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RATIONAL SOLUTIONS OF CERTAIN CLASSES OF NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Resonance method commonly used for Painlevé classification of ordinary differential equations often detects negative resonances, other than a trivial one of –1 (which should always be present). However, it is asserted by some researchers that the nature of these negative resonances is not fully understood. The problem arises as how to use negative nontrivial resonances to obtain information about the analytical properties of solutions of non-linear differential equations, in particular, to construct rational solutions. In the present paper, the method for constructing rational solutions of certain classes of autonomous non-linear ordinary differential equations is presented.

Key words: non-linear differential equations, resonances, rational solutions.

SOLUȚII RAȚIONALE ALE UNOR CLASE DE ECUAȚII DIFERENȚIALE NELINIARE

Rezumat. Metoda de rezonanță folosită frecvent pentru clasificarea Painlevé a ecuațiilor diferențiale ordinare detectează adesea rezonanțe negative, altele decât una trivială de –1 (care ar trebui să fie întotdeauna prezentă). Cu toate acestea, unii cercetători afirmă că natura acestor rezonanțe negative nu este pe deplin înțeleasă. Problema apare în modul de utilizare a rezonanțelor nontriviale negative pentru a obține informații despre proprietățile analitice ale soluțiilor ecuațiilor diferențiale neliniare, în special, pentru a construi soluții raționale. În lucrarea de față este prezentată metoda de construire a soluțiilor raționale ale unor clase de ecuații diferențiale neliniare ordinare autonome.

Cuvinte cheie: ecuații diferențiale neliniare, rezonanțe, soluții raționale.

Introduction

There has been a number of studies on rational solutions of the ordinary differential equations. Paper [1] provides certain general conditions for algebraic differential equations to have rational solutions. Rational solutions of the second-sixth Painlevé equations were considered in [2-4]. According to Backlund transformations [3, 5-6] rational solutions of higher analogues of the Painlevé equations were obtained and represented by special polynomials (generalized polynomials Yablonsky–Vorobiev). A recent paper [7] proves that for differential equations with nontrivial negative resonances the families of rational solutions always exist.

We consider the methods of using negative resonances for building rational solutions for algebraic autonomous ordinary differential equations of the form

$$P\left(y^{(n)}, y^{(n-1)}, \ldots, y', y\right) = 0, \qquad (1)$$

where P is polynomial in y and its derivatives with constant coefficients. If the solutions of equations (1) are given in the form of series

$$y = h_0(z-z_0)^{-s} + \dots + h_r(z-z_0)^{r-s}, s \in \mathbb{Z}, \quad (2)$$

substituting (2) into equation (1) gives the expression

$$A(h_0)(z-z_0)^{-d} + \ldots + B(r,h_0)h_r(z-z_0)^{r-d} + \ldots = 0,$$

where $d \geq n+s$, $A(h_0)$ are polynomials in h_0, and $B(r,h_0)$ are polynomials of *n-th* degree in r with coefficients depending on h_0. The number d is called the weight of dominant terms of polynomial P. Finding h_0 from the equation $A(h_0)=0$ and substituting the obtained h_0 into the equation $B(r,h_0)=0$, allows to find the corresponding values r, called resonance numbers (resonances).

In order to have single-valued solutions of differential equations (1), it is necessary that the resonances r are distinct integers [8-11], moreover, one of them is equal to –1 [9]. For solutions of the form (2), we will extract the set

$$(s, h_0; -1, r_1, r_2, \ldots, r_{n-1}). \quad (3)$$

The n-order's equations can be divided into three classes: equations for which the set (3) has positive integers $r_1, r_2, \ldots, r_{n-1}$; equations having both positive and negative resonances (among numbers $r_1, r_2, \ldots, r_{n-1}$); equations having only negative resonances. Much attention has been paid recently to the existence of the rational solutions of nonlinear differential equations (see, for example, [6, 12, 13]). And the equations with such solutions often have negative resonances, i.e. belong to the last two classes from above. In this paper we ask a question of how the presence of non-trivial (except –1) negative resonances could be used for constructing rational solutions of equations (1).

The paper [14] argued that negative resonances "are still not fully understood and at the present stage are of great interest". In [15], eight equations having only one set of all negative resonances are allocated in a separate section. However, analytical properties of solutions of the eight obtained equations are not indicated [15]. The work [16] is devoted to this issue. According to [17], a formula is given for constructing rational solutions of equation (1) by using negative resonances.

It is shown in [18] that all nonlinear equations

$$y^{(n)} = P\left(y^{(n-1)}, \ldots, y', y\right) \quad (4)$$

of arbitrary order n, which have polynomial right-hand side of degree higher than 2 with constant coefficients, which possess strong Painlevé property [11] (all solutions of the equation are meromorphic) admit nontrivial negative resonances. In [7] it has been proven that from the existence of such nontrivial negative resonances in equations with strong Painlevé property follows the presence of families of rational partial solutions in a simplified equation corresponding to equation (4). Such simplified equations (4) with solutions of the form (2) where $s=1$, have the following form

$$y^{(n)}=ayy^{(n-1)}+\sum_{\chi_j:\sum(j+1)\chi_j=n+1}a_\chi\left(y^{(n-2)}\right)^{\chi_{n-2}}\cdot\ldots\cdot\left(y'\right)^{\chi_1}y^{\chi_0}. \qquad (5)$$

In this paper, a general method is provided for constructing rational solutions of equations of the form (1) using negative resonances, as well as a special method applicable to simplified equations (5) corresponding to equations (4).

Rational solutions of equations (5)

In this section, families of rational solutions of simplified equations (5) can be found by means of uncomplicated algebraic transformations and direct integration of resulting equations.

For equations (5) possessing the strong Painlevé property if $a\neq 0$, it implies that without loss of generality (up to scale transformation $y=ku$) all roots h_0 of the equation $A(h_0)=0$ can be considered rational numbers with the smallest absolute value of 1. Assume all these roots are positive integers (a case common among known higher order equations). Let q be the largest of these numbers. In this case, the variable change $y=\upsilon'/\upsilon$ reduces equation (5) to a homogeneous equation with entire solutions. The function $\upsilon=C(z-z_0)^q$ will satisfy the resulting equation on υ, where C is an arbitrary constant. If the equation $B(r,h_0)=0$ admits nontrivial negative resonances (according to [7] this holds for equations of degree higher than 2), then it can be shown [18] that there exists $m+2$-parametric family of polynomial solutions of the equation on υ of degree no higher than q, where m is the number of nontrivial negative resonances within the set (3). This family can be found from the system composed by the equation $\upsilon^{(q+1)}=0$ and the equation on υ, obtained from (5) by substituting $y=\upsilon'/\upsilon$. From this system we obtain the homogeneous equation of degree $m+2$ on function υ. The general solution of the last equation is polynomial and the corresponding $m+1$-parametric family of rational solutions of equation (5) is defined by $y=\upsilon'/\upsilon$.

For example, for the equation

$$y^{(4)}+2yy'''+11y'y''+2y^2y''+7y\left(y'\right)^2+y^3y'=0 \qquad (6)$$

sets of the form (3) will be as following:

$$(1,1;-1,1,3,5),\ (1,4;-1,-4,1,6),\ (1,6;-1,-2,-5,6).$$

The corresponding equation on function υ will be

$$\upsilon^2\upsilon^{(5)}-3\upsilon\upsilon'\upsilon^{(4)}+\upsilon\upsilon''\upsilon'''+3\upsilon'''\left(\upsilon'\right)^2-2\upsilon'\left(\upsilon''\right)^2=0. \qquad (7)$$

The largest root $h_0=q=6$ of the equation $A(h_0)=0$ has the two corresponding nontrivial negative resonances $r=-2, r=-5$, so $m=2$. The system from equation (7) and $\upsilon^{(7)}=0$ is equivalent to the equation

$$6\upsilon\upsilon''\upsilon^{(4)}-3\upsilon\left(\upsilon'''\right)^{2}-6\upsilon'\upsilon''\upsilon'''+4\left(\upsilon''\right)^{3}=0. \tag{8}$$

Equation (8) admits the following four-parameter family of polynomial solutions:

$$\upsilon=C_1\left(z-z_0\right)^6+C_1C_2\left(z-z_0\right)^4+\frac{3}{5}C_1C_2{}^2\left(z-z_0\right)^2+C_3\left(z-z_0\right)-\frac{1}{25}C_1C_2{}^3 \tag{9}$$

And besides, equation (8) has the three-parameter family of solutions

$$\upsilon=C_2\left(\left(z-z_0\right)^4-C\right), \tag{10}$$

corresponding to the root $h_0=4$ as well as the family of singular solutions $\upsilon=C\left(z-z_0\right)$. Then the equation (6) has rational solutions of the form $y=\upsilon'/\upsilon$ where υ belongs to one of those three families – last one, (9) or (10).

Construction of rational solutions of the equations (1)

In general, rational solutions of differential equations can be constructed in the form of converging series on negative powers of an independent variable with direct participation of negative resonances.

Let in set of the form (3) $r_1=-v, v\in\mathbb{N}\backslash\{1\}$. The resonance r_1 corresponds to the series

$$y=\sum_{k=0}^{\infty}\frac{h_k}{\left(z-z_0\right)^{kv+s}}. \tag{11}$$

For series (11) the Hankel matrix [19, p. 465] will be

$$H=\left\|h_{k+j}\right\|_0^{\infty}. \tag{12}$$

We represent the rank of matrix (12) in terms of $p, p\in\mathbb{N}\{1\}$.

There are numbers $\alpha_n, n=\overline{1,p},$ such that

$$h_{k+p}=\sum_{n=1}^{p}\alpha_n h_{k+p-n},\ k=0,1,2,\ldots. \tag{13}$$

Under the condition (13), the formula

$$y=\frac{h_0t^{pv}+\sum\limits_{k=1}^{p-1}\left(h_k-\sum\limits_{j=1}^{k}\alpha_jh_{k-j}\right)t^{(p-k)v}}{t^s\left(t^{pv}-\sum\limits_{k=1}^{p}\alpha_kt^{(p-k)v}\right)},\ t=z-z_0 \tag{14}$$

is given in [17], which represents rational solution of the equation (1). It is not difficult to draw that the solution (14) have *n* poles (taking into account their multiplicity): $n=pv$, if $v\ge s$, $\alpha_p\ne0$; $n=pv+s-v$, if $v<s$ or $\alpha_p=0$. According to [19] if $v=s=1$, the number of poles of the solution (14) will be $n=p$.

If the rank of matrix (12) is $p=1$, the coefficients h_k of series (11) represent a geometric progression, and then series (11) can easily be written as

$$y = \frac{h_0 (z - z_0)^{\nu}}{(z - z_0)^{s} \left((z - z_0)^{\nu} - C \right)}, \ \ \forall z_0, C. \tag{15}$$

Substitution of (14) or (15) into equation (1) confirms that these formulas indeed provide rational solutions of the equation (1).

Thus, using resonances $r = -4,\ r = -5,\ r = -2$ (where $p = 3$) for equation (6) we respectively get rational solutions

$$y = \frac{4(z - z_0)^3}{(z - z_0)^4 - C}, \ \ y = \frac{6(z - z_0)^5 - C}{(z - z_0)^6 - C(z - z_0)},$$
$$y = \frac{6(z - z_0)^5 - 20C(z - z_0)^3 + 30C^2(z - z_0)}{(z - z_0)^6 - 5C(z - z_0)^4 + 15C^2(z - z_0)^2 + 5C^3}, \tag{16}$$

where $z_0,\ C$ are arbitrary constants.

The first rational solution from (16) can be obtained by the formula $y = \upsilon'/\upsilon$, where the function υ is defined in (10); the second rational solution from (16) is obtained if counting $C_2 = 0$ in (9); the third rational solution is obtained if in (9) we count $C_2 = 0,\ C_3 \neq 0.$

Remark 1. Equation (6) has the first integral

$$2\left(y' + y^2\right)\left(y''' + 2yy'' + 2(y')^2\right) = (y'' + 2yy')^2 - \frac{10}{3}\left(y' + y^2\right)^3 + 3y^2\left(y' + y^2\right)^2 + C, \tag{17}$$

where the arbitrary constant C corresponds to the resonance $r = 6$.

For equation (7), there is the first integral

$$6\upsilon\upsilon''\upsilon^{(4)} - 3\upsilon(\upsilon''')^2 - 6\upsilon'\upsilon''\upsilon''' + 4(\upsilon'')^3 = 3C\upsilon^3. \tag{18}$$

When $C = 0$ (8) follows from (18).

Next, it will be shown how we can obtain rational solution (14) of equation (1), if looking for its solution in the form of series of non-negative powers of $z - z_0$.

According to (13), we assume

$$\alpha_p \beta_{k+p} + \alpha_{p-1}\beta_{k+p-1} + \ldots + \alpha_1 \beta_{k+1} - \beta_k = 0, \ \ k = 0,1,2,\ldots. \tag{19}$$

When $p = 2,\ \alpha_2 \neq 0$ from (19) we obtain

$$\beta_{k+2} + \frac{\alpha_1}{\alpha_2}\beta_{k+1} - \frac{1}{\alpha_2}\beta_k = 0,, \ \ k = 0,1,2,\ldots. \tag{20}$$

Theorem 1. Suppose that for differential equation (1) we have set (3), where $r = -\nu, \nu \in \mathbb{N}, \nu > 1,$ and the rank of matrix (12) is $p = 2.$ Then the rational solution (14) of equation (1) can be represented as series

$$y = \sum_{k=0}^{\infty} \beta_k (z - z_0)^{k\nu + \nu - s}, \tag{21}$$

where the coefficients β_k are subject to equality (20), and α_1, α_2 are taken from (13) for $p = 2$, while

$$\beta_0 = \frac{\alpha_1}{\alpha_2} h_0 - \frac{1}{\alpha_2} h_1,\ \beta_1 = -\frac{\alpha_1}{\alpha_2} \beta_0 - \frac{1}{\alpha_2} h_0. \tag{22}$$

Series (21) converges in the region

$$0 \neq |z - z_0| < \sqrt[2\nu]{|\alpha_2|}. \tag{23}$$

Proof. Assuming $\beta_k = \lambda^k$, the number λ will be the root of the equation

$$\lambda^2 + \frac{\alpha_1}{\alpha_2}\lambda - \frac{1}{\alpha_2} = 0. \tag{24}$$

Then, two cases are considered as follows.

a) When $\alpha_1{}^2 + 4\alpha_2 \neq 0$, the equation (24) will have two different roots λ_1, λ_2, herewith, $\lambda_1 + \lambda_2 = -\alpha_1/\alpha_2$, $\lambda_1 \cdot \lambda_2 = -1/\alpha_2$. Through initial conditions (22) and the linearity of equality (20), we find that

$$\beta_k = \frac{1}{\lambda_1 - \lambda_2}\Big((\beta_1 - \lambda_2\beta_0)\lambda_1{}^k + (\beta_0\lambda_1 - \beta_1)\lambda_2{}^k \Big),\ k = 0,1,2,\ldots. \tag{25}$$

$$\beta_1 - \lambda_2\beta_0 = \lambda_1{}^2(\lambda_2 h_1 - h_0),\ \lambda_1\beta_0 - \beta_1 = \lambda_2{}^2(h_0 - \lambda_1 h_1). \tag{26}$$

If the conditions $\left|\lambda_k (z - z_0)^\nu\right| < 1,\ k = 0,1,2$ are satisfied, then

$$y = \frac{\beta_1 - \lambda_2\beta_0}{\lambda_1 - \lambda_2} t^{\nu - s} \sum_{k=0}^{\infty} \left(\lambda_1 t^\nu\right)^k + \frac{\beta_0\lambda_1 - \beta_1}{\lambda_1 - \lambda_2} t^{\nu - s} \sum_{k=0}^{\infty} \left(\lambda_2 t^\nu\right)^k =$$

$$= \frac{\beta_1 - \lambda_2\beta_0}{\lambda_1 - \lambda_2} t^{\nu - s} \cdot \frac{1}{1 - \lambda_1 t^\nu} + \frac{\beta_0\lambda_1 - \beta_1}{\lambda_1 - \lambda_2} t^{\nu - s} \cdot \frac{1}{1 - \lambda_2 t^\nu} =$$

$$= \frac{t^{\nu - s}}{\lambda_1 - \lambda_2} \cdot \frac{(\beta_1 - \lambda_2\beta_0)(1 - \lambda_2 t^\nu) + (\beta_0\lambda_1 - \beta_1)(1 - \lambda_1 t^\nu)}{\lambda_1\lambda_2\left(t^{2\nu} - \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2} t^\nu + \frac{1}{\lambda_1\lambda_2} \right)},$$

where $t = z - z_0$. Taking into account (26), we get (14) while $p = 2$.

b) When $\alpha_1{}^2 + 4\alpha_2 = 0$, it occurs that $\lambda_1 = \lambda_2 = \lambda = 2/\alpha_1$. Instead of (25) acquire

$$\beta_k = \left(\beta_0 + \left(\frac{\beta_1}{\lambda} - \beta_0 \right) k \right) \lambda^k,\ k = 0,1,2,\ldots. \tag{27}$$

Thus $\beta_0 = \lambda^2 h_1 - 2\lambda h_0$, $\beta_1 = 2\lambda\beta_0 + \lambda^2 h_0$. Because of $k(\lambda t^\nu)^k = \frac{t}{\nu}\left(\left(\lambda t^\nu \right)^k \right)'$, if $\left|\lambda t^\nu\right| \le \delta < 1$, then

$$y=\frac{\beta_0 t^{\nu-s}}{1-\lambda t^{\nu}}+\left(\frac{\beta_1}{\lambda}-\beta_0\right)\frac{t^{\nu-s+1}}{\nu}\left(\frac{1}{1-\lambda t^{\nu}}\right)'=\frac{\beta_0 t^{\nu-s}}{1-\lambda t^{\nu}}+\left(\frac{\beta_1}{\lambda}-\beta_0\right)t^{\nu-s}\frac{\lambda t^{\nu}}{\left(1-\lambda t^{\nu}\right)^2}=$$

$$=\frac{t^{\nu}\left(\left(\frac{\beta_1}{\lambda^2}-2\frac{\beta_0}{\lambda}\right)t^{\nu}+\frac{\beta_0}{\lambda^2}\right)}{t^s\left(t^{\nu}-\frac{1}{\lambda}\right)^2}=\frac{h_0 t^{2\nu}+\left(h_1-\alpha_1 h_0\right)t^{\nu}}{t^s\left(t^{\nu}-\frac{\alpha_1}{2}\right)^2}=\frac{h_0 t^{2\nu}+\left(h_1-\alpha_1 h_0\right)t^{\nu}}{t^s\left(t^{2\nu}-\alpha_1 t^{\nu}-\alpha_2\right)},\ t=z-z_0.$$

The theorem is proved.

Remark 2. When $\alpha_1\neq 0,\alpha_2=0,p=2,$ from (19) obtain $\beta_{k+1}=\frac{\beta_k}{\alpha_1},k=1,2,\ldots.$ Assuming $h_1=l\alpha_1,\ \beta_0=h_0-l,\ \beta_1=-l/\alpha_1,$ we have $\beta_k=-l/\alpha_1^{\ k},\ k=1,2,\ldots.$ Then

$$y=\sum_{k=0}^{\infty}\beta_k t^{k\nu-s}=\frac{h_0-l}{t^s}-lt^{-s}\sum_{k=1}^{\infty}\left(\frac{t^{\nu}}{\alpha_1}\right)^k=$$

$$=\frac{h_0-l}{t^s}-\frac{lt^{\nu-s}}{\alpha_1\left(1-\frac{t^{\nu}}{\alpha_1}\right)}=\frac{h_0 t^{\nu}+\alpha_1\left(l-h_0\right)}{t^s\left(t^{\nu}-\alpha_1\right)}=\frac{h_0 t^{\nu}+h_1-\alpha_1 h_0}{t^s\left(t^{\nu}-\alpha_1\right)},$$

so

$$y=\frac{h_0 t^{\nu}+h_1-\alpha_1 h_0}{t^s\left(t^{\nu}-\alpha_1\right)},\ t=z-z_0. \tag{28}$$

Remark 3. Under the conditions of $\alpha_1\cdot\alpha_2\neq 0,\ \alpha_3=0,\ p=3,$ in order to find the coefficients β_k of series (21), the formulas (26) will be satisfied for $k=1,2,\ldots$, and the initial values β_1,β_2,β_3 must be specified.

Remark 4. The rational solution (15) can be represented as the series

$$y=h_0\sum_{k=0}^{\infty}\gamma^{k+1}\left(z-z_0\right)^{k\nu+\nu-s},\ \gamma a=1,$$

converging in the region $0\neq\left|z-z_0\right|<\left|a\right|^{\frac{1}{\nu}}$.

Remark 5. Theorem 1 can be generalized in case $p>2$. Then, in order to find the values of λ, instead of the equation (24) we will have an equation of degree p, which can greatly complicate the calculation of its roots.

Some cases

1). For the Chazy equation [20] $y'''=12\left(y'\right)^2+72y^2y'+54y^4$, sets of the form (3) will be as following: $(1,1;-1,-3,10)$, $\left(1,\frac{1\pm\sqrt{5}}{6};-1,2,5\right)$. Resonance $r=-3$ corresponds to the rational solution [17] $y=\frac{t^5+5at^2}{t^6-5at^3-5a^2},\quad t=z-z_0,\ \forall z_0,a.$ Herewith

$\alpha_1 = 5a,\ \alpha_2 = 5a^2$. Equation (24) in this case has roots $\lambda_1 = \frac{3-\sqrt{5}}{2a\sqrt{5}},\ \lambda_2 = -\frac{3+\sqrt{5}}{2a\sqrt{5}}$. Then series (21) has the form $y = \sum_{k=0}^{\infty} \beta_k (z-z_0)^{3k+2}$, where according to formulas (25) coefficients

$$\beta_k = \frac{1-\sqrt{5}}{\left(2a\sqrt{5}\right)^{k+1}}\left(\left(3-\sqrt{5}\right)^k + \frac{(-1)^k}{2}\left(3+\sqrt{5}\right)^{k+1}\right),\ k = 0,1,2\ldots.$$

2). For equation $y'y''' = \frac{2}{3}(y'')^2 + \frac{1}{3}yy'y'' + 3(y')^3 - \frac{1}{3}y^2(y')^2$ obtained from [21], sets of the form (3) will be $(1,1;-1,1,3),\ (1,-10;-1,-5,6)$. The resonance $r = -5$ corresponds to the rational solution $y = \frac{10t^9 - 90at^4}{t^{10} - 18at^5 + 6a^2},\ t = z - z_0,\ \forall z_0, a$. Herewith $\alpha_1 = 18a,\ \alpha_2 = -6a^2$. In this case, equation (24) has roots $\lambda_1 = \frac{3\sqrt{3}+5}{2a\sqrt{3}},\ \lambda_2 = \frac{3\sqrt{3}-5}{2a\sqrt{3}}$. And series (21) submits to $y = \sum_{k=0}^{\infty} \beta_k (z-z_0)^{3k+2}$, where

$$\beta_k = \frac{5}{(2a)^{k+1}}\left(\left(3+\frac{5}{\sqrt{3}}\right)^{k+1} + \left(3-\frac{5}{\sqrt{3}}\right)^{k+1}\right),\ k = 0,1,2\ldots.$$

3). The equation (see in [9], [20]) $y^{(4)} = 30yy'' - 60y^3$ has the corresponding sets $(2,1;-1,2,3,10),\ (2,2;-1,-2,5,12)$. And by resonance $r = -2$ it has the rational solution $y = \frac{2t^2 + 2a}{\left(t^2 - a\right)^2},\ t = z - z_0,\ \forall z_0, a$. Herewith $\alpha_1 = 2a,\ \alpha_2 = -a^2$. In this case, for equation (24) there are roots $\lambda_1 = \lambda_2 = 1/a$. Thus, finding coefficients β_k from (27) the series (21) tend to be $y = 2\sum_{k=0}^{\infty}(2k+1)\gamma^{k+1}(z-z_0)^{2k},\ a\gamma = 1,\ |z - z_0| < \sqrt{|a|}$.

4). For equation [9] $y^{(4)} = 20yy'' + 10(y')^2 - 40y^3$, sets of the form (3) correspond to $(2,1;-1,2,5,8),\ (2,3;-1,-3,8,10)$. Using resonance $r = -3$ its rational solution is found in the form $y = \frac{3t^4 + 6at}{\left(t^3 - a\right)^2},\ t = z - z_0,\ \forall z_0, a$. This solution can be represented in the form of series $y = 3\sum_{k=0}^{\infty}(3k+2)\gamma^{k+1}(z-z_0)^{3k+1},\ a\gamma = 1,\ |z - z_0| < \sqrt[3]{|a|}$, which is obtained from (21) with coefficients calculated by the formula (27), moreover $\beta_0 = \frac{6}{a},\ \beta_1 = \frac{15}{a^2},\ \lambda = \frac{1}{a}$.

5). The equation [20] $y''' = yy'' + 5(y')^2 - y^2 y'$ possess the corresponding sets $(1,-1;-1,1,5)$, $(1,-6;-1,-5,6)$. The rational solution $y = -\dfrac{6t^5 - a}{t(t^5 - a)}$ is found by resonance $r = -5$, where $t = z - z_0$, $\forall z_0, a$. Herewith $p = 2$, $\alpha_1 = a$, $\alpha_2 = 0$. This solution can be represented in the form of series

$$y = -\frac{1}{z - z_0} + 5\sum_{k=0}^{\infty} \gamma^{k+1}(z - z_0)^{5k+4}, \ \gamma a = 1, \ 0 \neq |z - z_0| < \sqrt[5]{|a|}.$$

6). For equation [15] $yy^{(4)} = 2y'y''' - (y'')^2 + 10y^3 y'' - 4y^6$, the corresponding sets are $(1,\pm1;-1,2,3,4)$, $(1,\pm2;-1,-3,4,8)$. By resonance $r = -3$ rational solution of the equation is $y = \pm\dfrac{2t^3 + a}{t(t^3 - a)}$, $t = z - z_0$, $\forall z_0$, a. In this case, $p = 2$, $\alpha_1 = a$, $\alpha_2 = 0$. This solution can be represented in the form of series

$$y = \mp\frac{1}{z - z_0} \mp 3\sum_{k=0}^{\infty} \gamma^{k+1}(z - z_0)^{3k+2}, \ \gamma a = 1, \ 0 \neq |z - z_0| < \sqrt[3]{|a|}.$$

7). For equation [15] $yy^{(4)} = y'y''' + 14y^2 y'' - 12y^4$, sets of the form (3) are $(2,1;-1,2,5,6)$, $(2,6;-1,-5,6,12)$. The Hankel matrix (12) corresponding to the resonance $r = -5$ has rank $p = 3$. Herewith, $\alpha_1 = 2a$, $\alpha_2 = -a^2$, $\alpha_3 = 0$, $h_0 = 6$, $h_1 = 30a$, $h_2 = 55a^2$. The corresponding rational solution is $y = \dfrac{6t^{10} + 18at^5 + a^2}{t^2(t^5 - a)^2}, t = z - z_0$, $\forall z_0, a$, which can be represented in the form of series

$$y = \frac{1}{(z - z_0)^2} + 5\sum_{k=0}^{\infty}(5k + 4)\gamma^{k+1}(z - z_0)^{5k+3}, \ \gamma a = 1, \ 0 \neq |z - z_0| < \sqrt[5]{|a|}.$$

8). For equation $y'' + 3yy' + y^3 = 0$, there is the general solution [3, p.122]

$$y = \frac{1}{z - C_1} + \frac{1}{z - C_2}. \tag{29}$$

This equation has the sets of the form (3) $(1,1;-1,1)$, $(1,2;-1,-2)$. The series (11), corresponding to the resonance $r = -2$, can be written as

$$y = \frac{2}{z - z_0} + 2\sum_{k=1}^{\infty}\frac{a^k}{(z - z_0)^{k+1}}, \ \forall z_0, \ a,$$

which defines rational solution

$$y = \frac{2(z - z_0)}{(z - z_0)^2 - a}. \tag{30}$$

Solution (30) matches (29) when $2z_0 = C_1 + C_2$, $a = {z_0}^2 - C_1 C_2$.

Rational solution (30) can be represented in the form of series $y = -2\sum_{k=0}^{\infty} \gamma^{k+1} \left(z - z_0\right)^{2k+1}$, $\gamma a = 1$, converging in the region $\left|z - z_0\right| < \sqrt{|a|}$. Assume $z_0 = C_1$, then we can also write

$$y = \frac{2}{z - C_1} + \sum_{k=1}^{\infty} \frac{h^k}{\left(z - C_1\right)^{k+1}}, \ h = C_2 - C_1, \ z - C_1 \in \overset{\circ}{V}\left(\infty\right);$$

$$y = \frac{2}{z - C_1} - \sum_{k=0}^{\infty} \frac{\left(z - C_1\right)^k}{h^{k+1}}, \ 0 \neq \left|z - C_1\right| < |h|;$$

$$y = -\sum_{k=0}^{\infty} \left(\frac{1}{C_1^{\,k+1}} + \frac{1}{C_2^{\,k+1}} \right) z^k, \ |z| < |C|, \ |C| = \min\left\{|C_1|, \ |C_2|\right\}.$$

Conclusion

This paper describes the relationship between the negative resonance numbers of nonlinear ordinary differential equations and their rational solutions. In particular, the relation determining the number of poles of possible rational solutions through the value of the negative resonance and the rank of the Hankel matrix is indicated. Two methods are presented for constructing such rational solutions. First, by means of direct algebraic transformations, which allow construction of directly integrable equations describing families of rational solutions for the simplified equations. And second, by means of converging series by negative powers of an independent variable, directly illustrating the role of negative resonances in the construction of rational solutions. In addition to the practical significance of this approach for building partial solutions of certain classes of non-linear equations, the results also shed light on the nature and significance of negative resonances, which were considered by many authors to be obscure until now.

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