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On the essential norm of singular operators with applications

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Abstract. The paper presents certain results devoted to the essential norms and norms of singular integral operators in spaces with weights. It is found that these rules in the case of Lyapunov-type contour on portions essentially also on the contour. The values of the essential norms are used to determine noetherian conditions for the characteristic single operators with measurable and marginal coefficients.

Keywords: singular integral operator, noetherian operators, piecewise Lyapunov contour.

Asupra normelor esențiale ale operatorilor singulari cu aplicații

Rezumat. În lucrare sunt prezentate anumite rezultate consacrate normelor și normnelor esențiale ale operatorilor integrali singulari în spații cu ponderi. Se constată că aceste norme în cazul conturului de tip Lyapunov pe porțiuni în mod essential depend și de contur. Valorile normelor esențiale sun utilizate pentru determinarea unor condiții noetheriene pentru operatorii singulari caracteristici cu coeficienți măsurabili și mărginiți.

Cuvinte cheie: operator integral singular, operatori noetherieni, contur de tip Lyapunov pe porțiuni.

1. Introduction

Let Γ be a contour made up of m closed lines $\gamma_1, \gamma_2, \ldots, \gamma_m$ of the Lyapunov type on portions that have a single point t_0 in common. We assume that the line γ_j has nothing in common with the domain F_k $(k \neq j)$ bounded by the line γ_k. We note by $L_p(\Gamma, \rho)$ the space L_p on the contour Γ with the weight

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\beta_k} \quad (t_k \in \Gamma,\ 1 < p < \infty,\ -1 < \beta_k < p - 1),$$

where by $L(L_p\,(\Gamma,\rho))$ we denote the algebra of all linear and bounded operators acting in the space $L_p\,(\Gamma,\rho)$, and by $\ddot{T}(=\ddot{T}(L_p\,(\Gamma,\rho))$ its maximum ideal formed by the set of all compact operators. The number

$$\inf_{T\in T}\|A+T\|_{L_p(\Gamma,\rho)} \qquad \left(A\in L\left(L_p(\Gamma,\rho)\right)\right) \tag{1}$$

is called the essential norm of the operator A and is denoted by $|A|_{L_p(\Gamma,\rho)}$.

In the case when the contour of integration is a closed Lyapunov contour, the essential norm of the operators

$$(S_\Gamma\phi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\phi(\tau)}{\tau-t}d\tau,$$

$$(P_\Gamma\phi)\,(t)=\frac{1}{2}\phi\,(t)+\frac{1}{2}(S_\Gamma\phi)(t)$$

and

$$(P_\Gamma\phi)\,(t)=\frac{1}{2}\phi\,(t)-\frac{1}{2}(S_\Gamma\phi)(t)$$

depends only on the numbers p and $\beta_k(k=1,2,\dots n)$ and does not depend on the contour Γ.

In the work [1] it was demonstrated that this property occurs no longer if the contour Γ possesses angular points.

In this paper are presented some lower estimates for the essential norms and, therefore, also for the norms of the operators S_Γ , P_Γ and Q_Γ in the case of the composite contour. It is shown that in some cases the obtained estimates are accurate. For this, it is studied the subalgebra of algebra $L(L_p\,(\Gamma,\rho))$ generated by the operators S_Γ and S_Γ^* in the when case Γ is the union of the coordinate axes and at the same time the results from the work [2] are essentially used.

The presented results show that the essential norms of the operators depend not only on the space $L_p\,(\Gamma,\rho)$ but also on the contour. With the help of the obtained estimates, noetherian conditions are established for singular singular equations with measurable and bounded coefficients. In particular, some results of I.Simonenko [3] are generalized for the weighted space L_2 and for the case of the composite contour

2. Estimates for the norms and essential norms of the operators S, P, Q in the case of a Lyapunov contour

Let Γ be a piecewise Lyapunov contour with a finite number of self-intersection points. In 1927 M.Riesz proved the boundedness of the operator

$$(S_\Gamma \phi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t} d\tau \quad (t \in \Gamma) \tag{2}$$

in the space $L_p(\Gamma_0)$ $(\Gamma_0 = \{z \in \mathbb{C} : |z| = 1\})$. Then G.Hardy and J.Littlewood and K.Babanko transferred this result to the spaces $L_p(R, \rho)$ with weight $\rho(x) = |x|^\alpha$ $(1 < p < \infty,\ -1 <\propto< p-1)$. In the work [4] B.Khvedelidze proved the boundedness of operator S_Γ in the space$L_p(\Gamma, \rho)$ for an arbitrary Lyapunov contour Γ and the weight

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \qquad (t_k \in \Gamma,\ 1 < p < \infty,\ -1 < \beta_k < p-1). \tag{3}$$

E.Gordadze transferred this result to an arbitrary piecewise Lyapunov contour. Using this result one can prove the boundedness of the operator S_Γ in the case of a composite contour with a finite number of self-intersection points in the space L_p with the weight (3). The condition $-1 < \beta_k < p-1$ is necessary for the boundedness of the operator S_Γ in the space $L_p(\Gamma, \rho)$. It is confirmed by the following lemma

Lemma 2.1. *Let S_Γ be bounded in $L_p(\Gamma, \rho)$, then $\rho^{1/p} \in L_p(\Gamma)$ and*

$$\rho^{-1/p} \in L_q(\Gamma)\left(p^{-1} + q^{-1} = 1\right).$$

Proof. The boundedness of the operator S_Γ in the space $L_p(\Gamma, \rho)$ implies the boundedness of the operator $R = \pi i \rho^{1/p}(RS - SR)\rho^{-1/p} I$ *in* $L_p(\Gamma)$, where $(R\phi)(t) = \frac{1}{t - z_0}\phi(t)$. and $z_0 \notin \overline{\Gamma}$. But

$$(R\phi)(t) = \rho^{1/p}(t)\frac{1}{t - z_0}\int_\Gamma \frac{\rho^{-1/p}(\tau)\phi(\tau)}{\tau - z_0} d\tau.$$

Therefore $\rho^{1/p} \in L_p(\Gamma)$ and $\rho^{-1/p} \in L_q(\Gamma)$. ■

Corollary 2.1. *If the operator S is bounded in the space $L_p(\Gamma, \rho)$, $\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$, then from the above relations $\rho^{1/p} \in L_p(\Gamma)$, and $\rho^{-1/p} \in L_q(\Gamma)$, it follows that the numbers β_k verify the inequalities*

$$-1 < \beta_k < p - 1, \quad k = 1, 2, \ldots n.$$

Remark. *If the contour of integration Γ is unbounded, then the operator S is continuous at $L_p(\Gamma, \rho)$ if and only if*

$$-1 < \beta_k < p - 1 \quad i - 1 < \beta + \sum_{k=1}^{n} \beta_k < p - 1.$$

Let Γ be a set of simple nonintersecting closed contours of Lyapunov type and S_Γ be the singular integral operator along Γ.

In this section we present lower bounds for the norms of the operators S_Γ, P_Γ and Q_Γ in $L_p(\Gamma)$. In addition, for some values of p $\left(p = 2^n \ and \ p = 2^n (2^n - 1)^{-1}\right)$, the exact value of the Hilbert transform norm will be calculated (see [3]).

Theorem 2.1. *For each $p > 2$, the following estimates are valid:*

$$\inf_{T \in L(\dot{T}_p)} \|P_\Gamma + T\|_p \geq \frac{1}{sin\pi/p}, \quad \inf_{T \in L(\dot{T}_p)} \|Q_\Gamma + T\|_p \geq \frac{1}{sin\pi/p}, \tag{4}$$

$$\inf_{T \in L(\dot{T}_p)} \|S_\Gamma + T\|_p \geq ctg\pi/2p, \tag{5}$$

where $L(\dot{T}_p)$ is the set of all linear operators compact in $L_p(\Gamma)$.

Proof. Suppose that for some

$$\inf_{T \in L(\dot{T}_p)} \|P_\Gamma + T\|_p < \frac{1}{sin\pi/p}.$$

Consider an operator $aP_\Gamma + Q_\Gamma$, where $a(t)$ is the function taking two values:

$$a(t) = \left(\cos\left(\frac{\pi}{p}\right)\exp\left(\pm i\frac{\pi}{p}\right)\right).$$

Since $|a(t) - 1| = \sin\left(\frac{\pi}{p}\right)$, then $\inf_{T \in L(\dot{T}_p)} \|(a-1)P_\Gamma + T\|_p < 1$, and therefore , the operator $I + (a-1)P_\Gamma = aP_\Gamma + Q_\Gamma$ is Noetherian, which is impossible, because the function $a(t)$ is (see [2]) $p-$singular.

To prove the second relation (4), we consider the function

$$a(t) = \left(sec\left(\frac{\pi}{p}\right)\exp\left(\pm i\frac{\pi}{p}\right)\right).$$

Then $|(a(t) - 1)/a(t)| = \sin\left(\frac{\pi}{p}\right)$. The operator $a(I + (1-a)/a)Q_\Gamma = aP_\Gamma + Q_\Gamma$ Noetherian in $L_p(\Gamma)$ because the function $a(t)$ is $p-$ singular. This implies the second of relations (4).

Relation (5) is proved similarly, if we use the function $a(t) = \exp\left(\pm i\frac{\pi}{p}\right)$ and the equality

$$aP_\Gamma + Q_\Gamma = \frac{a+1}{2}\left(I + \frac{a-1}{a+1}S\right).$$

Theorem 2.2. *Let $\Gamma_o = \{z : |z| = 1\}$, then (see [2]) for all $n = 1, 2, \ldots$*

$$\|S_o\|_p = \begin{cases} \operatorname{ctg}\frac{\pi}{2p}, & if \ p = 2^n, \\ \operatorname{tg}\frac{\pi}{2p}, & if \ p = \frac{2^n}{2^n - 1}. \end{cases} \quad \left(S_o = S_{\Gamma_0}\right). \tag{6}$$

Proof. Let $\varphi(t) = t^m \ \ (|t| = 1, \ \ m = 0, \ \pm 1, \ \ \pm 2, \ \ldots)$. Then

$$(S_o\varphi)(t) = \begin{cases} \varphi(t) \ \ for \ m \geq 0, \\ -\varphi(t) \ \ for \ m < 0 \,. \end{cases}$$

Since the system $\{\mathrm{t}^{\mathrm{m}}\}_{-\infty}^{+\infty}$ forms an orthogonal basis in the space $L_2(\Gamma_0)$, then the operator S_o defined on the linear span of this basis is bounded in $L_2(\Gamma_0)$ and $\|S_o\|_2 = 1$. Let

$$\varphi(t) = \sum_{k=-N}^{k=N} a_k t^k$$

be a trigonometric polynomial,

$$\varphi_+(t) = \sum_{k=0}^{k=N} a_k t^k \quad \text{and} \quad \varphi_-(t) = \sum_{k=-N}^{k=-1} a_k t^k.$$

Since $\varphi = \varphi_+ + \varphi_- \ S\varphi = \varphi_+ - \varphi_-$, then

$$\varphi^2 + (S_0\varphi)^2 = 2\left(\varphi_+^2 + \varphi_-^2\right) = 2S_0\left(\varphi_+^2 - \varphi_-^2\right) = 2S_0\left(\varphi S_0\varphi\right).$$

That is

$$(S_0\varphi)^2 = 2S_0\left(\varphi S_0\varphi\right) - \varphi^2.$$

This equality implies that

$$\left\|(S_0\varphi)^2\right\|_p \leq \|2S_0\left(\varphi S_0\varphi\right)\| + \left\|\varphi^2\right\|_p \ .$$

Since $\left\|\varphi^2\right\|_p \ = \|\varphi\|_{2p}^2$ and $\|\varphi h\|_p \ \leq \|\varphi\|_{2p} \ \|h\|_{2p}$, then

$$\|S_0\varphi\|_{2p}^2 \leq 2\|S_0\|_p\|\varphi\|_{2p}\|S_0\varphi\|_{2p} + \|\varphi\|_{2p}^2.$$

That is

$$\frac{\|S_0\varphi\|_{2p}}{\|\varphi\|_{2p}} \leq \|S_0\|_p + \sqrt{1 + \|S_0\|_p^2} \ ,$$

whence it follows that

$$\|S_0\|_{2p} \leq \|S_0\|_p + \sqrt{1 + \|S_0\|_p^2} \ .$$

From the last relation we obtain (by induction on n and using the equality $\|S_0\|_2 \ = 1$) that

$$\|S_o\|_{2^n} \leq ctg\frac{\pi}{2^{n+1}} \ . \tag{7}$$

Let $p = \frac{2^n}{2^n - 1}$, then $q = 2^n \ \left(\frac{1}{p} + \frac{1}{q} = 1\right)$. Since for any pair of trigonometric polynomials

$$\varphi(t) = \sum_{k=-N}^{k=N} a_k t^k, h(t) = \sum_{k=-N}^{k=N} \ b_k t^k$$

holds the equality

$$\int_{\Gamma_0} \varphi(t)\,\overline{(S_0 h)(t)}\;|dt| = \sum_{k=-N}^{k=N} \varepsilon_k a_k \overline{b_k} \;= \int_{\Gamma_0} (S_0\varphi)(t)\;\overline{(h)(t)}\;|dt|,$$

where $\varepsilon_k = 1$ for $k \geq 0$ and $\varepsilon_k = -1$ for $k < 0$, then the operator S_0^* , conjugate to the operator S_o , acting in the space $L_q(\Gamma_0)$, coincides with the operator S_o, acting in the space $L_p(\Gamma_0)$. Then from the first equality in (6) it follows that for $p = \frac{2^n}{2^n-1}$ we have $\|S_0\|_p \;= tg\frac{\pi}{2p}$. The theorem is proved. ∎

It is easy to show that for a fixed p , the norm $\|S_\Gamma\|_p$ depends on the contour Γ. In the next theorem we prove that the essential norm of the operator S_Γ,

$$|S_\Gamma|_p = \inf_{T\in\ddot{T}} \|S_\Gamma + T\|_{L_p(\Gamma)},$$

does not depend on the contour Γ.

Theorem 2.3. *The equality*

$$|S_\Gamma|_{L_p(\Gamma)} = \left|S_{\Gamma_0}\right|_{L_p(\Gamma_0)}, \tag{8}$$

holds, where $\Gamma_o = \{z : |z| = 1\}$ *is the unit circle and p is an arbitrary number from the interval* $(1 < p < \infty)$. *In particular, if* $p = 2^n$ $\left(p = 2^n\ (2^n - 1)^{-1}\right)$, *where* $n = 1, 2, \ldots,$ *then*

$$|S_\Gamma|_{L_p(\Gamma)} = ctg(\pi/2p(|S_\Gamma|_{L_p(\Gamma)} = tg(\pi/2p)). \tag{9}$$

Proof. First we consider the case when Γ consists of one closed curve. Let $t = \beta(z)$ be the function which conformally maps the circle $|z| < 1$ onto the set bounded by Γ. Since Γ is a contour of Lyapunov typ, the derivative $\beta'(z)$ satisfies a Hölder condition on Γ_0. We denote by B the bounded linear operator from $L_p\,(\Gamma)$ into $L_p\,(\Gamma_0)$, defined by

$$(B\varphi)\,(z) = |\beta'\,(z)|^{1/p}\varphi(\beta(z)).$$

It is easy to see that $BS_\Gamma - S_{\Gamma_0}B = T_1 + T_2$, where the operators T_1 and T_2 are defined by

$$(T_1\,\varphi)\,(z) = |\beta'\,(z)|^{1/p}\int_\Gamma (\frac{\beta'\,(\tau)}{\beta\,(\tau) - \beta\,(z)} - \frac{1}{\tau - z})\varphi(\beta\,(\tau)\,d\tau,$$

$$(T_2\,\varphi)\,(z) = |\beta'\,(z)|^{1/p}\int_\Gamma \frac{|\beta'\,(z)|^{1/p} - |\beta'\,(\tau)|^{1/p}}{\tau - z}\varphi(\beta\,(\tau)\,d\tau$$

and act form $L_p\,(\Gamma)$ into $L_p\,(\Gamma_0)$. Since the function $\beta'\,(z)$ satisfies a Hölder condition on Γ_0, it is easy to show that the kernel

$$\frac{\beta'\,(\tau)}{\beta\,(\tau) - \beta\,(z)} - \frac{1}{\tau - z}$$

of the operator T_1 has a weak singularity, and consequently T_1 is a completely continuous operator acting form $L_p(\Gamma)$ into $L_p(\Gamma_0)$. The operator T_2 is also completely continuous.

Really, T_2 can be represented in the form

$$T_2 = \left(\alpha(z) S_{\Gamma_0} - S_{\Gamma_0}\alpha(z)\right) V,$$

where $(V\varphi)(\tau) = \varphi(\beta(\tau)$ and $\alpha(z) = |\beta'(z)|^{1/p}$. The function $\alpha(z)$ ($|z| = 1$) is continuous; consequently, by virtue of well-known proposition of the theory of singular integral equations, the operator $\alpha(z) S_{\Gamma_0} - S_{\Gamma_0}\alpha(z)$ is completely continuous in $L_p(\Gamma_0)$. The operator B maps $L_p(\Gamma)$ isometrically onto $L_p(\Gamma_0)$, and, by virtue of what we have proved, $S_\Gamma = B^{-1}S_{\Gamma_0}B + T$, where T is compact. Hence it follows that

$$|S_\Gamma|_{L_p(\Gamma)} = \left|S_{\Gamma_0}\right|_{L_p(\Gamma_0)}.$$

We proceed to the proof of the theorem in the general case. We assume that Γ consists of a finite number of closed contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ without no self-intersection points and let P_k $(k = 1, 2, \ldots, m)$ be the projections defined in $L_p(\Gamma)$ by the equality

$$(P_k\varphi)(t) = \begin{cases} \varphi(t), & if \ \ t \in \Gamma_k \\ 0, & if \ \ t \in \Gamma \backslash \Gamma_k \,. \end{cases}$$

For the operator $R = S_\Gamma + T$, where T is an arbitrary operator compact in $L_p(\Gamma)$, we have

$$\|R\|_{L_p(\Gamma)} \geq \|P_1 R P_1\|_{L_p(\Gamma)} \geq \|S\|_{L_p(\Gamma_1)}.$$

By virtue of what we have proved, we have $|S_\Gamma|_{L_p(\Gamma)} \geq \left|S_{\Gamma_0}\right|_{L_p(\Gamma_0)}$.

Let us prove the reverse inequality. From (8) it follows that for each $k = 1, 2, \ldots, m$ and $\varepsilon > 0$ there exists an operator compact T_k, such that

$$\left|S_{\Gamma_0}\right|_{L_p(\Gamma_0)} + \varepsilon > max\left\|S_{\Gamma_k} + T_k\right\|_{L_p(\Gamma_k)}.$$

Denoting by T the operator compact, defined by $T = \sum_1^m P_k T_k P_k$, we have

$$\left|S_{\Gamma_0}\right|_{L_p(\Gamma_0)} + \varepsilon > max\left\|S_{\Gamma_k} + T_k\right\|_{L_p(\Gamma_k)} = \left\|\sum_1^m P_k(S_\Gamma + T)P_k\right\|_{L_p(\Gamma)}. \tag{10}$$

Since the operators $P_k S_\Gamma P_j$ $(j \neq k)$ are completely continuous in $L_p(\Gamma)$, we have

$$\left\|\sum_1^m P_k(S_\Gamma + T)P_k\right\|_{L_p(\Gamma)} = \left\|S_\Gamma + \tilde{T}\right\|_{L_p(\Gamma)},$$

where $\tilde{T}$ is some operator compact. From this and (10), it follows that $\left|S_{\Gamma_0}\right|_{L_p(\Gamma_0)} \geq |S_\Gamma|_{L_p(\Gamma)}$.

Relation (8) is proved, while (9) follows from (8) and from the results of [2]. ∎

3. On the essential norm of singular operators in the case of a contour with corner points

In the works [1] exact constants for the factor-norms of singular operators S_α, P_α, Q_α are established. If Γ_α has one corner point with angle $\pi\alpha(0 < \alpha \leq 1)$, then
$|S_\alpha|_2 = ctg\theta(\alpha)/2, |P_\alpha|_2 = |Q_\alpha|_2 = (\sin\theta(\alpha))^{-1}$, where $S_\alpha = S_{\Gamma_\alpha}$, and

$$ctg\theta(\alpha) = \frac{1}{2}\max_{-1\leq x\leq 1}\left|(1+x)\left(\frac{1-x}{1+x}\right)^{\alpha/2} - (1-x)\left(\frac{1+x}{1-x}\right)^{\alpha/2}\right|. \tag{11}$$

In particular,

$$\left|S_{\frac{1}{3}}\right|_2 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \left|S_{\frac{1}{3}}\right|_2 = \sqrt{2}\,.$$

Theorem 3.1. *Let Γ be a piecewise Lyapunov contour with corner points $t_1, ..., t_n$ and $\rho(t) = \prod_{k=1}^n |t-t_k|^{\beta_k}(-1 < \beta_k < 1)$, then $|S_\Gamma|_{L_2(\Gamma,\rho)} = \max_{1\leq k\leq n}\left|S_{\alpha_k}\right|_{L_2(\Gamma_{\alpha_k},|t|^{\beta_k})}$. Let $\min_{1\leq k\leq n}(\alpha_1, ..., \alpha_n) = \alpha_{k_0}$. If $\alpha_{k_0} = 1$, then*

$$|S_\Gamma|_{L_2(\Gamma,\rho)} = \max_{1\leq k\leq n} ctg\pi\frac{1-|\beta_k|}{4}.$$

If $\rho(t) \equiv 1$, then $|S_\Gamma|_{L_2(\Gamma)} = ctg\frac{\theta(\alpha_{k_0})}{2}$. For the operators P_Γ and Q_Γ the equalities hold

$$|P_\Gamma| = |Q_\Gamma| = \frac{1+|S_\Gamma|^2}{2\,|S_\Gamma|}. \tag{12}$$

In the space $L_p(\Gamma)$ the estimates

$$|S_\Gamma|_p \leq \begin{cases} ctg\frac{\theta(\alpha_{k_0})}{p}, & \text{if } p = 2^n, \\ ctg^t\frac{\theta(\alpha_{k_0})}{2^n}\cdot ctg^{1-t}\frac{\theta(\alpha_{k_0})}{2^{n+1}}, & \text{if } 2^n < p < 2^{n+1}, \end{cases}$$

where $t = (2^{n+1}-p)/p$, are valid.

Remark. Equality (12) confirms the following hypothesis of the mathematician S.Marcus: let $\mathcal{B}$ be some Banach space and L_1, L_2 subspaces from $\mathcal{B}$ such that $L_1 \bigcap L_2 = \emptyset$ and $\mathcal{B}$=$L_1 + L_2$, then equality

$$\|P\| = \|Q\| = \frac{1+\|S\|^2}{2\,\|S\|}$$

takes place, where P and Q are projectors projecting the space $\mathcal{B}$ onto L_1, respectively, on L_2 and $S = P - Q$.

4. Estimates of the essential norms of the operators S_Γ, P_Γ and Q_Γ in the case of complex contour

We consider a contour Γ, made up of closed lines $\gamma_1, ..., \gamma_m$ of Lyapunov type on portions that have a single common point t_0. Let be a function defined on Γ that has a finite number of points of discontinuity of the first case. We associate two numbers $a(\tau+0)$ and $a(\tau-0)$ the function a and the point $\tau \in \Gamma$ as follows. For $\tau = t_0$ we put

$$\begin{aligned} a(t_0+0) &= a_1(t_0+0)\cdots a_m(t_0+0), \\ a(t_0-0) &= a_1(t_0-0)\cdots a_m(t_0-0), \end{aligned} \tag{13}$$

where $a_j(t_0-0) = \lim\limits_{\gamma_j \ni t \to t_0-0} a(t), a_j(t_0+0) = \lim\limits_{\gamma_j \ni t \to t_0+0} a(t)$, $(j = 1, ..., m)$. If, however $\tau \neq t_0$, then $a(\tau+0)$ and $a(\tau-0)$ are the limits to the right and to the left of the function a at the point τ.

Theorem 4.1. *Let $t_0, t_1, ..., t_n$ be all points of discontinuity of the function a and*

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \ (-1 < \beta_k < 1), \delta_k = \frac{2\pi(1+\beta_k)}{p}. \tag{14}$$

The operator $A = aP + Q$ is Noetherian in space $L_p(\Gamma, \rho)$ if and only if the function a satisfies the conditions:

(1) $\inf_{t\in\Gamma} |a(t)| > 0,$

(2) $a(t_k+0) f_{\delta_k}(\mu) + a(t_k-0)\left(1 - f_{\delta_k}(\mu)\right) \neq 0$ *($0 \leq \mu \leq 1$, $k = 1, 2, \ldots, m$),*

where

$$f_\delta(\mu) = \begin{cases} \frac{\sin\theta\mu}{\sin\theta} \exp(i\theta(1-\mu)) \ (\theta = \pi - \delta), & if \delta \neq \pi, \\ \mu, \quad if \delta = \pi. \end{cases}$$

The proof of this theorem is given in [2].

We note that conditions (2) are equivalent to the fact, that the ratio $\frac{a(t_k-0)}{a(t_k+0)}$ can be expressed in the form, where

$$\frac{2\pi(1+\beta_k)}{p} - 2\pi < \operatorname{Re}\omega_k < \frac{2\pi(1+\beta_k)}{p}. \tag{15}$$

Let $\rho(t)$ be defined by equality (14). We introduce the following notations:

$$h_k = p(1+\beta_k)^{-1}, h_{n+1} = p, \overline{h}_k = \max\left(h_k, h_k(h_k-1)^{-1}\right) (k = 0, 1, ..., n),$$

$$h = \max\left(\overline{h}_1, \overline{h}_2, ..., \overline{h}_{n+1}\right).$$

Theorem 4.2. *The essential norms of the operators* S_{Γ}, P_{Γ} *and* Q_{Γ} *in the space* $L_p(\Gamma, \rho)$ *satisfies the relations*

$$P_\Gamma|_{L_p(\Gamma,\rho)} \geq \max\left((\sin\pi/h)^{-1}, \left(\sin\pi/m\overline{h}_0\right)^{-1}\right), \tag{16}$$

$$|Q_\Gamma|_{L_p(\Gamma,\rho)} \geq \max\left((\sin\pi/h)^{-1}, \left(\sin\pi/m\overline{h}_0\right)^{-1}\right), \tag{17}$$

$$|S_\Gamma|_{L_p(\Gamma,\rho)} \geq \max\left((\mathrm{ctg}\pi/2h)^{-1}, \left(\mathrm{ctg}\pi/2m\overline{h}_0\right)^{-1}\right). \tag{18}$$

Proof. Suppose, for example, that

$$|P_\Gamma|_{L_p(\Gamma,\rho)} < \max\left((\sin\pi/h)^{-1}, \left(\sin\pi/m\overline{h}_0\right)^{-1}\right).$$

We assume that

$$\max\left((\sin\pi/h)^{-1}, \left(\sin\pi/m\overline{h}_0\right)^{-1}\right) = \left(\sin\pi/m\overline{h}_0\right)^{-1}.$$

Denote by a the piecewise continuous function on Γ, which takes on each line $\gamma_j (j = 1, 2, \ldots, m)$ two values:

$$\cos\pi/m\overline{h}_0 \cdot exp(\pi i/m\overline{h}_0) \quad \text{and} \quad \cos\pi/m\overline{h}_0 \cdot exp(-\pi i/m\overline{h}_0)$$

and verify conditions:

$$a_j(t_0+0) = \begin{cases} \cos\pi/m\overline{h}_0 \cdot \exp\left(\pi i/m\overline{h}_0\right), & \text{if } \overline{h}_0 = h_0(h_0-1)^{-1}, \\ \cos\pi/m\overline{h}_0 \cdot \exp\left(-\frac{\pi i}{m\overline{h}_0}\right), & \text{if } \overline{h}_0 = h_0. \end{cases}$$

$$a_j(t_0-0) = \begin{cases} \cos\pi/m\overline{h}_0 \cdot \exp\left(-\pi i/m\overline{h}_0\right), & \text{if } \overline{h}_0 = h_0(h_0-1)^{-1}, \\ \cos\pi/m\overline{h}_0 \cdot \exp\left(\frac{\pi i}{m\overline{h}_0}\right), & \text{if } \overline{h}_0 = h_0. \end{cases}$$

Since

$$\prod_{j=1}^{m} a_j(t_0-0) / \prod_{j=1}^{m} a_j(t_0+0) = \exp\left(\frac{2\pi i}{\overline{h}_0}\right),$$

from Theorem 4.1, it follows that the operator $A = aP_\Gamma + Q_\Gamma$ is not Noetherian in the space $L_p(\Gamma, \rho)$. On the other hand, we have

$$|a(t) - 1| = \left|\cos^2\pi/m\overline{h}_0 \pm i\sin\frac{\pi}{m\overline{h}_0} \cdot \cos\frac{\pi}{m\overline{h}_0} - 1\right| = \sin\frac{\pi}{m\overline{h}_0}$$

and by the hypothesis we have

$$\inf_{T\in\mathrm{T}} \|(a-1)P_\Gamma + T\|_{L_p(\Gamma,\rho)} < 1.$$

From this relation it results that the operator $\hat{A} = a\hat{P}_\Gamma + \hat{Q}_\Gamma = (a-1)\,\hat{P}_\Gamma + \hat{I}$ is invertible in the space $L(L_p\,(\Gamma,\rho))/\ddot{T}(L_p\,(\Gamma,\rho))$. Then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian. Contradiction. Thus

$$|P_\Gamma|_{L_p(\Gamma,\rho)} \geq \left(\sin\pi/m\overline{h}_0\right)^{-1},$$

Let

$$\max\left((\sin\pi/h)^{-1}, \left(\sin\pi/m\overline{h}_0\right)^{-1}\right) = (\sin\pi/h)^{-1}.$$

Consider the function b that takes on Γ two values:

$$\cos\pi/h \cdot exp\,(\pi i/h)\,, \cos\pi/h \cdot exp(-\pi i/h)$$

and

$$b\,(t_r+0) = \begin{cases} \cos\pi/h \cdot \exp\,(\pi i/h)\,, & \text{if } \overline{h}_r = h_r(h_r-1)^{-1}, \\ \cos\pi/h \cdot \exp\left(-\frac{\pi i}{h}\right), & \text{if } \overline{h}_r = h_r; \end{cases}$$

$$b\,(t_r-0) = \begin{cases} \cos\pi/h \cdot \exp\left(-\frac{\pi i}{h}\right), & \text{if } \overline{h}_r = h_r(h_r-1)^{-1}, \\ \cos\pi/h \cdot \exp\left(\frac{\pi i}{h}\right), & \text{if } \overline{h}_r = h_r, \end{cases}$$

where t_r is the point, for which $h = \overline{h}_r$ $(1 \leq r \leq n)$. If $h = h_{n+1}$, then as t_r we can take any point other than the points $t_0, t_1, ..., t_n$. As

$$b\,(t_r-0)\,/b\,(t_r+0) = \exp\left(\frac{2\pi i}{h}\right)$$

from Theorem 4.1 it follows that the operator $B = bP_\Gamma + Q_\Gamma$ is not Noetherian in $L_p\,(\Gamma,\rho)$. But $|b\,(t)-1| = \sin\frac{\pi}{h}$ and by on the assumption we have

$$\inf_{T\in\ddot{T}} \|(b-1)\,P_\Gamma + T\| < 1,$$

where the operator results $B = bP_\Gamma + Q_\Gamma = (b-1)P_\Gamma + \mathrm{I}$ is Noetherian in $L_p\,(\Gamma,\rho)$. The obtained contradiction proves the relationship (16). In order to prove the relation (17), we consider two functions a and b that take respectively the values

$$\sec\pi/m\overline{h}_0 \cdot \exp\left(\pi i/m\overline{h}_0\right), \sec\pi/m\overline{h}_0 \cdot \exp(-\pi i/m\overline{h}_0)$$

$$\text{and} \quad \sec\pi/h \cdot \exp\,(\pi i/h)\,, \quad \sec\pi/h \cdot \exp\,(-\pi i/h)\,.$$

As

$$\prod_{j=1}^{m} a_j\,(t_0-0)\,/\prod_{j=1}^{m} a_j\,(t_0+0) = \exp\left(2\pi i/\overline{h}_0\right)$$

and $b(t_r-0)/b(t_r+0)=\exp\left(\frac{2\pi?i}{h}\right)$, it follows that the operators $A=aP_\Gamma+Q_\Gamma$ and $=bP_\Gamma+Q_\Gamma$ are not Noetherians in the space $L_p(\Gamma,\rho)$. On the other hand,

$$|a(t)-1/a(t)|=\sin\left(\frac{\pi}{m\overline{h}_0}\right),|b(t)-1/b(t)|=\sin\left(\frac{\pi}{h}\right),$$

$A=a\left(I+\frac{1-a}{a}Q_\Gamma\right)$ and $B=b\left(I+\frac{1-b}{b}Q_\Gamma\right)$.

It remains to repeat the previous reasoning and we obtain the relation (17). Finally, in order to prove the relation (18), we consider the function a that takes on each line γ_j $(j=1,\ 2,\ \ldots,m)$ the values $\exp\left(\pi i/m\overline{h}_0\right)$ and $\exp\left(-\pi i/m\overline{h}_0\right)$ and the function b that takes on the values $\exp(\pi i/h)$ and $\exp(-\pi i/h)$. In what follows, we will use the following equalities,

$$aP_\Gamma+Q_\Gamma=\frac{a+1}{2}\left(I+\frac{a-1}{a+1}S_\Gamma\right),bP_\Gamma+Q_\Gamma=\frac{b+1}{2}\left(I+\frac{b-1}{b+1}S_\Gamma\right),$$

$$\left|(a-1)(a+1)^{-1}\right|=tg\frac{\pi}{2m\overline{h}_0},\left|(b-1)(b+1)^{-1}\right|=tg\frac{\pi}{2h}.$$

The theorem is proved. ■

Theorem 4.2 can be generalized, if the contour Γ is made up of a finite number of contours $\Gamma_1,\ldots,\Gamma_s$, that satisfy the conditions of Theorem 4.2.

Let $\rho_j(t)=\rho(t)$ for $t\in\Gamma_j$ $(j=1,\ 2,\ \ldots,s)$. Theorem 4.2 implies

Theorem 4.3. *The essential norms of the operators P_Γ, Q_Γ and S_Γ in the space $L_p(\Gamma,\rho)$ satisfy the relations*

$$|P_\Gamma|_{L_p(\Gamma,\rho)}\geq\max\left|P_{\Gamma_j}\right|_{L_p(\Gamma_j,\rho_j)},|Q_\Gamma|_{L_p(\Gamma,\rho)}\geq\max\left|Q_{\Gamma_j}\right|_{L_p(\Gamma_j,\rho_j)},$$

$$|S_\Gamma|_{L_p(\Gamma,\rho)}\geq\max\left|S_{\Gamma_j}\right|_{L_p(\Gamma_j,\rho_j)}.$$

Theorem 4.4. *Operator*

$$(K\phi)(t)=\frac{1}{\pi\phi i}\int_\Gamma\left(\frac{\omega'(\tau)}{\omega(\tau)-\omega(t)}-\frac{1}{\tau-t}\right)\phi(\tau)\,d\tau$$

is compact in the space $L_p(\Gamma,\rho)$, if and only if $\sum_{k=1}^{l}\alpha_k=l$.

Theorem 4.5. *The operator S_Γ^* acting in the space $L_q\left(\Gamma,\rho^{1-q}\right)$ has the form (see [2])*

$$S_\Gamma^*=-VhSVhI,$$

where $(V\phi)(t)=\overline{\phi(t)}$ and h is a piecewise Holder function on Γ.

Theorem 4.6. *The operator $S_\Gamma^*-S_\Gamma$ is compact in the space $L_2(\Gamma)$ if and only if*

$$\sum_{k=1}^{l}\alpha_k=l.$$

5. Operator $A = aP_\Gamma + bQ_\Gamma$ with measurable and bounded coefficients

In this section the values of the essential operators norms will be used to determine noetherian conditions for operators of the form $aP_\Gamma + bQ_\Gamma$ with measurable and bounded coefficients. As *essinf* $|a\ (t)\ | \neq 0$ and *essinf* $|b\ (t)\ | \neq 0$ represent necessary conditions under which these operators are Noetherians, we will assume that $b\ (t) \equiv 1$.

In this and next items we asume that Γ consists of two curves Γ_1 and Γ_2, having one common point t_0, and moreover the tangents to Γ at this point are perpendicular and $\rho(t)$ is the function determined by the equality (14).

Let τ_0 be a point on Γ different from t_0. Denote by $\lambda(\tau_0)$ the closed halfplane which does not contain the origin. By $\triangle(t_0)$ we denote the angle with vertex at the origin and value of $\pi/2$. By $M_\rho(\Gamma)$ we denote the class of essentially bounded measurable functions $a\ (t)$, satisfying the conditions:

(1) $essinf\ |a(t)| \neq 0;\ t \in \ \Gamma;$

(2) for any point $t \in \ \Gamma \backslash \{t_0\}$ there exists a neighbourhood $u(\tau)(\ \Gamma \backslash \{t_0\})$ of the point τ and a pair of functions $g_\tau^\pm$ such that $(g_\tau^+(t))^{\pm 1} \in L_\infty^+(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in L_\infty^-(\Gamma)$ and the range of the function $g_\tau^+(t)h(t)a(t)g_\tau^-(t)$ at $t \in u(\tau)$ is contained inside $\lambda(\tau)$.

(3) for any point t_0 either there exists a neighbourhood $u(t_0)$ and a pair of functions $g_0^\pm$ such that $(g_0^{\pm 1}(t))^{\pm 1} \in L_\infty^{\pm 1}(\Gamma)$ and the range of the function $g_0^+(t)h(t)a(t)g_0^-(t)$ at $t \in u(t_0)$ is contained inside of $\triangle(t_0)$, or there exist finite limits $a(t_0 \pm 0)$ and

$$\frac{h_{t_0}\ (t_0 - 0)\ a\ (t_0 - 0)}{h_{t_0}\ (t_0 + 0)\ a\ (t_0 + 0)} \in C \backslash\ (-\infty, 0]\ .$$

Theorem 5.1. $M_\rho\ (\Gamma) \subset Fact_{2,\rho}\ (\Gamma)$.

Corollary 5.1. *Let $a \in M_\rho\ (\Gamma)$. Then the operator $A = aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2\ (\Gamma, \rho)$.*

Corollary 5.2. *$M_\rho\ (\Gamma) \bigcap PC(\Gamma) = Fact\ (\Gamma) \bigcap PC(\Gamma)$, where $PC(\Gamma)$ is a set of all piecewise continous functions on Γ.*

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