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The behaviour of the inverse operations in the class of preradicals in special cases

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Abstract. In [4], [5], [6] four new operations are introduced and studied in the class of preradicals $\mathbb{PR}$ of the category R-Mod of left R-modules, and is shown the behaviour of these operations in the case of some special types of preradicals as prime, coprime, $\wedge$-prime, $\vee$-coprime, irreducible and coirreducible. In this work we will present the behaviour of inverse operations in the case of semiprime and semicoprime preradicals.
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Comportamentul operaţiilor inverse din clasa preradicalilor în cazuri speciale

Rezumat. În [4], [5], [6] sunt introduse şi studiate patru operaţii noi în clasa preradicalilor $\mathbb{PR}$ a categoriei R-modulelor stângi R-Mod, şi este arătat comportamentul acestor operaţii în cazul unor preradicali de tipuri speciale, aşa ca primi, coprimi, $\wedge$-primi, $\vee$-coprimi, ireductibili şi coireductibili. În această lucrare vom prezenta comportamentul operaţiilor inverse în cazul preradicalilor semiprimi şi semicoprimi.
Cuvinte cheie: Inel, modul, categorie, latice, (pre)radical.

1. Introduction and preliminary facts

This work is devoted to the theory of radicals of modules ([1], [2], [9], [10]) and contains some investigations of new four operations defined and studied in [4 - 6] in the class of preradicals of a module category.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. We remind that a *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f : M \to M'$.

We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod. In this class four operation are defined [1], [2], [9]:

1) the *meet* $\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}}$:
$$\left(\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha\right)(M) \overset{def}{=} \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$
2) the *join* $\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}}$:
$$\left(\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha\right)(M) \overset{def}{=} \sum_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$
3) the *product* $r \cdot s$ of preradicals $r, s \in \mathbb{PR}$:
$$(r \cdot s)(M) \overset{def}{=} r(s(M)),\ M \in R\text{-Mod};$$
4) the *coproduct* $r \# s$ of preradicals $r, s \in \mathbb{PR}$:
$$[(r \# s)(M)]/s(M) \overset{def}{=} r(M/s(M)),\ M \in R\text{-Mod}.$$

In the class $\mathbb{PR}$ the partial order relation " $\leq$ " is defined by the rule:
$$r_1 \leq r_2 \overset{def}{\Leftrightarrow} r_1(M) \subseteq r_2(M) \text{ for every } M \in R\text{-Mod}.$$
The class $\mathbb{PR}$ is a large complete lattice with respect to the operations of meet and join.

We remark that in the book [1], [2], [9] the coproduct is denoted by $(r : s)$ and is defined by the rule $[(r : s)(M)]/r(M) = s(M/r(M))$, so in our notations $(r \# s) = (s : r)$.

The following properties of distributivity hold ([1], [2], [9]):

(1) $(\wedge r_\alpha) \cdot s = \wedge (r_\alpha \cdot s)$; (2) $(\vee r_\alpha) \cdot s = \vee (r_\alpha \cdot s)$;

(3) $(\wedge r_\alpha) \# s = \wedge (r_\alpha \# s)$; (4) $(\vee r_\alpha) \# s = \vee (r_\alpha \# s)$

for every family $\{r_\alpha\}_{\alpha\in\mathfrak{A}} \subseteq \mathbb{PR}$ and $s \in \mathbb{PR}$.

Using these relations in [4], [5], [6] four new operations are introduced and studied in the class of preradicals $\mathbb{PR}$ in modules, namely, the inverse operations of the product and of the coproduct with respect to meet and to join. They are defined as follows:

(1) the *left quotient with respect to join* $r \,{}^{\vee}\!/_{\cdot}\, s = \vee\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\}$, which exists for any preradicals $r, s \in \mathbb{PR}$;

(2) the *left coquotient with respect to meet* $r \,{}^{\wedge}\!/_{\#}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$, which exists for any preradicals $r, s \in \mathbb{PR}$;

(3) the *left quotient with respect to meet* $r \,{}^{\wedge}\!/_{\cdot}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r\}$, which exists for any preradicals $r, s \in \mathbb{PR}$ such that $r \leq s$;

(4) the *left coquotient with respect to join* $r \,{}^{\vee}\!/_{\#}\, s = \vee\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r\}$, which exists for any preradicals $r, s \in \mathbb{PR}$ such that $r \geq s$.

The similar questions are discussed in [3; 7; 8].

For each of defined operation we indicate a particular case, which coincides with a well known operator in $\mathbb{PR}$. Moreover, some properties of these operators are shown [4 - 6; 10 - 14].

For any preradical $r \in \mathbb{PR}$, these particular cases are:

(1) $0 \, {}^{\vee}\!/_{\cdot} \, r = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r = 0 \} = a(r)$ is the *annihilator* of r;
(2) $1 \, {}^{\wedge}\!/_{\#} \, r = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# r = 1 \} = t(r)$ is the *totalizer* of r;
(3) $r \, {}^{\wedge}\!/_{\cdot} \, r = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r = r \} = e(r)$ is the *equalizer* of r;
(4) $r \, {}^{\vee}\!/_{\#} \, r = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# r = r \} = c(r)$ is the *co-equalizer* of r.

These operators possess the following properties for any $r \in \mathbb{PR}$ ([10]):

(1) $a(r)$ is a radical;
(2) $t(r)$ is a Jansian pretorsion;
(3) $e(r)$ is an idempotent preradical;
(4) $c(r)$ is a radical.

Now we remind the some types of preradicals ([11 - 14]). A preradical $r \in \mathbb{PR}$ is called:

- *prime*, if $r \neq 1$ and for each $t_1, t_2 \in \mathbb{PR}$, $t_1 \cdot t_2 \leq r$ implies $t_1 \leq r$ or $t_2 \leq r$ [11];
- *coprime*, if $r \neq 0$ and for each $t_1, t_2 \in \mathbb{PR}$, $t_1 \# t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [12];
- *semiprime*, if $r \neq 1$ and for each $t \in \mathbb{PR}$, $t \cdot t \leq r$ implies $t \leq r$ [13];
- *semicoprime*, if $r \neq 0$ and for each $t \in \mathbb{PR}$, $t \# t \geq r$ implies $t \geq r$ [14];
- $\wedge$-*prime*, if for each $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 \leq r$ implies $t_1 \leq r$ or $t_2 \leq r$ [11];
- $\vee$-*coprime*, if for each $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [12];
- *irreducible*, if for each $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 = r$ implies $t_1 = r$ or $t_2 = r$ [11];
- *coirreducible*, if for each $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 = r$ implies $t_1 = r$ or $t_2 = r$ [12].

The operations of meet and join are commutative and associative, while the operations of product and coproduct are associative. For every $r, s \in \mathbb{PR}$ by means of these operations four preradicals are obtained which are arranged in the following order: $r \cdot s \leq r \wedge s \leq r \vee s \leq r \# s$.

During this work we will use the following facts and notions from general theory of preradicals (see [1], [2], [4]-[5], [9]).

Lemma 1.1. *(Monotony of the product) For any* $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ *implies that* $r \cdot s_1 \leq r \cdot s_2$ *and* $s_1 \cdot r \leq s_2 \cdot r$ *for every* $r \in \mathbb{PR}$.

Lemma 1.2. *(Monotony of the coproduct) For any* $s_1, s_2 \in \mathbb{PR}$, $s_1 \le s_2$ *implies that* $r \# s_1 \le r \# s_2$ *and* $s_1 \# r \le s_2 \# r$ *for every* $r \in \mathbb{PR}$.

Lemma 1.3. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

(1) $(r \cdot s) \# t \ge (r \# t) \cdot (s \# t)$;

(2) $(r \# s) \cdot t \le (r \cdot t) \# (s \cdot t)$.

Proposition 1.4. *Let* $r, s, t \in \mathbb{PR}$. *Then*

(1) $r \ge t \cdot s \Leftrightarrow r \,{}^{\vee}\!/_{\cdot}\, s \ge t$;

(2) $r \le t \# s \Leftrightarrow r \,{}^{\wedge}\!/_{\#}\, s \le t$;

(3) $r \le t \cdot s \Leftrightarrow r \,{}^{\wedge}\!/_{\cdot}\, s \le t$, *where* $r \le s$;

(4) $r \ge t \# s \Leftrightarrow r \,{}^{\vee}\!/_{\#}\, s \ge t$, *where* $r \ge s$.

The statements of Proposition 1.4 can be considered as another way of defining the inverse operations.

2. The behaviour of the inverse operations for some special types of preradicals

In [4], [5], [6] are shown the behaviour of the inverse operations in $\mathbb{PR}$ in the case of such types of preradicals as prime, coprime, $\wedge$-prime, $\vee$-coprime, irreducible and coirreducible. In continuation we will indicate these properties.

Proposition 2.1. *Let* $r, s \in \mathbb{PR}$. *The following statemens are true:*

(1) The preradical r *is prime if and only if for any preradical* s *we have* $r \,{}^{\vee}\!/_{\cdot}\, s = 1$ *or* $r \,{}^{\vee}\!/_{\cdot}\, s = r$.

(2) If r *is* $\wedge$*-prime, then* $r \,{}^{\vee}\!/_{\cdot}\, s$ *is a* $\wedge$*-prime preradical.*

(3) If $r = t \cdot s$ *for some preradical* $t \in \mathbb{PR}$ *and* r *is irreducible, then the preradical* $r \,{}^{\vee}\!/_{\cdot}\, s$ *is irreducible.*

Proposition 2.2. *For every* $r, s \in \mathbb{PR}$ *we have:*

(1) The preradical r *is coprime if and only if for any preradical* s *we have* $r \,{}^{\wedge}\!/_{\#}\, s = 0$ *or* $r \,{}^{\wedge}\!/_{\#}\, s = r$.

(2) If r *is* $\vee$*-coprime, then* $r \,{}^{\wedge}\!/_{\#}\, s$ *is a* $\vee$*-coprime preradical.*

(3) If $r = t \# s$ *for some preradical* $t \in \mathbb{PR}$ *and* r *is coirreducible, then the preradical* $r \,{}^{\wedge}\!/_{\#}\, s$ *is coirreducible.*

Proposition 2.3. *Let* $r \in \mathbb{PR}$. *The following statements hold:*

(1) If r *is coprime, then* $r \,{}^{\wedge}\!/_{\cdot}\, s$ *is a coprime preradical for any preradical* $s \ge r$.

(2) If r *is* $\vee$*-coprime, then* $r \,{}^{\wedge}\!/_{\cdot}\, s$ *is a* $\vee$*-coprime preradical for any preradical* $s \ge r$.

(3) If $r = t \cdot s$ for some preradical $t \in \mathbb{PR}$ and r is coirreducible, then the preradical $r \,{}^{\vee}\!/_{\cdot}\, s$ is coirreducible for any preradical $s \in \mathbb{PR}$.

Moreover, from Propositon 2.3 ([12]):

(1) if the preradical r is coprime, then its equalizer $e(r)$ is coprime;

(2) if the preradical r is $\vee$-coprime, then its equalizer $e(r)$ is $\vee$-coprime;

(3) if the preradical r is coirreducible, then its equalizer $e(r)$ is coirreducible.

Proposition 2.4. *Let $r \in \mathbb{PR}$. The following facts are true:*

(1) If r is prime, then $r \,{}^{\vee}\!/_{\#}\, s$ is a prime preradical for any preradical $s \le r$.

(2) If r is $\wedge$-prime, then $r \,{}^{\vee}\!/_{\#}\, s$ is a $\wedge$-prime preradical for any preradical $s \le r$.

(3) If $r = t \# s$ for some preradical $t \in \mathbb{PR}$ and r is irreducible, then the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is irreducible for any preradical $s \in \mathbb{PR}$.

Moreover, from Propositon 2.4 ([11]):

(1) if the preradical r is prime, then its co-equalizer $c(r)$ is prime;

(2) if the preradical r is $\wedge$-prime, then its co-equalizer $c(r)$ is $\wedge$-prime;

(3) if the preradical r is irreducible, then its co-equalizer $c(r)$ is irreducible.

Now we will show the behaviour of the inverse operations in the case of semiprime and semicoprime preradicals.

Proposition 2.5. *If the preradical r is semiprime, then the left quotient $r \,{}^{\vee}\!/_{\cdot}\, s$ is a semiprime preradical for every $s \in \mathbb{PR}$.*

Proof. Suppose that $r \neq 1$ and $t \cdot t \le r \,{}^{\vee}\!/_{\cdot}\, s$ for each $t \in \mathbb{PR}$. From the Proposition 1.4(1) we have $r \ge (t \cdot t) \cdot s$. Using the associativity of the product of preradicals we obtain $r \ge t \cdot (t \cdot s)$. Since $t \ge (t \cdot s)$, from the monotony of product of preradicals it follows that $t \cdot (t \cdot s) \ge (t \cdot s) \cdot (t \cdot s)$, i.e. $r \ge (t \cdot s) \cdot (t \cdot s)$. If r is semiprime, then $r \ge (t \cdot s)$. From the Proposition 1.4(1) we obtain that $r \,{}^{\vee}\!/_{\cdot}\, s \ge t$.

So for each preradical $t \in \mathbb{PR}$ with $t \cdot t \le r \,{}^{\vee}\!/_{\cdot}\, s$ we have $t \le r \,{}^{\vee}\!/_{\cdot}\, s$, which means that the preradical $r \,{}^{\vee}\!/_{\cdot}\, s$ is semiprime. ■

Proposition 2.6. *If the preradical r is semicoprime, then the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ is a semicoprime preradical for every $s \in \mathbb{PR}$.*

Proof. Assume that $r \neq 0$ and $t \# t \ge r \,{}^{\vee}\!/_{\#}\, s$ for each $t \in \mathbb{PR}$. Then from Proposition 1.4(2) we obtain $r \le (t \# t) \# s$. Applying the associativity of coproduct of preradicals we have $r \le t \# (t \# s)$. Because $t \le t \# s$, using the monotony of coproduct of preradicals we obtain $t \# (t \# s) \le (t \# s) \# (t \# s)$, therefore $r \le (t \# s) \# (t \# s)$. If r is semicoprime, then $r \le (t \# s)$. From Proposition 1.4(2) we obtain that $r \,{}^{\vee}\!/_{\#}\, s \le t$.

So for each preradical $t \in \mathbb{PR}$ with $t \# t \geq r \,{}^{\wedge}\!/_{\#}\, s$ we have $t \geq r \,{}^{\wedge}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is semicoprime. ■

Proposition 2.7. *If r is a semicoprime preradical, then the preradical $r \,{}^{\wedge}\!/_{\cdot}\, s$ is semicoprime for any preradical $s \geq r$.*

Proof. The condition $r \leq s$ ensures the existence of the left quotient $r \,{}^{\wedge}\!/_{\cdot}\, s$.

Let the preradical $r \neq 0$ be semicoprime and $t \# t \geq r \,{}^{\wedge}\!/_{\cdot}\, s$ for each preradical $t \in \mathbb{PR}$. Using Proposition 1.4(3) we obtain $r \leq (t \# t) \cdot s$. From Lemma 1.3(2) $(t \# t) \cdot s \leq (t \cdot s) \# (t \cdot s)$, therefore $r \leq (t \cdot s) \# (t \cdot s)$. Since r is semicoprime, it follows that $r \leq t \cdot s$. Applying Proposition 1.4(3) we obtain $r \,{}^{\wedge}\!/_{\cdot}\, s \leq t$.

So for each $t \in \mathbb{PR}$ with $t \# t \geq r \,{}^{\wedge}\!/_{\cdot}\, s$ we have $t \geq r \,{}^{\wedge}\!/_{\cdot}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\cdot}\, s$ is semicoprime. ■

Moreover, from Propositon 2.7 if the preradical r is semicoprime, then its equalizer $e(r)$ is a semicoprime preradical ([14]).

Proposition 2.8. *If r is a semiprime preradical, then the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is semiprime for any preradical $s \leq r$.*

Proof. The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

Let the preradical $r \neq 1$ be semiprime and $t \cdot t \leq r \,{}^{\vee}\!/_{\#}\, s$ for each preradical $t \in \mathbb{PR}$. From the Proposition 1.4(4) we have $r \geq (t \cdot t) \# s$. By Lemma 1.3(1) we have $(t \cdot t) \# s \geq (t \# s) \cdot (t \# s)$, so $r \geq (t \# s) \cdot (t \# s)$. Since r is semiprime, it follows that $r \geq t \# s$. Using Proposition 1.4(4) we obtain $r \,{}^{\vee}\!/_{\#}\, s \geq t$.

So for each $t \in \mathbb{PR}$ with $t \cdot t \leq r \,{}^{\vee}\!/_{\#}\, s$ we have $t \leq r \,{}^{\vee}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is semiprime. ■

Moreover, from Propositon 2.8 if the preradical r is semiprime, then its co-equalizer $c(r)$ is a semiprime preradical ([13]).

The Propositions 2.5 – 2.8 complete the previous studies in this domain and show new properties of indicated operations.

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