Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 10(2), 2020, Pages 47–53 E-ISSN: 2587-3644
DOI: 10.36120/2587-3644.v10i2.47-53

On the monoid of endomorphisms of a topological universal algebra

Mitrofan Choban and Ion Valuţă

Abstract. In the present article we study the monoid of continuous endomorphisms, in the topology of pointwise convergence, of a topological universl algebra. Theorem 3.1 affirms that any semi-topological monoid (semigroup with unity) is isomorphic to a semigroup of continuous endomorphisms $End_c(G)$ of some cyclic free topological universal algebra G.
2010 Mathematics Subject Classification: 3C05; 08A05; 08C35.
Keywords: universal algebra, free topological universal algebra, left ideal, semigroup of continuous endomorphisms.

Despre monoidul endomorfismelor a unei algebre universale topologice

Rezumat. În articolul prezent se studiază monoidul endomorfismelor continue, în topologia convergenței punctuale, a unei algebre universale topologice. Teorema 3.1 afirmă că orice monoid semi-topologic (semigrup cu unitate) este izomorfă cu un semigrup de endomorfisme continue $End_c(G)$ a unei algebre universale topologice libere ciclice G.
Cuvinte cheie: algebră universală, algebră universală topologică liberă, ideal de stânga, semigrup de endomorfisme continue.

1. Introduction

Let $\mathbb{N} = \{1, 2, ...\}$ be the set of natural numbers and $n \in \omega = \{0, 1, 2, ...\}$ be the set of non-negative integers. The n-ary Cartesian power of a set X is denoted by X^n. If the set X is empty, then the set X^n is empty too. If the set X is non-empty, then the set X^0 is a singleton. The discrete sum $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$ of the pairwise disjoint topological spaces $\{\Omega_n : n \in N\}$ is called a continuous signature. If the space Ω is a discrete space, then we say that Ω is a discrete signature.

A topological Ω-algebra or a topological universal algebra of the signature Ω is a family $\{G, e_{nG} : n \in N\}$, where G is a non-empty topological space and $e_{nG} : \Omega_n \times G^n \to G$

is a continuous mapping for each $n \in \omega$. The concept of universal algebra was created by Alfred North Whitehead in 1898 as a generalization of Boole's logical algebras. The term universal algebra was proposed by James Joseph Sylvester [19]. Between 1935 and 1950 important works were published by Garrett Birkhoff, in which he introduced the notions of variety, quasi-variety, free algebra, congruences and proved the homomorphism theorems [1, 2]. After 1950, due to applications in mathematical logic, model theory, geometric algebras, theoretical and computer physics, the theory of universal algebras began to develop fruitfully [2, 5, 6, 8, 9, 11, 12, 18]. As in [14, 15, 16, 17, 18] we continue the study of semigroups of endomorphisms of universal topological algebras.

Let A, B and C be three topological universal algebras of signature Ω. The function $f : A \longrightarrow B$ is called a morphism or homomorphism, if $f(u(x)) = u(f^n(x))$ for any $n \in \omega$, any $u \in \Omega_n$ and any element $x = (x_1, x_2, ..., x_n) \in G^n$, where $f^n(x) = (f(x_1), f(x_2), ..., f(x_n))$. The composition of the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is the function $h = f \cdot g$: $A \longrightarrow C$, where $h(x) = g(f(x))$ for any $x \in A$. The composition of two continuous morphisms is always a continuous morphism. A morphism that is a bijective function is called an isomorphism. An isomorphism which is a homeomorphismis called a topological isomorphism.

If a topological isomorphism can be established between two topological universal algebras, they are called topologic isomorphs. Two topologically isomorph topological universal algebras are identified. Morphisms, respectively isomorphisms, of a topological universal algebra in itself are called endomorphisms, respectively automorphisms.

Subalgebras and Cartesian products of, topological Ω-algebras are defined in traditional way [19, 1, 2, 7, 9, 4, 3, 18].

Let G be a topological space and $n \in N$. A continuous mapping $\lambda : G^n \rightarrow G$ is called an n-ary operation on G.

If G is a topological Ω-algebra and $u \in \Omega_n$, then $u : G^n \rightarrow G$, where $u(x) = e_{nG}(u, x)$ for every $x \in G^n$, is an n-ary operation on G.

A semigroup S equipped with a topology is called a semi-topological semigroup if the tranlations $\{u_a, \varphi_b : a, b \in G\}$, where $u_a(x)a \cdot x$ and $\varphi_b(x)$ for all $a, b, x \in S$, are continuous mappings of the space S into itself.

Monoids are semigroups with identity.

For any topological space S denote by $C_p(X, X)$ the space of all continuous mappings $f : X \longrightarrow X$ in the topology of pointwise convergence. The following assertion is well known.

Theorem 1.1. *For any topological space X the space $C_p(X,X)$ with the operation of composition $f \cdot g$ is a semi-topological monoid with unity.*

Proof. The identity mapping $e_X : X \longrightarrow X$ is the unity of the semigroup $C_p(X,X)$.

Each function $f : X \longrightarrow X$ can be thought as a point in the Cartesian product X^X: $f = (f(x) : x \in X) \in X^X$. The topological space $C_p(X,X)$ is a subspace of the Cartesian product X^X with the Tychonoff product topology, so $C_p(X,X)$ inherits a topology from the product topology on X^X. A net $\{f_\mu : \mu \in M\}$ in $(C_p(X,X)$ converges to some $f \in C_p(X,X)$ if and only if for each $x \in X$ the net net $\{f_\mu(x) : \mu \in M\}$ to converges to $f(x)$ in X.

Fix $g \in C_p(X,X)$. Consider the left and right translations $L_g, R_g : C_p(X,X) \longrightarrow C_p(X,X)$, where $L_g(f) = g \cdot f$ and $R_g(f) = f \cdot g$ for each $f \in C_p(X,X)$. In $C(X,X)$ fix a net $\{f_\mu : \mu \in M\}$ convergent to $f \in C_p(X,X)$. For any point $x \in X$ we have $lim_{\mu\in M} L_g(f_\mu)(x) = lim_{\mu\in M} f_\mu(g(x)) = f(g(x))$ and $lim_{\mu\in M} R_g(f_\mu)(x) = lim_{\mu\in M} g(f_\mu(x)) = g(f(x))$. Therefore $lim_{\mu\in M} L_g(f_\mu) = g \cdot f$ and $lim_{\mu\in M} R_g(f_\mu)(x) = f \cdot g$. Hnce the mappings L_g and R_g are continuous and $C_p(X,X)$ is a semi-topological semigroup. □

The family of all continuous endomorphisms $End_c(G)$ of a topological universal algebra G relatively to the operation of composition $f \cdot g$ is a monoid.

Corollary 1.2. *The monoid $End_c(G)$ in the topology of pointwise convergence is a semi-topological semigroup and a subsemigroup of the semi-topological semigroup $C_p(G,G)$.*

2. Free topological algrbras

Let Ω be a fixed signature. For any non-empty subset A of the universal algebra G denote by $s_G(A)$ the smalest subalgebra of G which contains the set A.

Let Ω be a fixed signature. A topological universal algebra G is a topological free universal algebra in some class of universal algebras if there is given a subspace $I = I_G \subset G$ with the properties:

(1) the algebra G is generate by the set I, i.e. G= $s_G(I)$, and I is called the space of generators of G;
(2) for any continuous mapping $f : I \longrightarrow G$ there exists a (unique) continuous endomorphism $\hat{f} : G \longrightarrow G$ such that $f(x) = \hat{f}(x)$ for each $x \in I$.

Let Ω be a fixed signature. A topological free universal algebra G is almost discrete if the space of generators I_G is a discrete subspace of the space G.

A universal algebra A is called cyclic if there exists a point $a \in G$ such that the set $\{a\}$ generate the algebra G.

Any topological free cyclic universal algebra is almost discrete.

Remark. Let G be a universal topological algebra of the signature Ω. If the set Ω is finite or countable and G is a cyclic algebra, then the set G is finite or countable too. If G is a cyclic algebra and the set Ω is infinite, then the $|G| \leq |\Omega|$, where $|G|$ sign the cardinality of the set G.

3. The monoid of continuous endomorphisms

G. Gratzer and E. T. Schmidt [10] proved that any complete lattice is isomorphic to the lattice of congruence of some universal algebra. The semigroup $End(G)$ of all endomorphisms and the semigroup $End_c(G)$ of all continuous endomorphisms of a topological universal algebra G are are semigroups with unity. In [13] A. I. Mal'cev describe the structure of a symmetrical groupoid (semigroup of all transformations of a set).

The following theorem is a generalization and conceptualization of the theorem from ([18], pag.98).

Theorem 3.1. *For any semi-topological monoid S there exist a discrete signature Ω and a topological universal algebra G_S of signature Ω such that:*

(1) The semi-topological monoids S and $End(G_S)$ are topologically isomorphic.
(2) G_S is a free cyclic topological universal algebra of discrete signature Ω.
(3) $\Omega = \Omega_1$ and there exists a bijection $u : S \longrightarrow \Omega$ such tat $u(x) = u_x$ and $u_x \cdot u_y = u_{y \cdot x}$ for any $x, y \in S$. In particular, relatively to operation of composition the signature Ω is a semi-topological semigroup topologically anti-isomorphic with the semi-topological semigroup S.

Proof. Let e be the unity of S. We put $G_S = S$ and $\Omega = \Omega_1 = \{u_a : a \in S\}$. For any $a \in S$, $u_a \in \Omega_1$ and any $x \in G_S = S$ we put $u_a(x) = a \cdot x$. If $a, b \in S$, then $(u_a \cdot u_b)(x) = u_b(u_a(x)) = (b \cdot a) \cdot x = u_{b \cdot a}(x)$. Hence, if $u(a) = u_a$ for any $a \in S$, then $u : S \longrightarrow \Omega$ is a bijection and $u(a \cdot b) = u_b \cdot u_a$ for all $a, b \in S$. Therefore $u : S \longrightarrow \Omega$ is an anti-isomorphism.

For any $u_a \in \Omega$ we have $u_a(e) = a$ and $u_e(a) = a$. Hence the universal algebra G_S is generated by the element e and G_S is a cyclic algebra.

Since $e_{1G_S}(u_a, x) = a \cdot x$, where $u_a = a \in \Omega_1 = S$ and $x \in G_S = S$, and the space Ω_1 is a discrete space, the mapping $e_{1G_S} : \Omega_1 \times G_S \to G_S$ is continuous and G_S is a topological universal algebra.

For any $a \in S$ consider the function $\varphi_a : G_S \longrightarrow G_S$, where $\varphi_a(x) = x \cdot a$ for any $x \in G_S = S$. We have $\varphi_a(u_b(x)) = (b \cdot x) \cdot a = b \cdot (x \cdot a) = u_b(\varphi_a(x))$ for any $x \in G_S$. Therefore $\varphi_a \in End(G_S)$. Since G_S is a cyclic universal algebra, φ_a is the unique endomorphism of G_S for which $\varphi_a(e) = a$. Moreover if $\varphi : G_S \longrightarrow G_S$ is a continuous endomorphism, then $\varphi = \varphi_{\varphi(e)}$. Hence G_S is a free cyclic universal algebra of signature Ω and $End_c(G_S) = \{\varphi_a : a \in S\}$.

Consider the correspondence $\psi : S \longrightarrow End(G_S)$, where $\psi(a) = \varphi_a$ for each $a \in S$. We have $(\varphi_a \cdot \varphi_b)(x) = \varphi_b(\varphi_a(x)) = (x \cdot a) \cdot b = x \cdot (a \cdot b) = \varphi_{a \cdot b}(x)$. Therefore $\psi(a \cdot b) = \varphi_{a \cdot b} = \varphi_a \cdot \varphi_b = \psi(a) \cdot \psi(b)$ and ψ is an isomorphism of the semigroups S and $End(G_S)$. The proof is complete. □

In the case of continuous signature the above theorem is not true and we have the following theorem.

Theorem 3.2. *For any topological semigroup with unity S there exist a continuous signature Ω and a topological universal algebra G_S of signature Ω such that:*

(1) The topological semigroups S and $End(G_S)$ are topologically isomorphic.

(2) G_S is a free cyclic topological universal algebra of continuous signature Ω.

(3) $\Omega = \Omega_1$ and there exists a bijection $u : S \longrightarrow \Omega$ such that $u(x) = u_x$ and $u_x \cdot u_y = u_{y \cdot x}$ for any $x, y \in S$. In particular, relatively to operation of composition the signature Ω is a topological semigroup topologically anti-isomorphic with the topological semigroup S.

Proof. As in proof of Theorem 3.1, we consider that The spaces S, G_S and $\Omega_1 = \Omega$ are homeomorphic. Since $e_{1G_S}(u_a, x) = a \cdot x$, where $u_a = a \in \Omega_1 = S$ and $x \in G_S = S$, and the space Ω_1 is a topological space, the mapping $e_{1G_S} : \Omega_1 \times G_S \rightarrow G_S$ is continuous and G_S is a topological universal algebra. In this case $S \times S = \Omega_1 \times G_S$ and $End_c(G_S)$ is a topological group. □

Let $Aut(G)$ be the group of automorphisms of the universal topological algebra G of the signature Ω. Consider $Aut(G)$ as semitopological group of the semitopological monoid $End(G)$.

Example 3.3. Let G be an infinite additive group such that $(G \setminus F) + (G \setminus F) + G$ for every finite subset F of G. By instance, we may put $G = \mathbb{Z}$ is the group of integers. Assume that $\omega, \varepsilon \notin G$. We put $G_0 = G \cup \{\omega\}$, $S = G_0 \cup \{\varepsilon\}$, $\omega + x = x + \omega = \omega$ for any $x \in G_0$, $\varepsilon + x = x + \varepsilon = x$ for any $x \in S$ aand G is a subsemigroup of S. Then S is a monoid with the unity ε. If G is a commutative group, then S is a commutative monoid. On S consider the topology $\mathcal{T} = \{\emptyset, \{\varepsilon\} \cup \{\{x\} : x \in G\} \cup \{S \setminus F : F$ is a finite subset of $S\}$.

Then S with the topology $\mathcal{T}$ is a compact monoid. The mapping $+ : S \times S \longrightarrow S$ is not continuous in the point (ω, ω). Hence S is not a topological monoid. As in the proof of Theorems 3.1 and 3.2, we put $G_S = S$, $\Omega = \Omega_1 = \{u_a : a \in S\}$ and $e_{1G_S}(u_a, x) = a + x$.

If on Ω we consider the discrete topology, then G_S is a topological universal algebra and the semitopological semigroups S and $End(G_S)$ are topologically isomorphic. If on Ω we consider the topology $\mathcal{T}$ as on S, then G_S is not a topological universal algebra.

We observe that $Aut(G_S) = \{u_\varepsilon\}$ is the singleton group and G is a subgroup of $End(G_S)$.

We say that two topological universal algebras are H-equivalent if theirs semi-topological semigroups of continuous endomorphisms are topological isomorphic. In this case, any topological universal algebra has its imprints in the semigroup of endomorphisms.

References

[1] G. Birkhoff, *On the structure of abstract algebras*, Proc. Camb. Philos. Soc. 31 (1935), 433-454.

[2] G. Birkhoff, *Universal algebra*, Comptes Rendus du Premier Congres Canadien de Mathematiques, University of Toronto Press, Toronto, 1946, 310-326.

[3] Choban M. M. *Some topics in topological algebra.* Topology and Appl., 1993, **54**, 183–202.

[4] M. M. Ciobanu, *Algebre Universale Topologice*, Oradea: Editura Univ. din Oradea, 1999.

[5] A. N. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups, 1,* American Mathematical Society, 1961.

[6] A. N. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups, 2,* American Mathematical Society, 1967.

[7] P. M. Cohn, *Universal Algebra,* Harper and Row, New York, 1965.

[8] S. Eilenberg, *Automata, Languages and Machines,* Pure and Applied Mathematics, vol. 59-B, Academic Press, New York, 1976.

[9] G. Gratzer, *Universal Algebra,* D. Van Nostrand Company, Inc., 1968.

[10] G. Gratzer, E. T. Schmidt, *Characterizations of congruence lattices of abstract algebras,* Acta Sci. Math. (1963), no. 24, 34-59.

[11] A.I. Mal'cev, *On the general theory of algebraic systems.* Matem. Sb. N.S. 35(77), (1954). 3-20. English translation: Trans. Amer. Math. Soc., 27 (1963), 125-142.

[12] A. I. Mal'cev, *Free topological algebras*. Izvestia Akad. Nauk SSSR 21 (1957), vyp. 2, 171-198. English translation: Amer. Math. Soc., Transl., II. Ser. 17 (1961), 173-200.

[13] A. I. Mal'cev, *Symmetrical groupoids,* Matem. Sb. 31 (1952), 136-151.

[14] I. I. Valuţă (Valutsa), *Left ideals of the semigroup of endomorphisms of a free universal algebra,* Dokl. Akad. Nauk SSSR 150 (1963), 235-237.

[15] I. I. Valuţă, *Left ideals of the semigroup of endomorphisms of a free universal algebra,* Matem. Sb. 62 (1963), no. 3, 371-384.

[16] I. I. Valuţă, *Ideals of the algebra of endomorphisms of a free universal algebra,* Matematicheskie Issledovania, Kishinev, 3 (1968), no. 2, 104-112.

[17] I. I. Valuţă, *About ideals semigroups of transformations,* Issledovania po Algebre, Kishinev: Izd. AN MSSR, 1965, 67-80.

[18] I. I. Valuţă, *Mappings. Algebraical aspects of the theory,* Kishinev: Shtiitsa, 1976.

[19] A. N. Whitehead, *A Treatise on Universal Algebra,* Cambridge, 1898.

(Choban Mitrofan) Department of Mathematics, Tiraspol State University, Chişinău, Republic of Moldova
E-mail address: `mmchoban@gmail.com`

(Ion Valuţă) Department of Mathematics, Technical University of Moldova, Chişinău, Republic of Moldova