ϕ defined by Φ;

b) the system that we refer to is non-initialized, without input, non-deterministic, with a variable structure given by the coordinates that are computed independently on each other;

c) time is discrete.

In [2] many results are stated, connecting several types of systems with several invariance/stability properties, as well as boundedness properties. The common part of these results is the existence of the Lyapunov-Lagrange functions. The purpose of our paper is to give a Lyapunov-Lagrange type invariance theorem, that is analogue with Theorem 3.3.1 from page 93, and Theorem 4.1.3 from page 151 in [2], in the case of the Boolean asynchronous dynamical systems.

2 Preliminaries

Notation 1 *We denote with* $\mathbf{B} = \{0, 1\}$ *the binary Boole algebra, endowed with the usual laws: complement* $\overline{}$, *product* $\cdot$, *union* $\cup$, *exclusive or* $\oplus$.

Definition 2 *Let* $\alpha : \mathbf{N} \to \mathbf{B}^n$ *a sequence* $\alpha^0, \alpha^1, ...$ *of binary* $n-$*tuples. It is said to be* ***progressive*** *if* $\forall i \in \{1, ..., n\}$, *the set* $\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$ *is infinite. The set of the progressive sequences* α *is denoted by* Π_n.

Definition 3 *The* $\lambda-$***iterate*** *of* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, *where* $\lambda \in \mathbf{B}^n$, *is the function* $\Phi^\lambda : \mathbf{B}^n \to \mathbf{B}^n$ *defined by* $\forall \mu \in \mathbf{B}^n, \forall i \in \{1, ..., n\}$,

$$\Phi_i^\lambda(\mu) = \begin{cases} \mu_i, \ if \ \lambda_i = 0, \\ \Phi_i(\mu), \ if \ \lambda_i = 1. \end{cases}$$

Definition 4 *Function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *defines the* ***flow*** $\phi : \Pi_n \times \mathbf{B}^n \times \mathbf{N} \to \mathbf{B}^n$ *by* $\forall \alpha \in \Pi_n, \forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}$,

$$\phi^\alpha(\mu, k) = \begin{cases} \mu, \ if \ k = 0, \\ \Phi^{\alpha^{k-1}}(\phi^\alpha(\mu, k-1)), \ if \ k \geq 1. \end{cases}$$

Φ *is called* ***generator function*** *(of* ϕ*),* α *is the* ***computation function*** *(of* Φ*),* μ *is the* ***initial state****, and* k *is the* ***time*** *variable.*

Remark 5 *The flows model the asynchronous computations, which are made in the absence of an input (equivalently: under constant input). We notice in Definition 4 that the coordinates* $\Phi_1, ..., \Phi_n$ *are computed independently on each other, asynchronously. The computation functions show when and how the iterations of* Φ *are made. Progressiveness of* α *asks that each coordinate* Φ_i *is computed infinitely many times.*

Definition 6 *We define the* ***Hamming distance*** *between* $\mu \in \mathbf{B}^n$ *and the set* $A \subset \mathbf{B}^n, A \neq \emptyset$ *by*

$$d(\mu, A) = \min_{\nu \in A} card(\{i | i \in \{1, ..., n\}, \mu_i \neq \nu_i\}).$$

3 Definitions of invariance

Lemma 7 *If* $\Phi : A \to A$ *and* A *is a finite set, then the following statements are equivalent:*

a) Φ *is injective,*

b) Φ *is surjective,*

c) Φ *is bijective.*

Proof. This result is elementary and its proof is omitted. ■

Theorem 8 *Let the function* $\Phi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ *and the set* $A \subset \mathbf{B}^n, A \neq \emptyset$. *The relations*

$$\forall \mu \in A, \exists \alpha \in \Pi_n, \forall k \in \mathbf{N}, \phi^\alpha(\mu, k) \in A, \quad (1)$$

$$\exists \alpha \in \Pi_n, \forall \mu \in A, \forall k \in \mathbf{N}, \phi^\alpha(\mu, k) \in A, \quad (2)$$

$$\forall \alpha \in \Pi_n, \forall \mu \in A, \forall k \in \mathbf{N}, \phi^{\alpha}(\mu, k) \in A, \tag{3}$$

$$\forall \lambda \in \mathbf{B}^n, \Phi^{\lambda}(A) \subset A, \tag{4}$$

$$\forall \lambda \in \mathbf{B}^n, \Phi^{\lambda}(A) = A \tag{5}$$

fulfill

$$(5) \Longrightarrow (4) \Longleftrightarrow (3) \Longrightarrow (2) \Longrightarrow (1).$$

Proof. The implications are obvious in general, we prove two of them.

(3)$\Longrightarrow$(4) Let $\lambda \in \mathbf{B}^n, \mu \in A$ arbitrary, fixed. We take $\alpha \in \Pi_n$ arbitrary, with $\alpha^0 = \lambda$. Then:

$$\Phi^{\lambda}(\mu) = \phi^{\alpha}(\mu, 1) \overset{(3)}{\in} A.$$

(4)$\Longrightarrow$(3) We take $\alpha \in \Pi_n, \mu \in A$ arbitrary, fixed and we prove (3) by induction on k. For $k = 0, \mu = \phi^{\alpha}(\mu, 0) \in A$ and we suppose that (3) is true for k. Then:

$$\phi^{\alpha}(\mu, k+1) = \Phi^{\alpha^k}(\phi^{\alpha}(\mu, k)) \overset{(4)}{\in} A.$$

■

Definition 9 *Relations (1),...,(5) are called of* ***invariance*** *of A. We say that A is a* $k'-$***invariant set*** *(or* $k'-$***stable set***), $k' \in \{(1), ..., (5)\}$ *(relative to* Φ*).*

Remark 10 *The invariance of a set A is the property of a system that, if it starts from an initial value* $\mu \in A$*, all the other values are in A. The different behaviours of the system in this situation make us have several definitions of invariance.*

Remark 11 *The concepts of invariance (of a set) and of stability (of a set) are borrowed from the real numbers systems theory, where they are distinct and present in several versions. Their brought in the Boolean context has produced a single concept.*

Remark 12 *Lemma 7 shows that property (5) has the following meaning:* $\forall \lambda \in \mathbf{B}^n$*, the restriction of* Φ^{λ} *to A has the values in A, and the resulted* $\Phi^{\lambda} : A \to A$ *function is bijective.*

4 Examples

Example 13 *We consider the state portrait from Figure 2,*

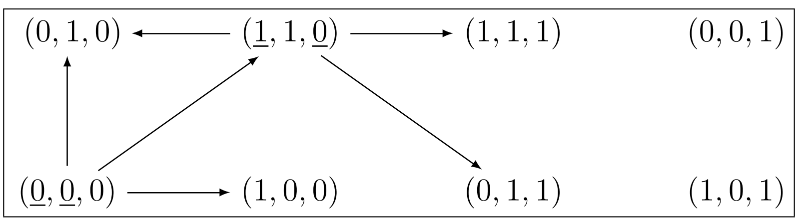

Figure 2. Function $\Phi(\mu_1, \mu_2, \mu_3) = (\overline{\mu_2}\ \overline{\mu_3} \cup \mu_1\mu_3, \overline{\mu_1}\ \overline{\mu_3} \cup \mu_2\mu_3 \cup \mu_1\mu_2, \mu_1\mu_2 \cup \mu_3)$

where the set $A = \{(0,0,0), (1,1,0), (1,1,1)\}$ *is* $(1)-$*invariant. In order to see this, we notice that for* $\alpha \in \Pi_3$ *with* $\alpha^0 = (1,1,0), \alpha^1 = (0,0,1)$ *we have*

$$\phi^{\alpha}((0,0,0), \cdot) = (0,0,0), (1,1,0), (1,1,1), (1,1,1), ... \in A,$$

for $\beta \in \Pi_3$ *with* $\beta^0 = (0,0,1)$ *we have*

$$\phi^{\beta}((1,1,0),\cdot) = (1,1,0),(1,1,1),(1,1,1),... \in A,$$

and for arbitrary $\gamma \in \Pi_3$ *we have*

$$\phi^{\gamma}((1,1,1),\cdot) = (1,1,1),(1,1,1),... \in A.$$

We prove now that (2) *is false. Indeed, if at time instant* $k \geq 1$ *the system runs through* A *(invariance), then* $(0,0,0) \to (1,1,0)$, $(1,1,0) \to (1,1,1)$ *are the only possibilities, giving the contradiction:*

$$(\underline{0},\underline{0},0) = \phi^{\alpha}(\mu,k-1) \neq \Phi^{\alpha^{k-1}}(\phi^{\alpha}(\mu,k-1)) = (1,1,0) \text{ implies } \alpha_1^{k-1} = 1,$$

$$(\underline{1},1,\underline{0}) = \phi^{\alpha}(\mu,k-1) \neq \Phi^{\alpha^{k-1}}(\phi^{\alpha}(\mu,k-1)) = (1,1,1) \text{ implies } \alpha_1^{k-1} = 0.$$

We have obtained that the choice of $\alpha \in \Pi_3$ *such that* (1) *holds depends on* $\mu \in A$.

Example 14 *Let in Figure 3 the set* $A = \{(0,0),(0,1)\}$.

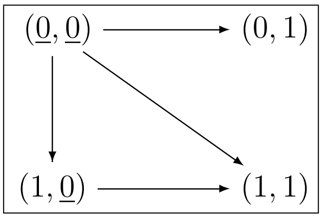

Figure 3. Function $\Phi(\mu_1,\mu_2) = (\mu_1 \cup \overline{\mu_2}, 1)$

A *is* (2)$-$*invariant: for this, it is enough to choose* $\alpha \in \Pi_2$ *with* $\alpha^0 = (0,1)$ *and see that*

$$\phi^{\alpha}((0,0),\cdot) = (0,0),(0,1),(0,1),... \in A.$$

A *is not* (3)$-$*invariant since if, for* $\alpha \in \Pi_2$, *we take* $\alpha^0 = (1,1)$, *we get* $\forall k \geq 1$, $\phi^{\alpha}((0,0),k) = (1,1) \notin A$. $A' = \{(0,1)\}$ *is (5)-invariant.*

For this system, $B = \{(0,0),(1,0),(1,1)\}$ *is (2)-invariant,* $B' = \{(1,0),(1,1)\}$ *is (3)-invariant, and* $B'' = \{(1,1)\}$ *is (5)-invariant.*

Example 15 *In Figure 4,*

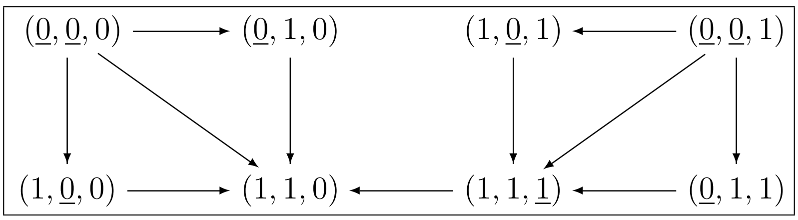

Figure 4. Function $\Phi(\mu_1,\mu_2,\mu_3) = (1,1,\overline{\mu_1}\mu_3 \cup \overline{\mu_2}\mu_3)$

the set $A = \{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$ *is* (3)$-$*invariant.*
The subsets $A' = \{(1,0,0),(1,1,0)\}$, $A'' = \{(0,1,0),(1,1,0)\}$ *are (3)-invariant and* $A''' = \{(1,1,0)\}$ *is (5)-invariant.*

We see that $B = \{(0,0,1),(1,0,1),(0,1,1),(1,1,1),(1,1,0)\}$ *is (3)-invariant also, together with some of its subsets.*

Example 16 *We have the example from Figure 5,*

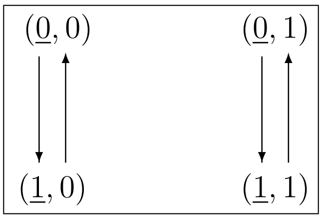

Figure 5. Function $\Phi(\mu_1, \mu_2) = (\overline{\mu_1}, \mu_2)$.

where the whole space $\mathbf{B}^2$ *is* $(5)-$*invariant. Indeed, for different values of* $\lambda \in \mathbf{B}^2$, *from* $\Phi^{\lambda}(\mu_1, \mu_2) = (\mu_1 \oplus \lambda_1, \mu_2)$ *we get two functions,* $1_{\mathbf{B}^2}(\mu_1, \mu_2) = (\mu_1, \mu_2)$ *and* $\Phi(\mu_1, \mu_2) = (\mu_1 \oplus 1, \mu_2)$. *Functions* $1_{\mathbf{B}^2}$ *and* Φ *are both bijective.*

In this example the sets $\{(0,0),(1,0)\}, \{(0,1),(1,1)\}$ *are (5)-invariant too.*

Example 17 *The identity* $1_{\mathbf{B}^2}$

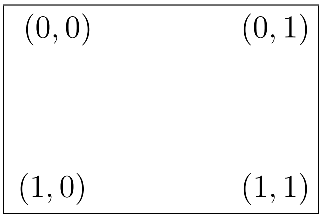

Figure 6. Function $1_{\mathbf{B}^2}$.

has the property that any nonempty subset of $\mathbf{B}^2$ *is* $(5)-$*invariant.*

5 A Lyapunov-Lagrange Type Invariance Theorem

Definition 18 *We use to say about a function* $\varphi : \mathbf{N} \to \mathbf{N}$ *that it is* ***strictly increasing*** *if* $\forall k \in \mathbf{N}, \forall k' \in \mathbf{N}, k < k' \Longrightarrow \varphi(k) < \varphi(k')$ *and* ***decreasing*** *if* $\forall k \in \mathbf{N}, \forall k' \in \mathbf{N}, k < k' \Longrightarrow \varphi(k) \geq \varphi(k')$.

Theorem 19 *Let* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and the nonempty set* $A \subset \mathbf{B}^n$. *We consider the following statements involving the functions* $V : \mathbf{B}^n \to \mathbf{N}$ *and* $\varphi_1, \varphi_2 : \mathbf{N} \to \mathbf{N}$*:*

$$\varphi_1(0) = \varphi_2(0) = 0, \tag{6}$$

$$\varphi_1, \varphi_2 \text{ are strictly increasing,} \tag{7}$$

$$\forall \mu \in \mathbf{B}^n, \varphi_1(d(\mu, A)) \leq V(\mu) \leq \varphi_2(d(\mu, A)), \tag{8}$$

$$\forall \mu \in A, \exists \alpha \in \Pi_n, V(\phi^{\alpha}(\mu, \cdot)) \text{ is decreasing,} \tag{9}$$

$$\exists \alpha \in \Pi_n, \forall \mu \in A, V(\phi^{\alpha}(\mu, \cdot)) \text{ is decreasing,} \tag{10}$$

$$\forall \mu \in A, \forall \alpha \in \Pi_n, V(\phi^{\alpha}(\mu, \cdot)) \text{ is decreasing,} \tag{11}$$

$$\forall \mu \in A, \forall \mu' \in A, \forall \lambda \in \mathbf{B}^n, \mu \neq \mu' \Longrightarrow \Phi^{\lambda}(\mu) \neq \Phi^{\lambda}(\mu'). \tag{12}$$

a) The functions V, φ_1, φ_2 exist such that (6), (7), (8), (9) hold if and only if A is $(1)-$*invariant;*

b) the functions V, φ_1, φ_2 exist such that (6), (7), (8), (10) hold if and only if A is $(2)-$*invariant;*

c) the functions V, φ_1, φ_2 exist such that (6), (7), (8), (11) hold if and only if A is $(3)-$*invariant.*

d) the functions V, φ_1, φ_2 exist such that (6), (7), (8), (11), (12) hold if and only if A is $(5)-$*invariant.*

Proof. a) If. We prove the truth of the implication for $V(\mu) = d(\mu, A)$ and $\varphi_1 = \varphi_2 = 1_{\mathbf{N}}$. Indeed, (6), (7), (8) are true. For (9), we take an arbitrary $\mu \in A$. The hypothesis states the existence of $\alpha \in \Pi_n$ with the property that $\forall k \in \mathbf{N}, \phi^{\alpha}(\mu, k) \in A$, in other words $\forall k \in \mathbf{N}, d(\phi^{\alpha}(\mu, k), A) = 0$. This implies $V(\phi^{\alpha}(\mu, 0)) \geq V(\phi^{\alpha}(\mu, 1)) \geq V(\phi^{\alpha}(\mu, 2)) \geq ...$

Only if. We suppose against all reason that the $(1)-$invariance property of A is false, i.e. $\mu' \in A$ exists such that

$$\forall \alpha' \in \Pi_n, \exists k' \in \mathbf{N}, \phi^{\alpha'}(\mu', k') \notin A. \tag{13}$$

We can write

$$0 \overset{(6)}{=} \varphi_1(0) = \varphi_1(d(\mu', A)) \overset{(8)}{\leq} V(\mu') \overset{(8)}{\leq} \varphi_2(d(\mu', A)) = \varphi_2(0) \overset{(6)}{=} 0,$$

therefore $V(\mu') = 0$. We get from (9) the existence of $\alpha \in \Pi_n$ with

$$0 = V(\mu') = V(\phi^{\alpha}(\mu', 0)) \geq V(\phi^{\alpha}(\mu', 1)) \geq V(\phi^{\alpha}(\mu', 2)) \geq ... \geq 0, \tag{14}$$

wherefrom

$$\forall k \in \mathbf{N}, V(\phi^{\alpha}(\mu', k)) = 0. \tag{15}$$

The truth $\forall k \in \mathbf{N}$ of

$$0 \leq \varphi_1(d(\phi^{\alpha}(\mu', k), A)) \overset{(8)}{\leq} V(\phi^{\alpha}(\mu', k)) \overset{(15)}{=} 0 \tag{16}$$

proves that

$$\forall k \in \mathbf{N}, \varphi_1(d(\phi^{\alpha}(\mu', k), A)) = 0 \tag{17}$$

i.e., from (6), (7), $\forall k \in \mathbf{N}, d(\phi^{\alpha}(\mu', k), A) = 0$, in other words $\forall k \in \mathbf{N}, \phi^{\alpha}(\mu', k) \in A$. We have obtained a contradiction with (13). A is $(1)-$invariant.

c) If. We prove similarly with the If part of the proof of a) that $V(\mu) = d(\mu, A)$ and $\varphi_1 = \varphi_2 = 1_{\mathbf{N}}$ fulfill (6), (7), (8), (11).

Only if. The proof is similar with the Only if part of the proof of a), with slight differences concerning the use of the quantifiers. We take $\alpha \in \Pi_n$ and $\mu \in A$ arbitrary, fixed. We have

$$0 \overset{(6)}{=} \varphi_1(0) = \varphi_1(d(\mu, A)) \overset{(8)}{\leq} V(\mu) \overset{(8)}{\leq} \varphi_2(d(\mu, A)) = \varphi_2(0) \overset{(6)}{=} 0,$$

thus $V(\mu)=0$. From (11), function V satisfies

$$0=V(\mu)=V(\phi^{\alpha}(\mu,0))\geq V(\phi^{\alpha}(\mu,1))\geq V(\phi^{\alpha}(\mu,2))\geq ...\geq 0,$$

therefore

$$\forall k\in\mathbf{N}, V(\phi^{\alpha}(\mu,k))=0. \tag{18}$$

But the truth $\forall k\in\mathbf{N}$ of

$$0\leq\varphi_1(d(\phi^{\alpha}(\mu,k),A))\overset{(8)}{\leq} V(\phi^{\alpha}(\mu,k))\overset{(18)}{=}0$$

shows us that

$$\forall k\in\mathbf{N}, \varphi_1(d(\phi^{\alpha}(\mu,k),A))=0. \tag{19}$$

As α,μ were arbitrary, (19) is true for any α and any μ. We suppose now against all reason that the $(3)-$invariance property of A is false, thus $\alpha\in\Pi_n, \mu\in A, k\in\mathbf{N}$ exist such that $\phi^{\alpha}(\mu,k)\notin A$, in other words $d(\phi^{\alpha}(\mu,k),A)>0$. We infer from (7) that $\varphi_1(d(\phi^{\alpha}(\mu,k),A))>0$, representing a contradiction with (19). We have obtained that A is $(3)-$invariant.

d) If. Let $\lambda\in\mathbf{B}^n$ arbitrary. From $\Phi^{\lambda}(A)=A$ we have $\Phi^{\lambda}(A)\subset A$ and this, from Theorem 8, is equivalent with the (3)-invariance of A thus, taking into account item c), V,φ_1,φ_2 exist such that (6), (7), (8), (11) take place. In addition, the surjectivity of the function $\Phi^{\lambda}:A\to A$ implies, from Lemma 7, its injectivity, i.e. (12) is true.

Only if. The existence of V,φ_1,φ_2 such that (6), (7), (8), (11) be true shows due to c) that (3) is fulfilled, thus (4) is fulfilled also, see Theorem 8. From (4), from the injectivity of Φ^{λ} with arbitrary $\lambda\in\mathbf{B}^n$, and from Lemma 7, we have that $\Phi^{\lambda}:A\to A$ is surjective, therefore (5) holds. ■

Example 20 *(Example 13 revisited) The functions* $\varphi_1,\varphi_2:\mathbf{N}\to\mathbf{N}$,$\varphi_1=\varphi_2=1_{\mathbf{N}}$ *and* $V:\mathbf{B}^3\to\mathbf{N}$, $\forall\mu\in\mathbf{B}^3, V(\mu)=d(\mu,A)$ *satisfy (6), (7), (8), (9) for* $A=\{(0,0,0),(1,1,0),(1,1,1)\}$. *In (9) for any* $\mu\in A$, *we choose* $\alpha\in\Pi_3$ *such that* $\forall k\in\mathbf{N}, d(\phi^{\alpha}(\mu,k),A)=0$, *i.e.* $V(\phi^{\alpha}(\mu,k))=0$.

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