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Submetrizable spaces and open mappings

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Abstract. The present article characterizes the continuous open complete images of submetrizable spaces as the spaces with pseudo-base of countable order. Some theorems about selections of set-valued mappings into submetrizable spaces are obtained.
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Spații submetrizabele și aplicații deschise

Rezumat. În acest articol se caracterizează imaginile continue deschise şi complete a spaţiilor submetrizabile ca spaţii cu pseudo-baza de ordine numărabil. Sunt demonstrate unele teoreme despre selecţiile pentru aplicaţii cu valori complete în spaţii submetrızabile.
Cuvinte cheie: spaţiu submetrizabil, aplicaţie deschisă, set-valued aplicaţie multivocă.

1. Introduction

For notation and terminology the reader is referred to [12] and [22]. Space is used here to mean topological T_1-space.

A pseudo-metric on a space M is a function $d : M \times M \longrightarrow \mathbb{R}$ such that for any $x, y, z \in M$ the following holds:

(1) $d(x, x) = 0$ (identity of indiscernibles);
(2) $d(x, y) = d(y, x)$ (symmetry);
(3) $d(x, z) \leq d(x, y) + d(y, z)$ (subadditivity or triangle inequality).

If $d(x, y) = 0$ if and only if $x = y$, d is called a metric on M.

Any pseudometric is non-negative: $d(x, y) \geq 0$ for any $x, y \in M$. Let d be a pseudometric on a topological space M. For a point $x \in M$ and a real number $r > 0$ we define the open ball of radius r about x as the set $B(x, d, r) = \{y \in M : d(x, y) < r\}$. The pseudometric d is continuous on M if the balls $B(x, d, r)$ are open in the space M.

A space X is called to be submetrizable if on X there exists a continuous metric. A topological space is submetrizable if and only if it admits a continuous bijection onto a metric space. Let $C_p(X)$ be the space of continuous functions on the space X.

The density of X, denoted $d(X)$, is defined by $d(X) = min\{|L| : l \subset X, cl_X(L) = X\}$. The pseudocharacter of a space X at a subset A, denoted by $\psi(A, X)$, is defined as the smallest infinite cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\cap\mathcal{U} = A$. If $A = \{x\}$ is a singleton, then we put $\psi(x, X) = \psi(\{x\}, X)$. The pseudocharacter of a space X is defined to be $\psi(X) = sup\{\psi(x, X) : x \in X\}$. The superior pseudocharacter of a space X is defined to be $\Psi(X) = sup\{\psi(A, X) : A \subset X, cl_X A = A\}$. If $\Psi(X) = \aleph_0$, then the space X is called a perfect space. The diagonal number $\Delta(X)$ of a space X is the pseudocharacter of the square $X \times X$ at its diagonal $\Delta_X = \{(x, x) : x \in X\}$.

The class of spaces with countable pseudocharacter is large and important. For example, any T_1-space with countable pseudocharacter is a Moscow space. A space X is called Moscow if the closure of every open subset U of X is the union of a family of G_δ-subsets of X (see [12]). Moreover, it is wellknown that any topological group with countable pseudocharacter is submetrizable.

In the works [5, 6, 23] were obtained the following assertions.

Theorem 1.1. *For any Tychonoff space X we have* $d(X) = \Delta(C_p(X)) = \psi(C_p(X))$.

Corollary 1.2. *For any Tychonoff space X the following assertions are equivalent:*

(1) X is a separable space.

(2) $C_p(X)$ *is a space of countable pseudocharacter.*

(3) $C_p(X)$ *is a submetrizable space.*

In [32] V. I. Ponomarev has proven that a space is a first countable space if and only if it is an open continuous image of a metric space. Naturally arises the question: Is the analogical theorem of Ponomarev's theorem valid for spaces with countable pseudocharacters? The answer of this question is negative. In [24, 25], in particular it was shown the following fact.

Theorem 1.3. *For any topological space X there exist a space Z and an open continuous mapping* $g : Z \longrightarrow X$ *of Z onto X with the next properties:*

(1) Z is a paracompact submetrizable space.

(2) Z is union of a sequence of closed discrete subspaces.

(3) Z is a perfectly normal space.

So the following problem arises: To study the mappings that preserve the property of being a spaces with countable pseudocharacter and to characterize the open continuous images of the submetrizable spaces via such mappings. In the present article this problem has been solved. Various images of metric spaces and complete metric spaces have been studied in [3, 4, 19, 30, 32, 33].

2. On uniformly complete mappings

Let ρ be a continuous pseudometric on a space X. For every non-empty subset A of X the number $diam_\rho(A) = \sup\{\rho(x,y) : x, y \in A\}$ is the ρ-diameter of the set A. Consider that $diam_\rho(\emptyset) = 0$. There exists a set X/ρ, a metric d on X/ρ and a mapping $p_\rho : X \longrightarrow X/\rho$ such that $d(p_\rho(x), p_\rho(y)) = \rho(x,y)$ for all $x, y \in X$. The metric space $(X/\rho, d)$ is called the quotient space of the space X relatively to the pseudometric ρ.

Denote by ωX the Wallman compactification of a space X.

Definition 2.1. *A family $\mathcal{F}$ of subsets of a T_1-space X is complete if there exists a G_δ-subset Y of the Wallman compactification ωX of the space X such that $X \subseteq Y$ and any subset $F \in \mathcal{F}$ is closed in Y.*

Any family of compact subsets of a regular space is complete.

Proposition 2.1. *Let $g : X \longrightarrow Y$ be an open continuous mapping of a regular space X onto a space Y and $\mathcal{F} = \{g^{-1}(y) : y \in Y\}$ be a complete family of subsets of the space X. Then:*

(1) $\psi(Y) \leq \psi(X)$.

(2) $\psi(g(x), Y) \leq \psi(x, X)$ for every $x \in X$.

(3) $\psi(x, X) = \psi(x, g^{-1}(g(x))) + \psi(g(x), Y)$ for every point $x \in X$.

Proof. There exists a sequence $\{H_n : n \in \mathbb{N}\}$ of open subsets of space ωX such that $X \subset Z = \cap\{H_n : n \in \mathbb{N}\}$ and any subset $F \in \mathcal{F}$ is closed in Z. We assume that $H_{n+1} \subset H_n$ for every $n \in \mathbb{N}$.

Fix a point $a \in X$. There exists a family $\mathcal{U}$ of open subsets of the space X with the conditions:

- $\cap\mathcal{U} = \{a\}$ and $|\mathcal{U}| = \psi(a, X)$;

- for any $U, V \in \mathcal{U}$ and $n \in \mathbb{N}$ there exists $W = w(U, V, n) \in \mathcal{U}$ such that $cl_X W \subseteq U \cap V$ and $cl_Z W \subset H_n$.

We put $\mathcal{V} = \{V = g(U) : U \in \mathcal{U}\}$. By construction $|\mathcal{V}| \leq |\mathcal{U}|$. We affirm that $\cap\mathcal{V} = \{g(a)\}$. Fix $b \in Y$ such that $b \neq f(a)$.

By construction, $\xi = \{cl_{\omega X} f(U) : U \in \mathcal{U}\}$ is a centered family of closed subsets of the compact space ωX and $\cap\xi \subseteq Z$. The set $F(b) = cl_{\omega X} g^{-1}(b)$ is a compact subset of ωX and $F(b) \cap Z \subset X$.

By construction, $\xi(b) = \{F(b) \cap cl_{\omega X} f(U) : U \in \mathcal{U}\}$ is a family of closed subsets of the compact space ωX and $\cap\xi(b) = \emptyset$. Since for any two sets $A, B \in \xi(b)$ there exists $C \in \xi(b)$ such that $C \subset A \cap B$. Therefore, there exists $U \in \mathcal{U}$ such that $cl_{\omega X} U \cap F(b) = \emptyset$, $U \cap g^{-1}(b) = \emptyset$ and $b \notin g(U)$. Therefore $\cap\mathcal{V} = \{g(a)\}$. Assertion 2 is proved. Assertions 1 and 3 follow from assertion 2. ■

Corollary 2.1. *Let $g : X \longrightarrow Y$ be an open continuous mapping of a regular space X onto a space Y and $\mathcal{F} = \{g^{-1}(y) : y \in Y\}$ be a family of compact subsets of the space Y. Then:*

(1) $\psi(Y) \leq \psi(X)$.
(2) $\psi(g(x), Y) \leq \psi(x, X)$ for any point $x \in X$.
(3) $\psi(x, X) = \psi(x, g^{-1}(g(x))) + \psi(g(x), Y)$ for any point $x \in X$.

Definition 2.2. [11]. *A family $\mathcal{F}$ of subsets of a T_1-space X is said to be jointly metrizable if there is a metric d on the set X such that d metrizes all subspaces of X which belong to $\mathcal{F}$, that is, the restriction of d to A generates the subspace topology on A for every $A \in \mathcal{F}$.*

Definition 2.3. *A family $\mathcal{F}$ of subsets of a T_1-space X is said to be jointly continuous (complete) metrizable if there is a continuous metric d on the space X such that d (complete) metrizes all subspaces of X which belong to $\mathcal{F}$. A jointly continuous complete metrizable family of subsets is called uniformly complete.*

The spaces with metrizable familes of sets were studied in [9, 10, 11, 20, 21, 31, 30].

Proposition 2.2. *Let $\mathcal{F}$ be a uniformly complete family of subspaces of the space X. Then the family $\mathcal{F}$ is complete.*

Proof. There exists a continuous metric d which complete metrizes all subspaces of X which belong to $\mathcal{F}$. Let (Y, ρ) be the metric completion of the metric space (X, d). The mapping $f : X \longrightarrow Y$, where $f(x) = x$ for each $x \in X$, is a continuous injection: if $x, y \in X$, then $d(x, y) = \rho(f(x), f(y))$ and $x = y$ provided $f(x) = f(y)$.

Since Y is a metric space and any metric space is normal, then there exists a continuous mapping $g : \omega X \longrightarrow \beta Y = \omega X$ such that $g(x) = f(x)$ for each $x \in X$. As a complete metric space the space Y is a G_δ-subset of βY. Hence $Z = g^{-1}(Y)$ is a G_δ-subset of ωX and $X \subseteq Z$. Fix $A \in \mathcal{F}$. Since $(A, d) = (f(A), \rho)$ is a complete metric space, then $A = f(A)$ is a closed subset of the space Y, A is a closed subset of the space X and $g^{-1}(A) \cap Z$ is a closed subspace of the space Z. Since A is a closed subspace of the space X we have $cl_{\omega X} A = \omega A$ and $cl_{\omega Y} f(A) = \omega f(A)$. Since $f|A : A \longrightarrow f(A)$

is a homeomorphism, $g|cl_{\omega X}A : cl_{\omega X}A \longrightarrow cl_{\omega Y}f(A)$ is a homeomorphism and $A = g^{-1}(f(A))$ is a closed subset of the space Z. The proof is complete. ■

Definition 2.4. *A mapping $f : X \longrightarrow Y$ of a space X into a space Y is called:*

- a complete mapping if the family $\{f^{-1}(y) : y \in Y\}$ is a complete family of subspaces of the space X;

- a uniformly complete mapping if the family $\{f^{-1}(y) : y \in Y\}$ is a complete family of subspaces of the space X.

3. On pseudocharacter of rectifiable spaces

A rectification on a space X is a homeomorphism $\varphi : X \times X \longrightarrow X \times X$ with the following two properties:

- $\varphi(\{x\} \times X) = \{x\} \times X$ for every $x \in X$;

- there exists a point $e \in X$ such that $\varphi(x, x) = (x, e)$ for every point $x \in X$.

The point $e \in X$ is called the neutral element of the space X. A space with a rectification is called a rectifiable space. Every rectifiable space is homogeneous (see [7, 16, 17, 18]).

A topological space S is rectifiable (see [16, 17, 18]) if and only if there are two continuous mappings $p, q : S \times S \longrightarrow S$ such that for any $x, y \in S$ and some fixed $e \in S$ the next identities hold: $p(x, q(x, y)) = q(x, p(x, y)) = y$, $q(x, x) = e$.

Any rectifiable T_0-space is a Hausdorff space. Fix a point $s \in S$. Then the mappings $P_s(x) = p(s, x)$ and $Q_s(x) = q(s, x)$ are homeomorphisms of the space S, $P_s^{-1} = Q_s$, $Q_s(e) = e$ and $P_s(e) = s$. Hence S is a homogeneous space and $\psi(S) = \psi(e, S)$. Obviously, $\psi(S) \leq \Psi(S)$ and $\psi(S) \leq \Delta(S)$.

Any topological quasigroup is a rectifiable space..

Theorem 3.1. *Let S be a rectifiable T_0-space. Then $\psi(e, S) = \psi(S) = \Psi(S) = \Delta(S)$.*

Proof. Any rectifiable T_0-spac is a Hausdorff space. Fix a rectification $\varphi : S \times S \longrightarrow S \times S$ on a space S with the neutral point $e \in S$.

Let $\mathcal{U}$ be a family of open subsets of S and $\cap\mathcal{U} = \{e\}$. For any set $U \in \mathcal{U}$ we put $V(U) = S \times U$ and $W(U) = \varphi^{-1}(V(U))$. Since $\varphi(\Delta_S) = S \times \{e\} \subset V(U)$ and φ is a homeomorphism, we have $\Delta_S \subset W(U)$ for any $U \in \mathcal{U}$ and the sets $V(U)$, $W(U)$ are open in $S \times S$.

Since $\cap\mathcal{U} = \{e\}$ and $\cap\{V(U) : U \in \mathcal{U}\} = S \times \{e\}$, we have $\cap\{W(U) : U \in \mathcal{U}\} = \Delta_S$. Therefore $\Delta(S) \leq \psi(S)$. Obviously, $\psi(X) \leq \Psi(S)$ and $\psi(S) \leq \Delta(S)$. The proof is complete. ■

Example 3.1. Let $M = \mathbb{R}$ be the real numbers, $\mathbb{Q}$ be the rational numbers and $T_M = \{U : U \textit{ is open in } \mathbb{R}\} \cup \{A \subset \mathbb{R} : A \cap \mathbb{Q} = \emptyset\}$ be the topology on M. Then (M, T_M) is a topological space called the Michael line. The space M is a paracompact submetrizable space with first axiom of countability and the set $\mathbb{Q}$ is closed and a non G_δ-subset of M. The free topological group $F(M)$ and the Abelian free topological group $A(M)$ are submetrizable groups. Any topological group is a rectifiable space. Since M is a closed subspace of the free groups, the set $\mathbb{Q}$ is closed and is not a G_δ-subset in $F(M)$ and $A(M)$. Hence, if $G \in \{F(M), A(M)\}$, then $\psi(G) = \Delta(G) = \aleph_0$, $\Psi(G) > \aleph_0$ and G is a submetrizable group.

4. Spaces with monotonical pseudo-bases

Let X be a space.

Let $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a sequence of open families of X, and let $\pi = \{\pi_n : A_{n+1} \longrightarrow A_n : n \in \mathbb{N}\}$ be a sequence of mappings. A sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called a c-sequence if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbb{N}$. A c-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called an mc-sequence if $\cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is nonempty.

Consider the following conditions:

$(S1)$ $\cup\{U_\beta : \beta \in A_n\} = X$ for each $n \in \mathbb{N}$.

$(S2)$ $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$.

The sequence $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \longrightarrow A_n : n \in \mathbb{N}\})$ is called an A-sieve on the space X.

Definition 4.1. *A space X is called a space with a pseudo-base of countable order if there exists an A-sieve $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \longrightarrow A_n : n \in \mathbb{N}\})$ such that if $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is a c-sequence, then $\cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is empty or a singleton set.*

Definition 4.2. *A space X is called a space with a uniform pseudo-base if there exists an A-sieve $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \longrightarrow A_n : n \in \mathbb{N}\})$ such that:*

- any cover γ_n is point-finite;

- if $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is a c-sequence, then $\cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is empty or a singleton set.

Example 4.1. A base $\mathcal{B}$ of a space X is called a uniform base if for any open set V of X and each point $x \in V$ the set $\{U \in \mathcal{B} : x \in U, U\ V \neq \emptyset\}$ is finite. The concept of a uniform base was introduced by P. S. Alexandroff [1]. A. V. Arhangel'skii has proved (see [3, 4]) that a space is an open continuous image with compact fibers of a metric space if and only if X is a space with a uniform base. Any regular space with a uniform base is a space with a uniform pseudo-base.

Example 4.4. Recall that a collection of sets is said to be perfectly decreasing if and only if each of its elements properly includes an element of the collection. A base $\mathcal{B}$ of a space X is called a base of countable order if for any perfectly decreasing collection of sets $\xi \subset \mathcal{B}$ and any point $x \in \cap\xi$ the family ξ is a base of X at the point x. The concept of a uniform base was introduced by A. V. Arhangel'skii [2, 3, 4]. H.H. Wicke and J. M. Worrell has proved (see [33]) that a space is an open continuous image with a complete family of fibers of a metric space if and only if X is a space with a base of countable order. Any regular space with a base of countable order is a space with a pseudo-base of countable order.

Fix a T_1-space X with a pseudo-base of countable order $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\})$.

On any set A_n we consider the discrete topology. Denote by A the family of all mc-sequences $\alpha = (\alpha_n \in A_n : n \in \mathbb{N})$. If $\alpha = (\alpha_n \in A_n : n \in \mathbb{N})$ and $\beta = (\beta_n \in A_n : n \in \mathbb{N})$, then we put $\rho(\alpha, \beta) = \Sigma\{2^{-n} : \alpha_n \neq \beta_n, n \in \mathbb{N}\}$. Then (A, ρ) is a metric space. If $m \in \mathbb{N}$ and $\mu \in A_m$, then we put $V(\mu, m) = \{\alpha = (\alpha_n : n \in \mathbb{N}) \in A : \alpha_m = \mu\}$. If $\alpha = (\alpha_n : n \in \mathbb{N}) \in A$, then $\{V(\alpha_n, n) : n \in \mathbb{N}\}$ is a base of the point α in A in the topology $T(\rho)$. The metric ρ is the Baire metric on the product of a sequence of discrete spaces.

For every point $\alpha = (\alpha_n \in A_n : n \in \mathbb{N}) \in A$ we put $f(\alpha) = \cap(U_{\alpha_n} : n \in \mathbb{N})$.

Property 1. *$f : A \longrightarrow X$ is a single-valued mapping of the set A onto the space X.*

Fix a point $\alpha = (\alpha_n \in A_n : n \in \mathbb{N}) \in A$. Since α is an mc-sequence, $f(\alpha) \neq \emptyset$. Fix $x \in f(\alpha)$ and $y \in X \setminus \{x\}$. The set $U = X \setminus \{x\}$ is open in X. Then there exist an open subset V of X and a natural number $m \in \mathbb{N}$ such that $x \in V \subset X\ U$ and $U_{\alpha_m} \cap M \subseteq U$ provided $M \in \mathcal{A}$ and $M \cap V \neq \emptyset$. Hence $f(\alpha) \subset U_{\alpha_m} \subset U$ and $y \notin f(\alpha)$. Property 1 is proved.

Let $T(\rho)$ be the topology on A generated by the metric ρ. On A consider the topology T generated by the open base $\mathcal{B} = \{U \cap f^{-1}(V) : U \in T(\rho), V \textit{ is an open subset of } X\}$. Obviously, (A, T) is a Hausdorff space and ρ is a continuous metric on the space A.

Property 2. *The mapping $f : A \longrightarrow X$ is an open continuous mapping of the space A onto the space X.*

Since $f^{-1}(V) \in \mathcal{B} \subset T$ for any open subset V of the space X, the mapping f is continuous. Let U be an open subset of A and $\alpha = (\alpha_n \in A_n : n \in \mathbb{N}) \in U$. Then there exist an open subset V of X and $m \in \mathbb{N}$ such that $V_{\alpha,n} \cap f^{-1}(V) \subset U$. Then $f(V(\alpha, n))$ $= U_{\alpha_n}$ and the set $f(V_{\alpha,n} \cap f^{-1}(V)) = U_{\alpha_n} \cap V$ is open. Therefore, the set $f(U)$ is open in X as the union of open sets. Property 2 is proved.

Property 3. *If $a \in X$, then on $f^{-1}(a)$ the topologies $T(\rho)$ and T coincide and the set $f^{-1}(a)$ is complete metrizable by the metric ρ. Hence f is a uniformly complete mapping.*

By construction, the topologies $T(\rho)$ and T coincide on $f^{-1}(a)$. We put $A_n(a) = \{\alpha \in A_n : a \in U_\alpha\}$. In this case the set $A(a) = f^{-1}(a) = A \cap \Pi\{A_n(a) : n \in \mathbb{N}\}$ is a closed subset of the space $\Pi\{A_n : n \in \mathbb{N}\}$ with the Baire metric ρ. Hence $(A(a), \rho)$ is a complete metric space. Property 3 is proved.

Property 4. *Assume that the cover γ_n is point-finite for any $n \in \mathbb{N}$. Then the fibers $f^{-1}(x)$, $x \in X$, are compact subsets of the space A.*

In this case the sets $A_n(a)$, $a \in X$, are finite, the space $\Pi\{A_n(a) : n \in \mathbb{N}\}$ is compact and $f^{-1}(a)$ is compact as a closed subset of the space $\Pi\{A_n(a) : n \in \mathbb{N}\}$. Property 4 is proved.

Property 5. *If X is a T_i-space and $i \in \{2, 3, 3\frac{1}{2}\}$, then (A, T) is a T_i-space too.*

By construction, (A, T) is a subspace of the product of a metric space A and a T_i-space X. Property 5 is proved.

From properties 1 - 5 it follows the next two theorems.

Theorem 4.1. *Let X be a space with a pseudo-base of countable order. Then there exist a submetrizable space S and an open continuous uniformly complete mapping $f : S \longrightarrow X$ of the space S onto the space X. If the space X is regular or completely regular, then the space S is regular or complete regular too.*

Theorem 4.2. *Let X be a space with a uniform pseudo-base. Then there exist a submetrizable space S and an open continuous compact mapping $f : S \longrightarrow X$ of the space S onto the space X. If the space X is regular or completely regular, then the space S is regular or complete regular too.*

Theorem 4.3. *Let X be a regular space with a pseudo-base of countable order and $\mathcal{F}$ be a complete family of subsets of X. Then there exist a submetrizable space A, a continuous pseudometric d on A and an open continuous uniformly complete mapping $f : A \longrightarrow X$ of the space A onto the space X such that for any $F \in \mathcal{F}$ the subpace $f^{-1}(F)$ is complete metrizable by the metric d. In particular, each subspace $F \in \mathcal{F}$ has a complete base of countable order.*

Proof. Fix a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of the Wallman compactification ωX such that $X \subset Z = \cap\{V_n : n \in \mathbb{N}\}$ and the set F is closed in the space Z for each $F \in \mathcal{F}$.

Since X is a regular space, on X there exists a pseudo-base of countable order $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \rightarrow A_n : n \in \mathbb{N}\})$ such that:

$(S3)$ $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$;

$(S4) U_\alpha \subseteq cl_{\omega X} U_\alpha \subset V_n$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$.

Fix a set $F \in \mathcal{F}$ and a c-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ such that $F \cap U_{\alpha_n} \neq \emptyset$ for any $n \in \mathbb{N}$. Then, since F is a closed subset of Z, $\cap\{F \cap U_{\alpha_n} : n \in \mathbb{N}\} = \cap\{cl_{\omega X}(F \cap U_{\alpha_n}) : n \in \mathbb{N}\}$ is a singleton set $\{b\} \subset F$ and $\{F \cap U_{\alpha_n} : n \in \mathbb{N}\}$ is a base of the subspace F at the point $b \in F$. Hence the subspace F has a complete base of countable order (see [8]).

Now we consider the space $A = \prod\{A_n : n \in \mathbb{N}\}$ with the Baire metric d and the open continuous projection $f : A \longrightarrow X$. In this case $f^{-1}(F)$ is complete metrizable by the metric d as a closed subset of $\prod\{A_n : n \in \mathbb{N}\}$ and $f|f^{-1}(F) : f^{-1}(F) \longrightarrow F$ is a continuous open mapping of $(f^{-1}(F), d)$ onto F. The proof is complete. ■

5. Open images of submetrizable spaces

Theorem 5.1. *Let $f : X \longrightarrow Y$ be an open continuous complete mapping of a regular space X with a pseudo-base of countable order onto a space Y. Then Y is a space with a pseudo-base of countable order.*

Proof. Fix a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of the Wallman compactification ωX such that $X \subset Z = \cap\{V_n : n \in \mathbb{N}\}$ and the set $f^{-1}(y)$ is closed in the space Z for each point $y \in Y$.

Since X is a regular space, on X there exists a pseudo-base of countable order $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\})$ such that:

$(S3)$ $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$;

$(S4) U_\alpha \subseteq cl_{\omega X} U_\alpha \subset V_n$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$.

Fix a point $b \in Y$ and a c-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ such that $b \in \cap\{W_{\alpha_n} = f(U_{\alpha_n}) : n \in \mathbb{N}\}$.

Let $a \in X$ and $f^{-1}(f(a)) \cap U_{\alpha_n} \neq \emptyset$ for each $n \in \mathbb{N}$. We affirm that $f(a) = b$.

The sets $H(a) = cl_{\omega X} f^{-1}(f(a))$ and $F_n = cl_{\omega X} U_{\alpha_n}$ are closed in a compact space ωX and $F_{n+1} \subset F_n$, $F_n \cap H(a) \neq \emptyset$ for each $n \in \mathbb{N}$. Hence $F = \cap\{F_n \cap H(a) : n \in \mathbb{N}\} \neq \emptyset$. Since $F_n \subset V_n$ for each $n \in \mathbb{N}$, we have $F \subset Z$. Since $f^{-1}(f(a))$ is a closed subset of Z and $H(a) \cap Z = f^{-1}(f(a))$, we have $F \subseteq f^{-1}(f(a))$. Then $a \in \cap\{U_{\alpha_n} : n \in \mathbb{N}\} = \cap\{cl_X U_{\alpha_n} : n \in \mathbb{N}\} = F$ and $f(a) = b$.

Therefore for any c-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ we have:

- the set $\cap\{W_{\alpha_n} = f(U_{\alpha_n}) : n \in \mathbb{N}\}$ is empty or a singleton set;
- the set $\cap\{W_{\alpha_n} : n \in \mathbb{N}\}$ is a singleton set if and only if the set $\cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a singleton set;
- $\cap\{W_{\alpha_n} : n \in \mathbb{N}\} = f(\cap\{U_{\alpha_n} : n \in \mathbb{N}\})$.

Therefore $(f(\gamma), \pi) = (\{f(\gamma_n) = \{W_\alpha = f(U_\alpha) : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\})$ is a pseudo-base of countable order of the space Y. ■

Since any space with a G_δ-diagonal is a space with a pseudo-base of countable order, from Theorem 5.1 it follows

Corollary 5.1. *Let $f : X \longrightarrow Y$ be an open continuous complete mapping of a regular space X with a G_δ-diagonal onto a space Y. Then Y is a space with a pseudo-base of countable order.*

Corollary 5.2. *For a regular space X the following assertions are equivalent:*

(1) X is a space with a pseudo-base of countable order.
(2) X is an open continuous complete image of some regular space with a G_δ-diagonal.
(3) X is an open continuous complete image of some regular submetrizable space.

Theorem 5.2. *Let $f : X \longrightarrow Y$ be an open continuous uniformly complete mapping of a submetrizable space X onto a space Y. Then Y is space with a pseudo-base of countable order. If the mapping f is compact, then Y is a space with a uniform pseudo-base.*

Proof. Assume that the mapping f is uniformly complete relatively to the continuous metric d and T be the topology of the space X.

Since d is a continuous metric on X, on X there exists an A-sieve $(\gamma, \pi) = (\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\})$ with the following properties:

1. $U_\alpha \in T(d)$ and $diam_d(U_\alpha) < 2^{-n}$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$.

2. $\gamma_n = \{U_\alpha : \alpha \in A_n\}$ is an open locally finite cover of the metric space (X, d) for each $n \in \mathbb{N}$.

By construction, since d is a continuous metric, $\gamma_n = \{U_\alpha : \alpha \in A_n\}$ is an open locally finite cover of the topological space X for each $n \in \mathbb{N}$.

Obviously, (γ, π) is a uniform pseudo-base of the space X.

For any subset A of X the closures $cl_{(X,T)}A$ and $cl_{(X,d)}A$ are closed in X, $cl_{(X,T)}A \subset cl_{(X,d)}A$ and $diam_d(cl_{(X,d)}A) = diam_d(A)$.

Fix a point $b \in Y$ and a c-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ such that $b \in \cap\{W_{\alpha_n} = f(U_{\alpha_n}) : n \in \mathbb{N}\}$.

Let $a \in X$ and $f^{-1}(f(a)) \cap cl_{(X,d)}U_{\alpha_n} \neq \emptyset$ for each $n \in \mathbb{N}$. We affirm that $f(a) = b$.

The sets $H(a) = f^{-1}(f(a))$ and $F_n = cl_{(X,d)}U_{\alpha_n}$ are closed in the space (X, d) and $F_{n+1} \subset F_n$, $diam_d(F_n) \leq 2^{-n}$, $F_n \cap H(a) \neq \emptyset$ for each $n \in \mathbb{N}$. Since $(H(a), d)$ is a complete metric space, $F = \cap\{F_n \cap H(a) : n \in \mathbb{N}\} \neq \emptyset$. Hence $F = \{a\}$. If $f(a) \neq b$, then $a \notin f^{-1}(b)$, $f^{-1}(b)$ is a closed subset of (X, d) and $d(a, f^{-1}(b)) = 2r > 0$. Assume that $2^{-m} < r$. Then $F_n \cap f^{-1}(b) = \emptyset$ and $b \notin f(F_n)$ and $b \notin f(U_{\alpha_n})$, a contradiction. Hence $f(a) = b$.

Therefore for any c-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ we have:

- the set $\cap\{W_{\alpha_n} = f(U_{\alpha_n}) : n \in \mathbb{N}\}$ is empty or a singleton set;

- the set $\cap\{f(cl_{(X,d)}U_{\alpha_n}) : n \in \mathbb{N}\}$ is a singleton set if and only if the set $\cap\{cl_{(X,d)}U_{\alpha_n} : n \in \mathbb{N}\}$ is a singleton set;

- $\cap\{W_{\alpha_n} : n \in \mathbb{N}\} \subseteq f(\cap\{cl_{(X,d)}U_{\alpha_n} : n \in \mathbb{N}\})$.

Therefore $(f(\gamma), \pi) = (\{f(\gamma_n) = \{W_\alpha = f(U_\alpha) : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \longrightarrow A_n : n \in \mathbb{N}\})$ is a pseudo-base of countable order of the space Y.

If f is a compact mapping, then the covers $f(\gamma_n)$ are point-finite and $(f(\gamma), \pi)$ is a uniform pseudo-base. The proof is complete. ■

Now let us mention the following assertion

Proposition 5.1. *Any metacompact space with a pseudo-base of countable order is a space with a uniform pseudo-base.*

6. On set-valued mappings

We say that $\theta : X \longrightarrow Y$ is a set-valued mapping of a space X into a space Y if $\theta(x)$ is a closed non-empty subset of Y for any point $x \in X$. If $A \subset X$ and $B \subset Y$, then $\theta(A) = \cup\{\theta(x) : x \in A\}$ and $\theta^{-1}(B) = \{x \in X : \theta(x) \cap B \neq \emptyset\}$. The set-valued mapping θ is called lower (upper) semi-continuous if for any open (closed) subset H of the space Y the set $\theta^{-1}(H)$ is open (closed) in the space X.

Theorem 6.1. *Let $\theta : X \longrightarrow Y$ be a lower semi-continuous set-valued mapping of a paracompact k-space X into a submetrizable space Y and for any compact subset F of X there exist a compact subset $e(F)$ such that $\theta(F) \subset e(F)$. Then there exists an upper semi-continuous mapping $g : X \longrightarrow Y$ and a lower semi-continuous mapping $f : X \longrightarrow Y$ such that:*

(1) The sets $g(x)$ and $f(x)$ are non-empty compact subsets of Y and $f(x) \subset g(x) \subset \theta(x)$ for each point $x \in X$.

(2) If $dim X = 0$, then $f = g$ is a continuous single-valued mapping.

Proof. Assume that d is a continuous metric on Y. For any point $x \in X$ the set $\theta(x)$ is closed in Y, $e(\{x\})$ is a compact subset of Y and $\theta(x) \subseteq e(\{x\})$. Hence $\theta(x)$ is a metrizable compact subset of Y for any point $x \in X$. In particular, the family $\mathcal{F} = \{\theta(x); x \in X\}$ is uniformly complete relatively to the continuous metric d. Let T be the topology of the space Y. By virtue of the E. Michael's selection theorems [28, 29] there exist an upper semi-continuous mapping $g : X \longrightarrow (Y, d)$ and a lower semi-continuous mapping $f : X \longrightarrow (Y, d)$ such that:

1. The sets $g(x)$ and $f(x)$ are non-empty compact subsets of Y and $f(x) \subset g(x) \subset \theta(x)$ for each point $x \in X$.

2. If $dimX = 0$, then $f = g$ is a continuous single-valued mapping of the space X into the space (Y, d).

Fix a compact subset F of X. Then the topologies T and $T(d)$ coincide on the compact set $e(F)$. Hence the restrictions $g|F$ and $f|F$ are respectively upper and lower semi-continuous mappings of F into (Y, T). Since X is a k-space, the mappings g and f are respectively upper and lower semi-continuous of X into (Y, T). The proof is complete. ■

Theorem 6.2. *Let $\theta : X \longrightarrow Y$ be a lower semi-continuous mapping of a paracompact k-space X into a locally convex linear space Y of countable pseudocharacter. Assume that for any point $x \in X$ the set $\theta(x)$ is convex and for any compact subset F of X there exist a compact subset $e(F)$ such that $\theta(F) \subset e(F)$. Then there exists a single-valued continuous mapping $g : X \longrightarrow Y$ such that $g(x) \in \theta(x)$ for each point $x \in X$.*

Proof. For any point $x \in X$ the set $\theta(x)$ is closed in Y, $e(\{x\})$ is a compact subset of Y and $\theta(x) \subseteq e(\{x\})$. Hence $\theta(x)$ is a compact convex subset of Y for any point $x \in X$.

If $A \subset Y$, $B \subset Y$ and $C \subset \mathbb{R}$, then $A + B = \{x + y : x \in A, y \in B\}$ and $C \cdot A = \{t \cdot x : x \in A, t \in C\}$. There exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open convex subsets of the space Y such that $U_{n+1} + U_{n+1} + U_{n+1} \subset 2^{-2}U_n \subset U_n = -U_n$ for each $n \in \mathbb{N}$ and $\cap\{U_n : n \in \mathbb{N}\} = \{0\}$.

For any $n \in \mathbb{N}$ we define the Minkowski functional $p_n : Y \longrightarrow [0, \infty)$, defined by $p_n(y) = inf\{r \in [0, \infty) : y \in r \cdot U_n\}$. Let $q_n(y) = min\{1, p_n(y)\}$ and $q(y) = \Sigma\{2^{-n}q_n(y) : n \in \mathbb{N}\}$ for each $y \in Y$. From the properties of the Minkowski functionals and constructions follows the next properties:

- $p_n(u+v) \leq p_n(u)+p_n(v)$ and $q(u+v) \leq q(u)+q(v)$ for each $n \in \mathbb{N}$ and all $u, v \in Y$;
- $q(y) = 0$ if and only if $y = 0$;
- if $n \in \mathbb{N}$ and $y \in U_{n+1}$, then $q_{n+1}(y) = p_{n+1}(y) \leq 1$, $q_n(y) = p_n(y) \leq 2^{-2}$ and $q(y) \leq 2^{-n}$.

Hence $d(u, v) = q(u - v)$, $u, v \in Y$, is an invariant continuous metric on the space Y and $T(d)$ is a locally convex topology on the linear space Y. Therefore $\theta : X \longrightarrow (Y, d)$ is a lower semi-continuous compact and convex-valued mapping of the paracompact space X into a metrizable locally convex topological linear space (Y, d). By virtue of E. Michael's selection theorem [26, 27], there exists a single-valued continuous mapping $g : X \longrightarrow (Y, d)$ a such that $g(x) \in \theta(x)$ for each point $x \in X$. Since X is a k-space and the restriction $g|F : F \longrightarrow e(F)$ is continuous for any compact subset F of X, the mapping $g : X \longrightarrow (Y, T)$ is continuous too. The proof is complete. ■

For regular spaces Theorem 6.1 follows from the next theorem.

Theorem 6.3. *Let $\theta : X \longrightarrow Y$ be a lower semi-continuous mapping of a paracompact k-space X into a regular space Y with a pseudo-base of countable order. Assume that for any compact subset F of X there exist a compact subset $e(F)$ such that $\theta(F) \subset e(F)$. Then there exist an upper semi-continuous mapping $g : X \longrightarrow Y$ and a lower semi-continuous mapping $f : X \longrightarrow Y$ such that:*

(1) The sets $g(x)$ and $f(x)$ are non-empty compact subsets of Y and $f(x) \subset g(x) \subset \theta(x)$ for each point $x \in X$.
(2) If $dimX = 0$, then $f = g$ is a continuous single-valued mapping.

Proof. For any point $x \in X$ the set $\theta(x)$ is closed in Y, $e(\{x\})$ is a compact subset of Y and $\theta(x) \subseteq e(\{x\})$. Hence $\theta(x)$ is a compact subset of Y for any point $x \in X$. Hence $\mathcal{F} = \{F \subset Y : F \textit{ is a compact subset of } Y\}$ is a complete family of subsets of the space Y and $\theta(x) \in \mathcal{F}$ for each point $x \in X$.

By virtue of Theorem 4.7, there exist a submetrizable space A, a continuous metric d on A and an open continuous uniformly complete mapping $h : A \longrightarrow Y$ of the space A onto the space Y such that for any $F \in \mathcal{F}$ the subspace $h^{-1}(F)$ is complete metrizable by the metric d. Let T be the regular topology on the space A.

Consider the set-valued mapping $\Theta : X \longrightarrow A$, where $\Theta(x) = h^{-1}(\theta(x))$ for each $x \in X$. Since the mapping h is open, the mapping Θ is lower semi-continuous and the images $\Theta(x)$ are complete metrizable by the metric d.

By virtue of the E. Michael's selection theorems [28, 29] there exist an upper semi-continuous mapping $G : X \longrightarrow (A, d)$ and a lower semi-continuous mapping $\Phi : X \longrightarrow (A, d)$ such that:

1. The sets $G(x)$ and $\Phi(x)$ are non-empty compact subsets of (A, d) and $\Phi(x) \subset G(x) \subset \Theta(x)$ for each point $x \in X$.

2. If $dimX = 0$, then $\Phi = G$ is a continuous single-valued mapping of the space X into the space (A, d).

Fix a compact subset F of X. Then the topologies T and $T(d)$ coincide on the compact set $h^{-1}(e(F))$. Hence the restrictions $G|F$ and $\Phi|F$ are respectively upper and lower semi-continuous mappings of F into (A, T). Since X is a k-space, the mappings G and Φ are respectively upper and lower semi-continuous mappings of X into (A, T). Now we put $g(x) = h(G(x))$ and $f(x) = h(\Phi(x))$. The proof is complete. ∎

Similarly, with the respective results from [13, 14, 15], we can prove other results for selections. Let us mention the following theorem.

Theorem 6.4. *Let $\theta : X \longrightarrow Y$ be a lower semi-continuous mapping of a paracompact k-space X into a regular space Y with a pseudo-base of countable order. Assume that*

$dimX = m$, $m \in \mathbb{N}$, and for any compact subset F of X there exist a compact subset $e(F)$ such that $\theta(F) \subset e(F)$. Then there exists an upper semi-continuous mapping $g : X \longrightarrow Y$ such that $g(x) \subset \theta(x)$ and $|g(x)| \leq m + 1$ for each point $x \in X$.

Example 6.5. Let $Y = \{0\} \cup \{2^{-n} : n\mathbb{N}\}$ be a compact subset of the reals in the usual topology. In the article [19] was constructed a space S and an open continuous mapping $g : S \longrightarrow Y$ such that:

- $dimS = 0$ and any compact subset of S is finite;
- the fibers $g^{-1}(y)$, $y \in Y$ are finite.

Hence the space S is submetrizable, Y is a k-space as a compact space and g is a uniformly complete mapping. If $f : Y \longrightarrow S$ is a mapping and $f(y) \in g^{-1}(y)$ for each $y \in Y$, then the mapping f is not continuous. Thus, in the previous Theorems 6.1 - 6.4, the requirement that the image of a compact set is contained in a compact set is essential.

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