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Qualitative study of the cuartic system with maximal multiplicity of the line at the infinity

Vadim Repeşco

Abstract. We consider the generic cuartic differential system. In [7] it was proven that the class of differential cuartic systems that have the invariant straight line at the infinity of maximal multiplicity is affine equivalent with the system $\dot{x} = -3x + ay^4$, $\dot{y} = y$, $a > 0$. In this paper, by using several techniques, we obtain its phase portrait on Poincaré disk.
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Studiul calitativ al sistemului diferențial cuartic ce posedă dreapta de la infinit de multiplicitate maximală

Rezumat. Considerăm sistemul diferenţial cuartic general. În [7] a fost demonstrat că clasa sistemelor diferenţiale cuartice ce au dreapta de la infinit de multiplicitate maximală este echivalentă afin cu sistemul $\dot{x} = -3x + ay^4$, $\dot{y} = y$, $a > 0$. În această lucrare, prin intermediul mai multor tehnici, se obţine portretul fazic al acestui sistem pe discul Poincaré.
Cuvinte cheie: Sistem diferenţial cuartic, dreaptă invariantă, multiplicitatea curbei algebrice invariante.

1. Introduction

We consider a real polynomial differential system

$$\begin{cases} \dfrac{dx}{dt} = P(x,y) \\ \dfrac{dy}{dt} = Q(x,y) \end{cases}, \quad GCD(P,Q) = 1 \tag{1}$$

and the vectorial field associated with this system

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

Let $n = \max\{\deg(P), \deg(Q)\}$. If $n = 4$, then the system (1) is called cuartic.

Definition 1.1. *An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$, is called invariant algebraic curve for the system* (1)*, if there is polynomial $K_f \in \mathbb{C}[x, y]$, such that the following identity holds*

$$\mathbb{X}(f) = f(x, y) K_f(x, y). \tag{3}$$

The invariant algebraic curves are very important in the qualitative theory of dynamical systems. The problem of finding how many invariant straight lines a polynomial differential system can have is studied in [1]. The problem of coexistence of invariant straight lines and limit cycles and the problem of coexistence of invariant straight lines and singular points of center type are studied in [2, 3, 4, 5]. Another direction of applications of invariant straight lines is calculating the Darboux first integral. According to [6], we can calculate a Darboux first integral for a polynomial differential system, if the system has sufficiently many invariant straight lines considered with their multiplicities. In this paper we will obtain the phase portrait on the Poincaré disk for the system (4), obtained in [7].

$$\begin{cases} \dot{x} = -3x + ay^4, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{4}$$

This is a cuartic differential system with the invariant straight line at the infinity of the maximal multiplicity. There are different types of multiplicities for invariant algebraic curves, see for instance [8]. In [7], i.e. for the system (4), it was considered the algebraic multiplicity.

2. Qualitative study of the system (4)

The system (4) has a single singular point in the finite part of the phase plane - $O(0, 0)$, its eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 1$, therefore it follows that the coordinate's origin is of the saddle type. From the relation$yP_4 - xQ_4 \equiv ay^5$, it follows that this system has singular points only at the ends of the axis Ox. Therefore, we'll use the first Poincaré transformation of variables $x = \frac{1}{z}$, $y = \frac{u}{z}$ and the system will take the following form:

$$\begin{cases} \dot{z} = z\left(3z^3 - au^4\right), \\ \dot{u} = u\left(4z^3 - au^4\right). \end{cases} \tag{5}$$

The singular point situated at the origin of coordinate of the system (5) it's multiple, using the blow-up method, we'll decompose this point into 4 singular points, their cooordinates and their corresponding eigenvalues are shown in Table 1.

Table 1. Decomposition of the singular point $O(0,0)$

S.P.	$M_1(0,0)$	$M_2\left(0,\frac{\pi}{2}\right)$	$M_3\left(0,\pi\right)$	$M_4\left(0,\frac{3\pi}{2}\right)$
λ_1,λ_2	$1,3$	$0,0$	$-3,-1$	$0,0$

The singular point $M_1(0,0)$ is an unstable node and $M_3(0,\pi)$ – stable node. The singular points $M_2\left(0,\frac{\pi}{2}\right)$ and $M_4\left(0,\frac{3\pi}{2}\right)$ are multiple. To study the singular point $M_2\left(0,\frac{\pi}{2}\right)$, we'll translate this point to the origin of the coordinates and will compute the Taylor series in θ for the functions $P(\rho,\theta)$ and $Q(\rho,\theta)$:

$$\begin{cases}\dot{\rho} = -a\rho^2 + 2a\rho^2\theta^2 - 4\rho\theta^3 + \ldots\\ \dot{\theta} = \theta^4 - \dfrac{7}{6}\theta^6 - \dfrac{23}{40}\theta^8 + \ldots\end{cases}$$

The singular point at the origin of coordinates corresponds to the point M_2. Using the blow-up method, we'll decompose this point into 4 singular points, see Table 2.

Table 2. Decomposition of the singular point $M_2\left(0,\frac{\pi}{2}\right)$

S.P.	$N_1(0,0)$	$N_2\left(0,\frac{\pi}{2}\right)$	$N_3\left(0,\pi\right)$	$N_4\left(0,\frac{3\pi}{2}\right)$
λ_1,λ_2	$-a,a$	$0,0$	$-a,a$	$0,0$

The singular points $N_1(0,0)$ and $N_3(0,\pi)$ are of saddle type and the singular points $N_2\left(0,\frac{\pi}{2}\right)$ and $N_4\left(0,\frac{3\pi}{2}\right)$ are multiple. Again, using the blow-up method, the singular point $N_2\left(0,\frac{\pi}{2}\right)$ can be decomposed into 4 singular points, see Table 3.

Table 3. Decomposition of the singular point $N_2\left(0,\frac{\pi}{2}\right)$

P.S.	$R_1(0,0)$	$R_2\left(0,\frac{\pi}{2}\right)$	$R_3\left(0,\pi\right)$	$R_4\left(0,\frac{3\pi}{2}\right)$
λ_1,λ_2	$0,0$	$-a,a$	$0,0$	$-a,a$

The singular points $R_2\left(0,\frac{\pi}{2}\right)$ and $R_4\left(0,\frac{3\pi}{2}\right)$ are of saddle type and the singular points $R_1(0,0)$ and $R_3(0,\pi)$ are multiple. Via the blow-up method, the singular point $R_1(0,0)$ is decomposed into 6 singular points, see Table 4. As we can see from the Table 4, all these singular points are hyperbolic, therefore we can find the qualitative behavior of the trajectories near these points, see Figure 1.a).

Table 4. Decomposition of the singular point $R_1(0,0)$

P.S.	$T_1\ (0,0)$	$T_2\left(0, \text{arctg}\,\frac{7}{a}\right)$	$T_3\left(0, \frac{\pi}{2}\right)$	$T_4\ (0,\pi)$	$T_5\left(0, \pi\text{arctg}\,\frac{7}{a}\right)$	$T_6\left(0, \frac{3\pi}{2}\right)$
λ_1, λ_2	$-7, 1$	$\frac{1}{\sqrt{49+a^2}}, \frac{7a}{\sqrt{49+a^2}}$	$-a, a$	$-6, 1$	$-\frac{1}{\sqrt{49+a^2}}, -\frac{7a}{\sqrt{49+a^2}}$	$-a, a$
Tipul	S	N^u	S	S	N^s	S

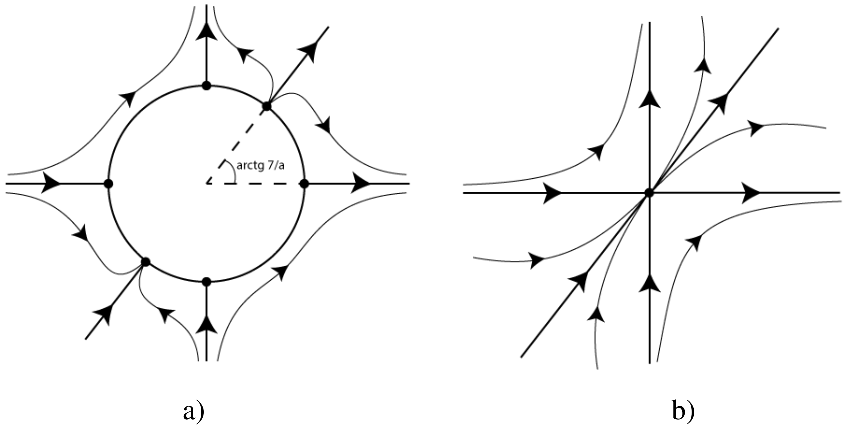

Figure 1. The singular point $R_1(0,0)$

Compressing the unit circle to the origin of coordinate, we get the behavior of the trajectories in the vicinity of the singular point $R_1(0,0)$, see Fig. 1.b).

In a similar way, using the blow-up method for the singular point $R_3(0,\pi)$, then getting the behavior of the trajectories near the singular points and compressing the unit circle to the origin, we get the qualitative behavior in the vicinity of the singular point $R_3(0,\pi)$, i.e. we get Fig. 2.

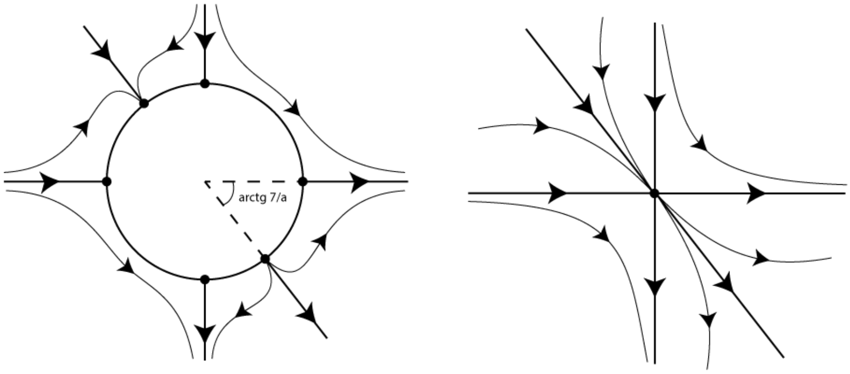

Figure 2. The singular point $R_3(0,\pi)$

Using the qualitative behavior of trajectoires from the Fig. 1 and Fig. 2, also the type of singular points from the Table 3, we obtain the qualitative behavior of the trajectorie in the vicinity of the multiple singular point $N_2\left(0, \frac{\pi}{2}\right)$, see Figure 3.

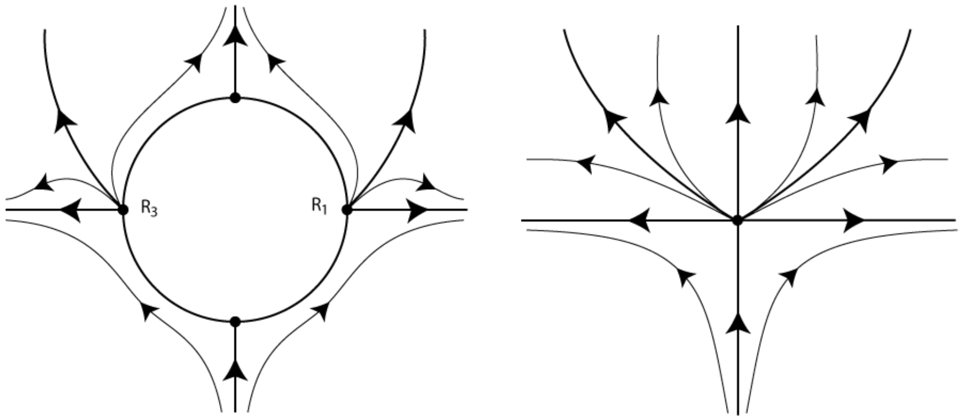

Figure 3. The singular point $N_2\left(0, \frac{\pi}{2}\right)$

Using the information from the Figure 3 to obtain the the behavior near the singular point $M_2\left(0, \frac{\pi}{2}\right)$, we can see that some information about singular point $R_3(0, \pi)$ is redundant and it will not be used. Therefore, for the other multiple singular point in which $M_2\left(0, \frac{\pi}{2}\right)$ decompose to, i. e. $N_4\left(0, \frac{3\pi}{2}\right)$, we can ignore some information.

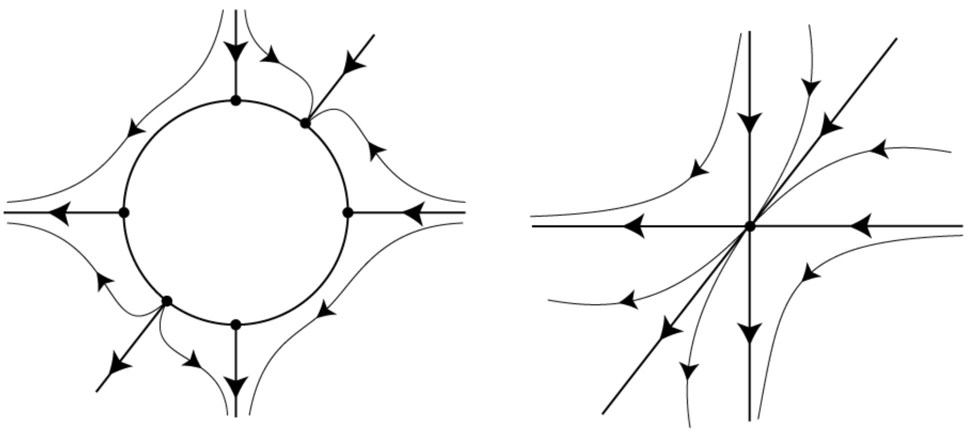

Figure 4. The singular point $R'_1\ (0, 0)$

This point decompose into 4 singular points: $R'_2\left(0, \frac{\pi}{2}\right)$ and $R'_4\left(0, \frac{3\pi}{2}\right)$ are of saddle type and the points $R'_1(0, 0)$ and $R'_3(0, \pi)$ are multiple. We'll examine only the singular point $R_1'(0,0)$, which decompose into 6 singular points, their behavior is illustrated in Figure 4.

Using Figure 4 and the fact that $R_2'\left(0, \frac{\pi}{2}\right)$ and $R_4'\left(0, \frac{3\pi}{2}\right)$ are of saddle type, we get Figure 5.

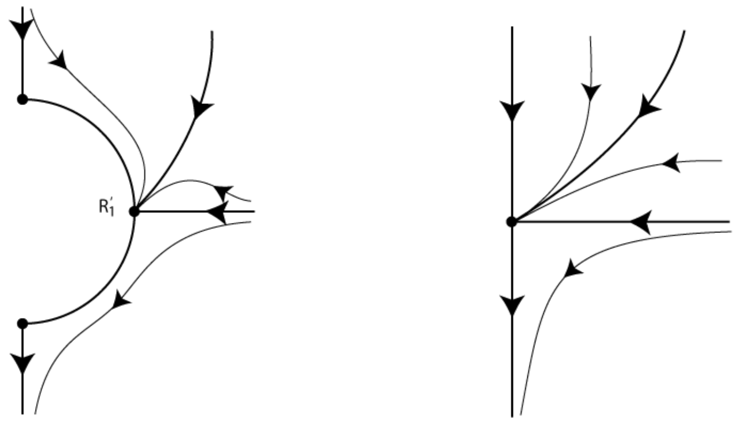

Figure 5. The singular point $N_4\left(0, \frac{3\pi}{2}\right)$

Using Table 2 and Figure 2 and Figure 5, we obtain the qualitative behavior of the trajectories in the vicinity of the singular point $M_2\left(0, \frac{\pi}{2}\right)$, see Figure 6.

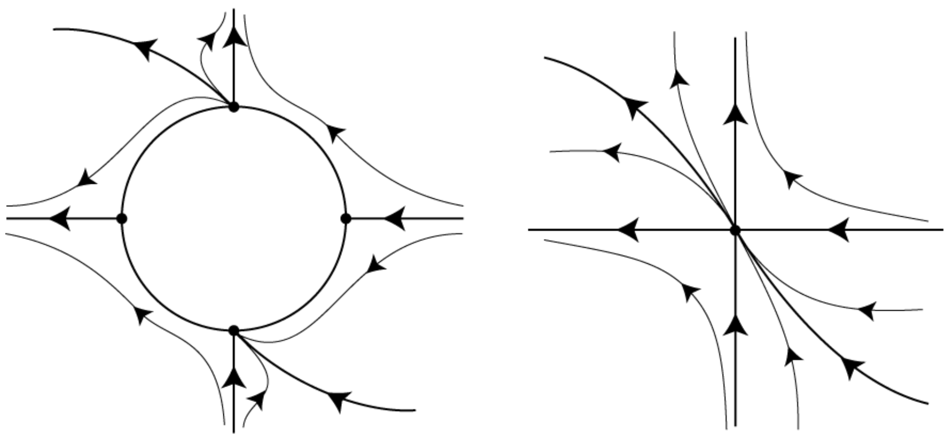

Figure 6. The singular point $M_2\left(0, \frac{\pi}{2}\right)$

Similarly, by performing all the above steps, we obtain the qualitative behavior of the trajectories from the vicinity of the multiple singular point $M_4\left(0, \frac{3\pi}{2}\right)$, see Fig. 7:

Finally, putting the singular point $M_1(0,0)$, $M_2\left(0, \frac{\pi}{2}\right)$, $M_3(0,\pi)$ and $M_4\left(0, \frac{3\pi}{2}\right)$ on the unitary circle, then compressing this circle into the origin, we obtain the qualitative

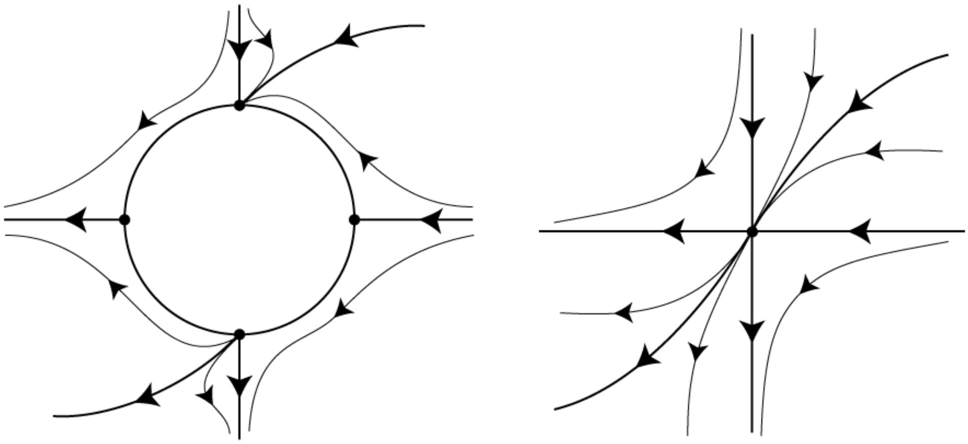

Figure 7. The singular point $M_4\left(0, \frac{3\pi}{2}\right)$

behavior of the trajectories in the vecinity of the singular point of the system (4) situated at the ends of the Ox axis.

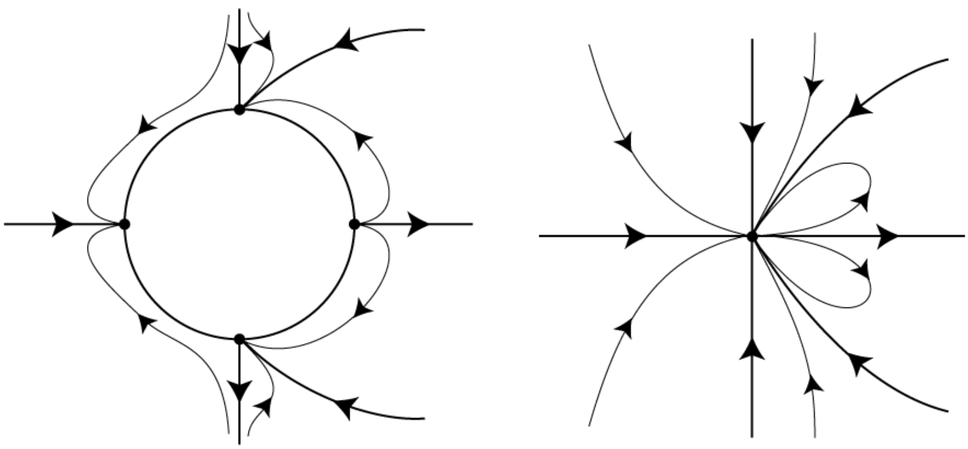

Figure 8. The singular point $O_\infty\ (\pm 1, 0, 0)$

3. The main result

Taking into account that the origin of coordinates for the system (4) is a singular point of saddle type, we can easy schetch on the Poincaré disk the global phase portrait of this system:

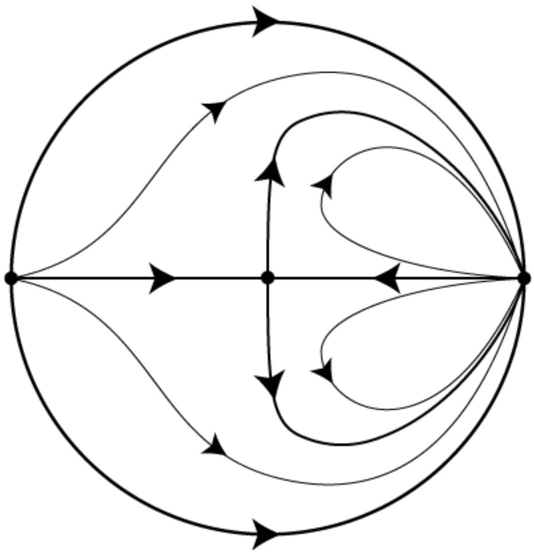

Figure 9. Phase portrait of the system (4)

References

[1] Artes J., Grunbaum B., Llibre J., On the number of invariant straight lines for polynomial differential systems. Pacific Journal of Mathematics, 1998, 184, No. 2, 207–230.

[2] Suo Guangjian, Sun Jifang, The n-degree differential system with $(n-1)(n+1)/2$ straight line solutions has no limit cycles. Proc. of Ordinary Differential Equations and Control Theory, Wuhan, 1987, 216-220 (in chinese).

[3] Kooij R., Cubic systems with four line invariants, including complex conjugated lines. Math., Proc. Camb. Phil. Soc., 1995, 118, No. 1, 7–19

[4] Cozma D., Șubă A., The solution of the problem of center for cubic differential systems with four invariant straight lines. Mathematical analysis and applications, (Iași, 1997) 44, suppl., 517–530.

[5] Șubă A., Cozma D., Solution of the problem of the center for cubic differential system with three invariant straight lines in generic position. Qualitative Theory of Dynamical Systems, Universitat de Lleida. Spaine, 2005, 6, 45–58.

[6] Llibre, Jaume; Xiang Zhang, On the Darboux Integrability of Polynomial Differential Systems, Qual. Theory Dyn. Syst., 2012,

[7] Repeşco, Vadim, The canonical form of all cuartic systems with maximal multiplicity of the line at the infinity. In: Sychasni problemi diferentzial'nyx rivnean' ta ix sastosuvannja (ukr), Cernăuţi, 16 - 19 veresnja, 2020 roku, pp 66-67

[8] Christopher C., Llibre J., Pereira J. V., Multiplicity of invariant algebraic curves in polynomial vector fields. Pacific Journal of Mathematics, 329, 2007, nr. 1, p. 63-117.

(Repeşco Vadim) Tiraspol State University, str. Ghenadie Iablocikin 5, Chişinău, Moldova
E-mail address: repescov@ust.md, repescov@gmail.com