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On integrability of homogeneous rational equations

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Abstract. We study the integrability of homogeneous rational linear and homogeneous rational quadratic differential equations. We prove that these equations can be integrated by using their algebraic solutions which are invariant straight lines.
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Integrabilitatea ecuaţiilor diferenţiale raţionale omogene

Rezumat. Se studiază integrabilitatea ecuaţiilor diferenţiale raţionare liniare omogene şi raţionare pătratice omogene. Se demonstrează că aceste ecuaţii pot fi integrate folosind soluţiile algebrice ale ecuaţiilor care reprezintă drepte invariante.
Cuvinte cheie: ecuaţii diferenţiale omogene, soluţii algebrice, integrabilitate.

1. Algebraic solutions and Darboux method of integrability

In this paper we deals with rational ordinary differential equations

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real and coprime polynomials in the variables x and y. We associate to this rational equation a planar polynomial differential system

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y) \tag{2}$$

by introducing an independent variable t usually called time, where $(x, y) \in \mathbb{R}^2$.

We define $\mathbb{C}[x, y]$ as the ring of polynomials in two variables with complex coefficients and denote by $n = \max\{degP,\ degQ\}$. One of the most evident questions to ask is whether the solutions to (2) would be algebraic [1]. By this we mean whether trajectories of (2) can be described by an algebraic formula, for example, $\Phi(x, y) = 0$, where Φ is a polynomial.

Definition 1.1. *An algebraic curve* $\Phi(x,y) = 0$ *is an invariant algebraic curve of a polynomial system* (2) (*a differential equation* (1)), *if there exists a polynomial* $K(x,y)$ *such that*

$$P(x,y)\frac{\partial \Phi}{\partial x} + Q(x,y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x,y)\,K(x,y). \tag{3}$$

The polynomial $K(x,y)$ *is called the cofactor of the invariant algebraic curve* $\Phi(x,y) = 0$.

We will always assume that $\Phi(x,y)$ is an irreducible polynomial in $\mathbb{C}[x,y]$.

Definition 1.2. *We say that the invariant algebraic curve* $\Phi(x,y) = 0$ *is an algebraic solution of* (2) *if and only if* $\Phi(x,y)$ *is an irreducible polynomial in* $\mathbb{C}[\mathrm{x},\mathrm{y}]$.

We define the notion of an exponential factor, described in [2], as ”degenerate algebraic curve”. Exponential factors appear from the coalescence of invariant algebraic curves. If we have a differential system (2) with an exponential factor of the form $\exp(g/h)$, then there is a 1-parameter perturbation of (2), given by a small ε, with two invariant algebraic curves $h = 0$ and $h + \varepsilon g = 0$.

Definition 1.3. *Let h, g be two coprime polynomials. The function* $R(x,y) = \exp(g/h)$ *is called an exponential factor for system* (2) (*equation* (1)) *if for some polynomial* $K_R(x,y) \in \mathbb{C}[x,y]$ *of degree at most* $n-1$ *the following identity holds*

$$P(x,y)\frac{\partial R}{\partial x} + Q(x,y)\frac{\partial R}{\partial y} \equiv R(x,y)\,K_R(x,y). \tag{4}$$

Invariant algebraic curves and exponential factors can be used in order to find a first integral for the differential system.

Definition 1.4. *We say that system* (2) *is integrable on an open set D of* $\mathbb{R}^2$ *if there exists a nonconstant analytic function* $F : D \to \mathbb{R}$ *which is constant on all solution curves* (x,y) *in* D, *that is* $F(x,y) = C$, *where the solutions are defined. Such an F is called a first integral of the system on D.*

Theorem 1.1. *The function* $F(x,y)$ *is a first integral of* (2) *on D if and only if*

$$P(x,y)\frac{\partial F}{\partial x} + Q(x,y)\frac{\partial F}{\partial y} \equiv 0. \tag{5}$$

Definition 1.5. *An integrating factor for a system* (2) (*an equation* (1)) *on some open set D of* $\mathbb{R}^2$ *is a* C^1 *function* $\mu = \mu(x,y)$ *defined on D, not identically zero on D such that*

$$P(x,y)\frac{\partial \mu}{\partial x} + Q(x,y)\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \equiv 0. \tag{6}$$

Integrating factors turn inexact differential equations into exact ones. The question is, how do we find an integrating factor? In 1878, Darboux published his paper [3] in which he gives a method of integration of differential equations by using invariant algebraic curves. Darboux's idea consists in searching for a first integral or an integrating factor in the form

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}, \tag{7}$$

where $\Phi_1 = 0, \dots, \Phi_q = 0$ are invariant algebraic curves and exponential factors of (2) in $\mathbb{C}[x, y]$ and $\alpha_j \in \mathbb{C},\ j = 1, \dots, q$.

Definition 1.6. *A first integral (an integrating factor) of differential system* (2) (*differential equation* (1)) *of the form* (7) *is called a Darboux first integral (a Darboux integrating factor).*

We notice that the study of invariant algebraic curves let us better understand the integrability of differential equations [4, 5].

Theorem 1.2. *The system* (2) (*equation* (1)) *has a Darboux first integral of the form* (7) *if and only if there exists constants* $\alpha_j \in \mathbb{C}$, $j = 1, \dots, q$ *not all identically zero such that*

$$\alpha_1 K_1(x, y) + \alpha_2 K_2(x, y) + \cdots + \alpha_q K_q(x, y) \equiv 0. \tag{8}$$

Theorem 1.3. *The system* (2) (*equation* (1)) *has a Darboux integrating factor of the form* (7) *if and only if there exists constants* $\alpha_j \in \mathbb{C}$, $j = 1, \dots, q$ *such that*

$$\alpha_1 K_1(x, y) + \alpha_2 K_2(x, y) + \cdots + \alpha_q K_q(x, y) + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0. \tag{9}$$

The integrability of differential equation (1) with $P(x, y) = x(a_1x + b_1y + c_1)$ and $Q(x, y) = y(a_2x + b_2y + c_2)$, i.e. when (2) is the quadratic Lotka-Volterra differential system, was studied in [6, 7, 8] by determining algebraic solutions. So in [7, 8] it was obtained the first integrals and the integrating factors of (2) by using invariant straight lines and irreducible invariant conics, and in [6] it was found the first integrals and the integrating factors of (2) by using irreducible invariant cubics.

The integrability of Iacobi differential equation (1) with $P(x, y) = a_1x + b_1y + c_1 + x(a_3x + b_3y + c_3)$ and $Q(x, y) = a_2x + b_2y + c_2 + y(a_3x + b_3y + c_3)$, was studied in [9, 10] by determining algebraic solutions of the first degree.

In this paper we study the integrability of differential equation (1) when $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of degree one (called *homogeneous rational linear differential equation*) and of degree two (called *homogeneous rational quadratic differential equation*). We prove that such type of differential equations can be integrated by using their algebraic solutions which are invariant straight lines.

2. Homogeneous rational linear differential equations with integrating factors

Let us consider the homogeneous differential equation

$$\frac{dy}{dx} = \frac{a_2x + b_2y}{a_1x + b_1y}, \tag{10}$$

where a_1, b_1, a_2, b_2 are real coefficients. It is known that by substitution [11]

$$y = x \cdot t(x), \quad y' = t + x \cdot t'$$

the equation (10) can be reduced to an equation with separable variables

$$\frac{dx}{x} = \frac{(a_1 + b_1t)dt}{a_2 + (b_2 - a_1)t - b_1t^2}. \tag{11}$$

Investigating the possible cases, depending on the coefficients of (10), we can integrate the equation (11) and obtain the general solution (the first integral) of (10).

In [12] it was proved that if $b_2 \neq a_1$, then the equation (10) has an integrating factor of the form

$$\mu = \frac{1}{a_2x^2 + (b_2 - a_1)\, xy - b_1y^2}. \tag{12}$$

In this paper we show that the integrating factors of (10) can be constructed from algebraic solutions of the equation (10).

Theorem 2.1. *Let $b_1 \neq 0$. Then the equation* (10) *has an integrating factor*

$$\mu = \frac{1}{(y - k_1x)\,(y - k_2x)} \tag{13}$$

composed of two invariant straight lines $y = k_1x$, $y = k_2x$, where k_1, k_2 are the solutions of the equation

$$b_1k^2 + (a_1 - b_2)\, k - a_2 = 0. \tag{14}$$

Proof. We are looking for algebraic solutions of the equation (10) in the form of straight lines $y = kx$, where k is a constant. Substituting in (10) $y = kx$, we obtain the quadratic equation (14) with respect to k.

Denote by $\Delta = (a_1 - b_2)^2 + 4a_2b_1$ the discriminant of (14). The straight lines are real if $\Delta > 0$ and complex conjugated if $\Delta < 0$. When $\Delta = 0$, we have the straight line

$$b_2x - a_1x - 2b_1y = 0. \tag{15}$$

Let us show that the function (13) is an integrating factor for (10). By [13], the function (13) is an integrating factor for (10) if the following relation is satisfied

$$P(x, y)\frac{\partial \mu}{\partial x} + Q(x, y)\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = 0 \tag{16}$$

with $P(x,y) = a_1x + b_1y$ and $Q(x,y) = a_2x + b_2y$. The equation (16) yields

$$x^2[(k_1+k_2)a_2+k_1k_2(b_2-a_1)]-2xy(a_2+b_1k_1k_2)+y^2[a_1-b_2+b_1(k_1+k_2)] = 0. \quad (17)$$

As k_1, k_2 are the solutions of the equation (14), then by Vieta's formulas we have

$$a_2 = -b_1k_1k_2, \;\; b_2 = a_1 + b_1(k_1 + k_2).$$

Substituting this in the equation (17) we obtain an identity. It was proved that the function (13) is an integrating factor for the equation (10). □

3. Homogeneous rational quadratic differential equations with integrating factors

Let us consider the homogeneous differential equation

$$\frac{dy}{dx} = \frac{b_{20}x^2 + b_{11}xy + b_{02}y^2}{a_{20}x^2 + a_{11}xy + a_{02}y^2}, \quad (18)$$

where a_{ij}, b_{ij} are real coefficients. Assume that $(b_{20}, b_{11}, b_{02}) \neq 0$, $(a_{20}, a_{11}, a_{02}) \neq 0$ and the right hand side of (18) is not reducible. It is known that by substitution

$$y = x \cdot t(x), \;\; y' = t + x \cdot t'$$

the equation (18) can be reduced to an equation with separable variables

$$\frac{dx}{x} = \frac{(b_{20} + b_{11}t + b_{02}t^2)dt}{a_{20} + (a_{11} - b_{20})t + (a_{02} - b_{11})t^2 - b_{02}t^3}. \quad (19)$$

Investigating the possible cases, depending on the coefficients of (18), we can try to integrate the equation (19) and obtain the first integral of (18).

In [14] it was proved that if $a_{20} \neq b_{11}$, then the equation (18) has an integrating factor of the form

$$\mu = \frac{1}{b_{20}x^3 + (b_{11} - a_{20})\,x^2y + (b_{02} - a_{11})\,xy^2 - a_{02}y^3}. \quad (20)$$

In this paper we show that the integrating factors of (18) can be constructed from algebraic solutions of this equation.

Theorem 3.1. *Let $a_{02} \neq 0$. Then the equation* (18) *has an integrating factor*

$$\mu = \frac{1}{(y - k_1x)\,(y - k_2x)\,(y - k_3x)} \quad (21)$$

composed of three invariant straight lines $y = k_1x$, $y = k_2x$, $y = k_3x$, where k_1, k_2 and k_3 are the solutions of the equation

$$a_{02}k^3 + (a_{11} - b_{02})k^2 + (a_{20} - b_{11})k - b_{20} = 0. \quad (22)$$

Proof. We are looking for algebraic solutions of the equation (18) in the form of straight lines $y = kx$, where k is a constant. Substituting in (18) $y = kx$, we obtain the equation

$$k = \frac{b_{20} + b_{11}k + b_{02}k^2}{a_{20} + a_{11}k + a_{02}k^2}$$

with respect to k which implies the cubic equation (22). Denote by

$$\Delta = 4b_{20}(a_{11} - b_{02})^3 + (a_{11} - b_{02})^2(a_{20} - b_{11})^2 - 4a_{02}(a_{20} - b_{11})^3 +$$

$$-18a_{02}b_{20}\left(a_{11} - b_{02}\right)\left(a_{20} - b_{11}\right) - 27a_{02}^2b_{20}^2$$

the discriminant of (22).

If $\Delta > 0$, then (18) has three real and distinct invariant straight lines; if $\Delta < 0$, then (18) has one real straight line and two complex conjugated straight lines; if $\Delta = 0$ and $k_1 = k_2$, then (18) has two real invariant straight lines; if $\Delta = 0$ and $k_3 = k_2 = k_1$, then the equation (18) has a real invariant straight line of multiplicity three.

Let us show that the function (21) is an integrating factor for (18). By Definition 1.5, the function (21) is an integrating factor of (18) if the following relation is satisfied

$$P(x, y)\frac{\partial \mu}{\partial x} + Q(x, y)\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = 0 \tag{23}$$

with $P\left(x, y\right) = a_{20}x^2 + a_{11}xy + a_{02}y^2$ and $Q\left(x, y\right) = b_{20}x^2 + b_{11}xy + b_{02}y^2$.

Identifying the coefficient of the monomials $x^i y^j$ in (23), we obtain an algebraic system of five equations $\left\{U_{ij} = 0,\ \ i + j = 4\right\}$ for the unknowns k_1, k_2 and k_3 :

$$\begin{aligned}
&k_1k_2k_3(a_{20} - b_{11}) - b_{20}(k_1k_2 + k_1k_3 + k_2k_3) = 0,\\
&k_1k_2k_3(a_{11} - b_{02}) + b_{20}(k_1 + k_2 + k_3) = 0,\\
&a_{20} - b_{11} - a_{02}(k_1k_2 + k_1k_3 + k_2k_3) = 0,\\
&a_{11} - b_{02} + a_{02}(k_1 + k_2 + k_3) = 0,\\
&(b_{11} - a_{20})(k_1 + k_2 + k_3) + (b_{02} - a_{11})(k_1k_2 + k_1k_3 + k_2k_3) +\\
&+3a_{02}k_1k_2k_3 - 3b_{20} = 0.
\end{aligned} \tag{24}$$

Let k_1, k_2 and k_3 be the solutions of (22). Then by Vieta's formulas we obtain that

$$\begin{aligned}
b_{02} &= a_{11} + a_{02}\left(k_1 + k_2 + k_3\right),\\
b_{11} &= a_{20} - a_{02}(k_1k_2 + k_1k_3 + k_2k_3),\\
b_{20} &= a_{02}k_1k_2k_3.
\end{aligned} \tag{25}$$

Substituting (25) in (24) we obtain that $U_{ij} \equiv 0$. The function (21) is an integrating factor for the equation (18). □

4. On integrability of homogeneous rational linear differential equations

Let us consider the homogeneous differential equation

$$\frac{dy}{dx} = \frac{a_2 x + b_2 y}{a_1 x + b_1 y}, \tag{26}$$

where a_1, b_1, a_2, b_2 are real coefficients. We choose the algebraic solutions of (26) as straight lines of the form $y = kx$, where k is a constant. Substituting $y = kx$ in (26) we obtain a quadratic equation with respect to k :

$$b_1 k^2 + (a_1 - b_2)\, k - a_2 = 0. \tag{27}$$

Suppose that $b_1 \neq 0$ and denote by $\Delta = (a_1 - b_2)^2 + 4a_2 b_1$ the discriminant of this equation.

Theorem 4.1. *Let* $\Delta > 0$. *Then the equation* (27) *has two real solutions* k_1, k_2 *and the equation* (26) *has two invariant straight lines*

$$mx - a_1 x + b_2 x - 2b_1 y = 0, \;\; mx + a_1 x - b_2 x + 2b_1 y = 0.$$

The equation (26) *has a first integral*

$$\frac{(mx - a_1 x + b_2 x - 2b_1 y)^{a_1 + b_2 + m}}{(mx + a_1 x - b_2 x + 2b_1 y)^{a_1 + b_2 - m}} = C, \tag{28}$$

where $m = \sqrt{4a_2 b_1 + (b_2 - a_1)^2}$ *and* C *is an arbitrary constant.*

Proof. According to Definition 1.1, the lines

$$\Phi_1 \equiv mx - a_1 x + b_2 x - 2b_1 y = 0, \;\; \Phi_2 \equiv mx + a_1 x - b_2 x + 2b_1 y = 0$$

are invariant straight lines with cofactors

$$K_1(x, y) = \frac{a_1 + b_2 - m}{2}, \;\; K_2(x, y) = \frac{a_1 + b_2 + m}{2}.$$

By Theorem 1.2, the differential equation (26) has a Darboux first integral of the form

$$\Phi_1^{\alpha_1} \Phi_2^{\alpha_2} = C \tag{29}$$

if and only if there exist constants α_1 and α_2 such that

$$\alpha_1 (a_1 + b_2 - m) + \alpha_2 (a_1 + b_2 + m) \equiv 0.$$

The last identity implies

$$\alpha_1 = a_1 + b_2 + m, \;\; \alpha_2 = -a_1 - b_2 + m.$$

Substituting α_1 and α_2 in (29) we obtain the first integral (28). Theorem 4.1 is proved. □

Theorem 4.2. *Let* $\Delta < 0$. *Then the equation* (27) *has two complex conjugate solutions* $k_1 = \alpha + i\beta$, $k_2 = \alpha - i\beta$ *and the equation* (26) *has two complex conjugate invariant straight lines*

$$(a_1 - b_2 - im)x + 2b_1 y = 0,\ (a_1 - b_2 + im)x + 2b_1 y = 0.$$

The equation (26) *has a complex first integral*

$$\frac{(a_1x - b_2x + 2b_1y - imx)^{a_1+b_2+im}}{(a_1x - b_2x + 2b_1y + imx)^{a_1+b_2-im}} = C$$

where $m = \sqrt{-4a_2b_1 - (b_2 - a_1)^2}$ *and* C *is an arbitrary constant.*

Proof. Theorem 4.2 can be proved similarly as was proved Theorem 4.1. Using the well-known formula from complex analysis

$$f^{\lambda}\overline{f}^{\overline{\lambda}} = \left[(\mathrm{Re}f)^2 + (\mathrm{Im}f)^2\right]^{\mathrm{Re}\lambda} \cdot \exp\left(-2\mathrm{Im}\lambda \cdot \arg(\mathrm{Re}f + i \cdot \mathrm{Im}f)\right),$$

$$\frac{f^{\lambda}}{\overline{f}^{\overline{\lambda}}} = \left[(\mathrm{Re}f)^2 + (\mathrm{Im}f)^2\right]^{\mathrm{Im}\lambda} \cdot \exp\left(2\mathrm{Re}\lambda \cdot \arg(\mathrm{Re}f + i \cdot \mathrm{Im}f)\right),$$

we obtain the first integral of (26) in the real form

$$\left((a_1x - b_2x + 2b_1y)^2 + m^2x^2\right)^m \exp\left(2(a_1 + b_2) \cdot \mathrm{arctg}\,\frac{a_1x - b_2x + 2b_1y}{mx}\right) = C$$

or

$$\left(4b_1^2y^2 + 4b_1\left(a_1 - b_2\right)xy - 4b_1a_2x^2\right)^{\sqrt{-4a_2b_1-(b_2-a_1)^2}} \cdot$$

$$\exp\left(2(a_1 + b_2) \cdot \mathrm{arctg}\,\frac{a_1x - b_2x + 2b_1y}{\sqrt{-a_1^2x^2 + 2a_1b_2x^2 - b_2^2x^2 - 4a_2b_1x^2}}\right) = C. \tag{30}$$

The last first integral can be written as

$$\frac{a_1 + b_2}{\sqrt{-a_1^2 + 2a_1b_2 - b_2^2 - 4a_2b_1}}\,\mathrm{arctg}\,\frac{a_1x - b_2x + 2b_1y}{\sqrt{-a_1^2x^2 + 2a_1b_2x^2 - b_2^2x^2 - 4a_2b_1x^2}} +$$

$$+\frac{1}{2}\ln\left|4b_1^2y^2 + 4b_1\left(a_1 - b_2\right)xy - 4b_1a_2x^2\right| = C, \tag{31}$$

where C is an arbitrary constant. □

Theorem 4.3. *Let* $\Delta = 0$. *Then equation* (27) *has a real solution* $k = (a_1 - b_2)/(2b_1)$ *and the differential equation* (26) *has one real invariant straight line*

$$b_2x - a_1x - 2b_1y = 0.$$

The equation (26) *has the first integral*

$$(b_2x - a_1x - 2b_1y)e^{x(a_1+b_2)/(b_2x-a_1x-2b_1y)} = C, \tag{32}$$

where C is an arbitrary constant.

Proof. According to Definition 1.1, the line $\Phi_1 \equiv b_2x - a_1x - 2b_1y = 0$ is an invariant straight line for (26) with cofactor $K_1(x,y) = \frac{a_1+b_2}{2}$. It is easy to verify that the function

$$\Phi_2 = e^{x(a_1+b_2)/(b_2x-a_1x-2b_1y)}$$

satisfies relation (4) with cofactor $K_2(x,y) = -\frac{a_1+b_2}{2}$ and Φ_2 is an exponential factor for (26). By Theorem 1.2, the differential equation (26) has a Darboux first integral of the form

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2} = C \tag{33}$$

if and only if there exist constants α_1 and α_2 such that

$$\alpha_1(a_1 + b_2) - \alpha_2(a_1 + b_2) \equiv 0.$$

The last identity implies $\alpha_1 = \alpha_2 = 1$. Substituting Φ_1, Φ_2 and α_1, α_2 in (33) we obtain the first integral (32). Theorem 4.3 is proved. □

5. On integrability of homogeneous rational quadratic differential equations

We consider the homogeneous rational differential equation

$$\frac{dy}{dx} = \frac{b_{20}x^2 + b_{11}xy + b_{02}y^2}{a_{20}x^2 + a_{11}xy + a_{02}y^2}, \tag{34}$$

where a_{ij}, b_{ij} are real coefficients. Assume that $(b_{20}, b_{11}, b_{02}) \neq 0$, $(a_{20}, a_{11}, a_{02}) \neq 0$ and the right hand side of (34) is not reducible.

We choose the algebraic solutions of (34) as straight lines of the form $y = kx$, where k is a constant. Substituting $y = kx$ in (34) we obtain a cubic equation with respect to k :

$$a_{02}k^3 + (a_{11} - b_{02})k^2 + (a_{20} - b_{11})k - b_{20} = 0. \tag{35}$$

Suppose that $a_{02} \neq 0$ and denote by Δ the discriminant of this equation

$$\Delta = 4b_{20}(a_{11} - b_{02})^3 + (a_{11} - b_{02})^2(a_{20} - b_{11})^2 - 4a_{02}(a_{20} - b_{11})^3 -$$

$$-18a_{02}b_{20}\left(a_{11}-b_{02}\right)\left(a_{20}-b_{11}\right)-27a_{02}^2b_{20}^2.$$

Let $k_1,\ k_2$ and k_3 be the solutions of (35). Then by Vieta's formulas we can write

$$\begin{aligned} b_{02} &= a_{11}+a_{02}\left(k_1+k_2+k_3\right), \\ b_{11} &= a_{20}-a_{02}\left(k_1k_2+k_1k_3+k_2k_3\right), \\ b_{20} &= a_{02}k_1k_2k_3. \end{aligned} \tag{36}$$

Theorem 5.1. *Let* $\Delta{=}a_{02}^4(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2>0$. *Then the equation* (35) *has three real solutions* $k_1,\ k_2,\ k_3$ *and the equation* (34) *has three invariant straight lines* $y=k_1x,\ \ y=k_2x,\ \ y=k_3x$. *The equation* (34) *has a first integral*

$$(y-k_1x)^{h_1}(y-k_2x)^{h_2}\left(y-k_3x\right)^{h_3}=C, \tag{37}$$

where the exponents in (37) *look as*

$$\begin{aligned} h_1 &= (k_2-k_3)\left(a_{02}k_1^2+a_{11}k_1+a_{20}\right), \\ h_2 &= (k_3-k_1)\left(a_{02}k_2^2+a_{11}k_2+a_{20}\right), \\ h_3 &= (k_1-k_2)\left(a_{02}k_3^2+a_{11}k_3+a_{20}\right). \end{aligned} \tag{38}$$

Proof. Let $k_1,\ k_2$ and k_3 be distinct real solutions of (35). By Definition 1.1, the lines

$$\Phi_1\equiv y-k_1x=0,\ \Phi_2\equiv y-k_2x=0,\ \Phi_3\equiv y-k_3x=0$$

are invariant straight lines with cofactors

$$\begin{aligned} K_1\left(x,y\right) &= x\left(a_{20}-a_{02}k_2k_3\right)+y(a_{11}+a_{02}(k_2+k_3)), \\ K_2\left(x,y\right) &= x\left(a_{20}-a_{02}k_1k_3\right)+y(a_{11}+a_{02}(k_1+k_3)), \\ K_3\left(x,y\right) &= x\left(a_{20}-a_{02}k_1k_2\right)+y(a_{11}+a_{02}(k_1+k_2)). \end{aligned}$$

By Theorem 1.2, the differential equation (34) has a Darboux first integral of the form

$$\Phi_1^{h_1}\Phi_2^{h_2}\Phi_3^{h_3}=C \tag{39}$$

if and only if there exist constants h_1, h_2 and h_3 such that

$$h_1K_1\left(x,y\right)+h_2K_2\left(x,y\right)+h_3K_3\left(x,y\right)\equiv 0.$$

The last identity implies

$$\begin{aligned} h_1 &= (k_2-k_3)\left(a_{02}k_1^2+a_{11}k_1+a_{20}\right), \\ h_2 &= (k_3-k_1)\left(a_{02}k_2^2+a_{11}k_2+a_{20}\right), \\ h_3 &= (k_1-k_2)\left(a_{02}k_3^2+a_{11}k_3+a_{20}\right). \end{aligned}$$

Substituting h_1, h_2 and h_3 in (39) we obtain the first integral (37). Theorem 5.1 is proved. □

Theorem 5.2. *If* $\Delta{=}-4\beta^2 a_{02}^4(\alpha^2-2\alpha k_1+k_1^2+\beta^2)^2 < 0$, *then the equation* (35) *has a real solution* k_1 *and two complex conjugate solutions* $k_2=\alpha+i\beta$ *and* $k_3=\alpha-i\beta$. *The equation* (34) *has one real invariant straight line* $y=k_1x$, *two complex conjugate invariant straight lines* $y=(\alpha\pm i\beta)x$ *and a first integral*

$$(y-k_1x)^{h_1}(y-\alpha x-i\beta x)^{h_2}(y-\alpha x+i\beta x)^{h_3}=C, \tag{40}$$

where the exponents in (40) *look as*

$$\begin{aligned} h_1 &= 2\beta(a_{02}k_1^2+a_{11}k_1+a_{20}),\\ h_2 &= (-\beta-i\alpha+ik_1)(a_{02}(\alpha+i\beta)^2+a_{11}(\alpha+i\beta)+a_{20}),\\ h_3 &= (-\beta+i\alpha-ik_1)(a_{02}(\alpha-i\beta)^2+a_{11}(\alpha-i\beta)+a_{20}). \end{aligned} \tag{41}$$

Theorem 5.2 can be proved similarly as was proved Theorem 5.1. Using the well-known formula from complex analysis

$$f^{\lambda}\overline{f}^{\overline{\lambda}} = \left[(\mathrm{Re}f)^2+(\mathrm{Im}f)^2\right]^{\mathrm{Re}\lambda} exp\left(-2\,\mathrm{Im}\lambda\cdot\arg(\mathrm{Re}f+i\cdot\mathrm{Im}f)\right),$$

we obtain the first integral of (34) in the real form

$$(y-k_1x)^{e_1}(\alpha^2x^2+\beta^2x^2-2\alpha xy+y^2)^{e_2}\cdot\exp\left(2e_3\cdot\mathrm{arctg}\,\frac{y-\alpha x}{\beta x}\right)=C, \tag{42}$$

where the exponents in (42) are of the form

$$\begin{aligned} e_1 &= 2\beta\left(a_{02}k_1^2+a_{11}k_1+a_{20}\right),\\ e_2 &= \beta\left((\alpha^2+\beta^2-2k_1\alpha)a_{02}-a_{11}k_1-a_{20}\right),\\ e_3 &= \left(\alpha^3-\alpha^2k_1+\alpha\beta^2+\beta^2k_1\right)a_{02}+\left(\alpha^2-\alpha k_1+\beta^2\right)a_{11}+(\alpha-k_1)\,a_{20}). \end{aligned} \tag{43}$$

Theorem 5.3. *Let* $\Delta=0$ *and* $k_1=k_2$. *Then the equation* (35) *has two real solutions* $k_1=k_2$, k_3 *and the differential equation* (34) *has two real invariant straight lines* $y=k_1x$ *and* $y=k_3x$. *The equation* (34) *has a first integral*

$$(y-k_1x)^{h_1}\left(y-k_3x\right)^{h_3}e^{(h_2x)/(y-k_1x)}=C, \tag{44}$$

where the exponents in (44) *are of the form*

$$\begin{aligned} h_1 &= a_{02}k_1^2-2a_{02}k_1k_3-a_{11}k_3-a_{20},\\ h_2 &= (k_3-k_1)\left(a_{02}k_1^2+a_{11}k_1+a_{20}\right),\\ h_3 &= a_{02}k_3^2+a_{11}k_3+a_{20}. \end{aligned} \tag{45}$$

Proof. Let k_1 and k_3 be distinct real solutions of (35). By Definition 1.1, the lines

$$\Phi_1\equiv y-k_1x=0,\ \Phi_3\equiv y-k_3x=0,$$

are invariant straight lines with cofactors

$$K_1(x,y) = x(a_{20} - a_{02}k_2k_3) + y(a_{11} + a_{02}(k_2 + k_3)),$$
$$K_3(x,y) = x\left(a_{20} - a_{02}k_1^2\right) + y(a_{11} + 2a_{02}k_1). \tag{46}$$

Applying Definition 1.3, it is easy to verify that the function

$$\Phi_2 = e^{x/(y-k_1x)}$$

is an exponential factor for (34) with cofactor $K_2(x,y) = a_{02}(y - k_3x)$.

By Theorem 1.2, the differential equation (34) has a Darboux first integral

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\Phi_3^{\alpha_3} = C \tag{47}$$

if and only if there exist constants α_1, α_2 and α_3 such that

$$\alpha_1K_1(x,y) + \alpha_2K_2(x,y) + \alpha_3K_3(x,y) \equiv 0.$$

The last identity implies

$$h_1 = a_{02}k_1^2 - 2a_{02}k_1k_3 - a_{11}k_3 - a_{20},$$
$$h_2 = (k_3 - k_1)\left(a_{02}k_1^2 + a_{11}k_1 + a_{20}\right),$$
$$h_3 = a_{02}k_3^2 + a_{11}k_3 + a_{20}.$$

Substituting h_1, h_2 and h_3 in (47) we obtain the first integral (44). Theorem 5.3 is proved. □

Theorem 5.4. *Let* $\Delta = 0$ *and* $k_3 = k_2 = k_1$. *Then the equation* (35) *has a real solution* k_1 *of multiplicity three and the differential equation* (34) *has the real invariant straight line* $y = k_1x$. *The equation* (34) *has a first integral*

$$(y - k_1x)^{-2k_1a_{02}} \cdot \exp\left(\frac{\left(a_{20}k_1 - a_{02}k_1^3\right)x^2 + (a_{11} + 2a_{02}k_1)\,y^2}{(y - k_1x)^2}\right) = C. \tag{48}$$

Proof. Let $k_3 = k_2 = k_1$ and the equation (35) have a real solution k_1 of multiplicity three. We shall prove that the relation (48) is a first integral for equation (34).

By Definition 1.1, the line $\Phi_1 \equiv y - k_1x = 0$ is an invariant straight line with cofactor

$$K_1(x,y) = x\left(a_{20} - a_{02}k_1^2\right) + y(a_{11} + 2a_{02}k_1).$$

Applying Definition 1.3, it is easy to establish that the function

$$\Phi_2 = \exp\left[((a_{20}k_1 - a_{02}k_1^3)x^2 + (a_{11} + 2a_{02}k_1)y^2)/(y - k_1x)^2\right]$$

is an exponential factor for (34) with cofactor

$$K_2(x,y) = 2a_{02}k_1\left(\left(a_{20} - a_{02}k_1^2\right)x + (a_{11} + 2a_{02}k_1)\,y\right).$$

By Theorem 1.2, the differential equation (34) has a Darboux first integral of the form

$$\Phi_1^{h_1}\Phi_2^{h_2} = C \tag{49}$$

if and only if there exist constants h_1, h_2 such that

$$h_1 K_1(x, y) + h_2 K_2(x, y) \equiv 0.$$

The last identity implies

$$(2a_{02}k_1h_2 + h_1)\left(a_{02}k_1^2x - a_{20}x - 2a_{02}k_1y - a_{11}y\right) = 0$$

and we obtain

$$h_1 = -2a_{02}k_1,\ \ h_2 = 1.$$

Substituting

$$\Phi_1 = y - k_1x,\ \ \Phi_2 = \exp\left[((a_{20}k_1 - a_{02}k_1^3)x^2 + (a_{11} + 2a_{02}k_1)y^2)/(y - k_1x)^2\right]$$

and $h_1 = -2a_{02}k_1$, $h_2 = 1$ in (49), we obtain the first integral (48). Theorem 5.4 is proved. □

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