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Regularization of some perturbed integral operators

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Abstract. The paper presents some generalizations and specifications of the paper [1]. In particular, examples of integral operators with point-like singularities are constructed, which do not represent admissible disturbances for the characteristic singular integral operators. This means that the built operators can influence the noetherian conditions of the singular operators.
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Regularizarea unor operatori integrali perturbați

Rezumat. În lucrare sunt prezintate anumite generalizări și precizări ale lucrării [1]. În particular, în ea se construesc exemple de operatori integrali cu singularități punctiforme care nu reprezintă perturbări admisibile pentru operatorii integrali singulari caracteristici. Aceasta înseamnă că operatorii construiți pot influența condițiile noetheriene ale operatorilor singulari.
Cuvinte cheie: operatori singulari perturbați, condiții noetheriene.

1. Introduction

We remind that an operator $A \in L(B)$ admits regularization if there exist operators $M_1, M_2 \in L(B)$ such that $AM_1 = I + T_1$ (left regularization) and $M_2A = I + T_2$ (right regularization), where T_1 and T_2 are compact operators in space $\mathcal{B}$. The class of operators admitting regularization is of particular interest, since the operators of this class have the following properties (F. Noether's theorems):

(1) *The equation $Ax = y$ is solvable if and only if its right-hand side is orthogonal to all solutions of the equation $A^*\varphi = 0$. This condition is equivalent to the condition that the set of values of the operator A is a subspace, or the equality*

$$\operatorname{Im} A = \cap_{f \in \operatorname{Ker} A^*} \operatorname{Ker} f$$

is true.

(2) *The equations $Ax = 0$ and $A^*\varphi = 0$ have a finite number of linearly independent solutions.*

Operators with these 2 properties are called *Noetherian operators* and represent essential generalizations of the class of operators of the form $I + T$, where T is a compact operator for which the well-known Fredholm theorems hold. If conditions (1) and (2) are satisfied, then the number $\dim \operatorname{Ker} A - \dim \operatorname{Ker} A^*$ is called the index of a Noetherian operator A and is denoted by $\operatorname{Ind} A$.

Let us denote by the $N(\mathcal{B})$ the set of all Noetherian operators acting in a Banach space $\mathcal{B}$ and let $\mathcal{H}$ be a Hilbert space. It is well known that if an operator $K \in L(\mathcal{H})$ and has the property $A + K \in N(\mathcal{H})$ for every $A \in N(\mathcal{H})$, then K is completely continuous.

And what will be, if we require that the implication $A \in N(\mathcal{H})$ implies "$A+K \in N(\mathcal{H}) \Leftrightarrow A \in N(\mathcal{H})$", but say, for all singular integral Noetherian operators. Is K necessarily completely continuous in this case? It turns out that it is not necessary. Examples of such operators can be found in [1], [3-5], and such examples are given in this paper.

2. Preliminaries

In the monographs of N.I. Muskhelishvili and F.D. Gakhov, an operator is called complete singular integral operator if it has the form

$$(A\varphi)(t) = a(t) + \frac{1}{\pi i} \int_{\Gamma} \frac{k(\tau, t)}{\tau - t} d\tau \qquad (1)$$

where $a(t)$ and $k(\tau, t)$ are functions satisfying the Hölder condition on Γ and $\Gamma \times \Gamma$, respectively, and the integral is understood in the sense of the principal value. The operator A, defined by equality (1), can be represented in the form $A = aI + bS + T$, where $b(t) = k(t, t)$, and is the integral operator with kernel

$$k_0(\tau, t) = \pi i \frac{k(\tau, t) - k(t, t)}{\tau - t}. \qquad (2)$$

In the case when $k(\tau, t)$ satisfies the Hölder condition on $\Gamma \times \Gamma$, the kernel (2) has a weak singularity; therefore, the operator T is completely continuous in the space $L_p(\Gamma)$. Due to this, the operator A is Noetherian in the space $L_p(\Gamma)$, if and only if the operator

$$A_0 = aI + bS$$

is Noetheran. Operator A_0 is called the characteristic part of the operator A. In this connection, Noether's theory of singular operators was developed mainly for characteristic operators. Significant successes have been achieved in this direction: there are obtained criteria to be Noetherian for such operators with piecewise continuous coefficients, with coefficients having discontinuities of almost periodic type, with arbitrary coefficients from

$L_\infty(\Gamma)$. However, in many problems of mechanics, physics and other areas that lead to singular equations, not characteristic operators appear, but complete ones. In this regard, it becomes necessary to study the complete singular operators (1) with functions and $k(\tau,t)$ not necessarily satisfying the Hölder condition. The main difficulty here is that the operator T with kernel (2) may turn out to be not completely continuous (not compact) or (more importantly) ceases to be an Φ-admissible perturbation.

Let's show this on an example. Let Γ_0 be the unit circle, $\chi(t)$ be the characteristic function of the $\{\operatorname{Im} t > 0\} \cap \Gamma_0$; $k(\tau,t) = \chi(t) - \chi(\tau)$, $\lambda \in \mathbb{C}$,

$$(A\varphi)(t) = \lambda\varphi(t) + \frac{1}{\pi i}\int\limits_{\Gamma_0} \frac{k(\tau,t)\,\varphi(\tau)}{\tau - t}d\tau.$$

In this example, $k(t,t) = 0$, therefore, the characteristic part of the operator A is a scalar operator $(A_0\varphi)(t) = \lambda\varphi(t)$. The operator A in this example can be represented in the form $A = \lambda I + \chi S - S\chi I$, whence it follows that it belongs to the algebra A_p, generated by singular integral operators with piecewise continuous coefficients. It was shown in [2] that on the algebra A_p one can introduce the symbol $(\gamma_{t,\mu})$ $((t,\mu) \in \Gamma_0 \times [0,1])$, which on the generators of S and aI takes the form

$$\gamma_{t,\mu}(aI) = \begin{Vmatrix} a(t+0)f_p(\mu)+a(t-0)\left(1-f_p(\mu)\right) & (a(t+0)-a(t-0))h_p(\mu) \\ (a(t+0)-a(t-0))h_p(\mu) & a(t+0)\left(1-f_p(\mu)\right)+a(t-0)f_p(\mu) \end{Vmatrix} \tag{3}$$

where

$$f_p(\mu) = \begin{cases} \dfrac{\sin\theta\mu}{\sin\theta}e^{i\theta(\mu-1)}, \left(\theta = \dfrac{\pi(p-2)}{2}\right), & \text{for } p \neq 2, \\ \mu, & \text{for } p = 2 \end{cases} \tag{4}$$

and $h_p(\mu)$ is some fixed continuous branch of the function $\sqrt{f_p(\mu)\left(1 - f_p(\mu)\right)}$.

In particular, for the operator $A = \lambda I + \chi S - S\chi I$ with $p = 2$ we have: $\det\gamma_{t,\mu}(A) = \lambda^2$ for $t \neq \pm 1$ and $\det\gamma_{t,\mu}(A) = \lambda^2 + 4\mu(1-\mu)$ for $t = \pm 1$. An operator A is Noetherian in $L_2(\Gamma)$ if and only if $\lambda b^2 + 4\mu(1-\mu) \neq 0$ for all $\mu \in [0,1]$. This is equivalent to $\lambda \neq ti$, where $t \in [-1,1]$.

Thus, for $\lambda = \tau i$, where $\tau \in [-1,1] \setminus 0$, the operator A is not Noetherian, but its characteristic part A_0 is Noetherian. This implies that the operator $M = A - A_0$ is not a Φ-admissible perturbation of the characteristic part of the operator A. This also implies that M is not compact.

For this operator, we managed to obtain criteria for Noetherian property due to the fact that we embedded it in the algebra A_p (see [7]). You can do the same with some other complete operators. This work will describe one class of such operators.

In what follows, we will consider the perturbation of the characteristic operators by operators of the following form

$$(K\varphi)(t) = \sum_{k=1}^{m} c_k(t)(M_k\varphi)(t) \qquad (c_k \in L_\infty(\Gamma)), \tag{5}$$

where

$$(M_k\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t - \alpha_k} d\tau \ (t \in \Gamma) \tag{6}$$

and $\alpha_k (\neq 0)$ are some complex numbers. First of all, note that if the function $\tau - t - \alpha_k$ vanishes at some point $(\tau, t) \in \Gamma \times \Gamma$, then the corresponding operator M_k is not compact. This follows from the following theorem.

Theorem 2.1. *Let $\Gamma_k = \{z | z = t - \alpha_k, t \in \Gamma\}$ If $\Gamma \cup \Gamma_k \neq \emptyset$, the operator M_k is not compact in the space $L_p(\Gamma)$.*

Proof. Suppose that the operator M_k is compact in the space $L_p(\Gamma)$. Let $\gamma = \Gamma \cup \Gamma_k$ and t_0 be one of the intersection points of the contours Γ with Γ_k. In the space $L_p(\Gamma)$ consider the singular operator defined by the equality

$$A = aI + bS_\gamma,$$

where $a(t)$ and $b(t)$ are continuous at each point $t \in \gamma \setminus \{t_0\}$ and satisfy the conditions:

$$a(t \pm 0) \pm b(t \pm 0) \neq 0,$$

$$((a(t_0 - 0) + b(t_0 - 0)))/((a(t_0 - 0) - b(t_0 - 0))) = i$$

and

$$((a(t_0 + 0) + b(t_0 + 0)))/((a(t_0 + 0) - b(t_0 + 0))) = 1.$$

Under these conditions, the operator A is not Noetherian [2] in space $L_2(\gamma)$. Operator R, acting by rule

$$(R\varphi)(t) = (\varphi(t), \varphi(t - \alpha_k)), (t \in \Gamma),$$

is the reversible operator from $L\left(L_2(\gamma), L_2^2(\Gamma)\right)$. Let $\psi \in L_p(\Gamma)$ and consider the equation

$$A\varphi = a\varphi + bS_\gamma\varphi = \psi.$$

This equation can be rewritten as a system of two equations: in one equation $t \in \Gamma$, and in the second equation $t \in \Gamma_k$.

$$\begin{cases} a(t)\varphi(t) + \dfrac{b(t)}{\pi i}\displaystyle\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \frac{b(t)}{\pi i}\int_{\Gamma_k} \frac{\varphi(\tau)}{\tau - t}d\tau = \psi(t), t \in \Gamma \\ a(t)\varphi(t) + \dfrac{b(t)}{\pi i}\displaystyle\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \frac{b(t)}{\pi i}\int_{\Gamma_k} \frac{\varphi(\tau)}{\tau - t}d\tau = \psi(t), t \in \Gamma_k \end{cases}.$$

In the integral $\int_{\Gamma_k} \frac{\varphi(\tau)}{\tau-t} d\tau$ we change the variables $\tau \to \tau - \alpha_k$ and in the second equation of the resulting system, replace t by $t - \alpha_k$. We get

$$\begin{cases} a_1(t)\varphi_1(t) + \frac{b_1(t)}{\pi i}\int_\Gamma \frac{\varphi_1(\tau)}{\tau - t}d\tau + \frac{b_1(t)}{\pi i}\int_\Gamma \frac{\varphi_2(\tau)}{\tau - t - \alpha_k}d\tau = \psi_1(t), t \in \Gamma \\ a_2(t)\varphi_2(t) + \frac{b_2(t)}{\pi i}\int_\Gamma \frac{\varphi_2(\tau)}{\tau - t}d\tau + \frac{b_2(t)}{\pi i}\int_{\Gamma_k} \frac{\varphi_1(\tau)}{\tau - t + \alpha_k}d\tau = \psi_2(t), t \in \Gamma \end{cases}$$

where the notation $f_1(t) = f(t)$, $f_2(t) = f(t - \alpha_k)$ $t \in \Gamma$ is used. Thus, the operator RAR^{-1} has the form

$$RAR^{-1} = \left\| \begin{matrix} a_1 I + b_1 S_\Gamma & b_1 M_k \\ b_2 N_k & a_2 I + b_2 S_\Gamma \end{matrix} \right\|, \tag{7}$$

where

$$(S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau, \quad (N_\Gamma\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t + \alpha_k}d\tau, \quad t \in \Gamma.$$

As

$$\mu\frac{a_j(t-0) + b_j(t-0)}{a_j(t-0) - b_j(t-0)} + (1-\mu)\frac{a_j(t+0) + b_j(t+0)}{a_j(t+0) - b_j(t+0)} \neq 0$$
$$(t \in \Gamma, 0 \leqslant \mu \leqslant 1, j = 1, 2),$$

then the operators $a_j I + b_j S_\Gamma$ $(j = 1, 2)$ are Noetherian in the space $L_2(\Gamma)$. Then equality (8) (taking into account the compactness of the operators M_k and N_k) implies that the operator A is Noetherian in the space $L_2(\Gamma)$. The resulting contradiction proves that the operator M_k is not compact in the space $L_2(\Gamma)$. Since, in addition, the operator M_k is bounded in all spaces $L_p(\Gamma)$ $(1 < p < \infty)$, by virtue of M. Krasnoselsky's theorem [6] M_k is not compact in any space $L_p(\Gamma)$. ■

3. Main result

In order to present the main results, we need to introduce some notation.

Let $\Gamma_k = \{\zeta : \zeta = t - \alpha_k, t \in \Gamma\}$ and $\tilde{\Gamma}_k = \{\zeta : \zeta = t + \alpha_k t \in \Gamma\}$. If the contour Γ_k has no points in common with Γ, then, obviously, the operator M_k , defined by equality (6) is completely continuous in the space $L_p(\Gamma)$ and does not affect the Noetherian character of singular operators of the form $A_0 = aP + bQ + T$ $(P = 1/2(I + S), Q = 1/2(I - S))$. In this connection, in what follows we will assume that the numbers $\alpha_k (k = 1, \ldots, m)$ are such that $\Gamma \cup \Gamma_k \neq \emptyset$. For the sake of simplicity, we will assume that Γ is the unit circle: $\Gamma = \{t : |t| = 1\}$. We also note here that the results of the paper are valid for any closed Lyapunov contour Γ, with the property that Γ and Γ_k intersect at a finite number of points.

Let $t_k^{(1)}$, $t_k^{(2)}$ be the intersection points of the contours Γ and Γ_k $(k = 1, \ldots, m)$ and $t_k^{(3)}$, $t_k^{(4)}$ points of intersection of the contours Γ and $\tilde{\Gamma}_k$: $t_k^{(3)} = t_k^{(1)} + \alpha_k$ and $t_k^{(4)} = t_k^{(2)} + \alpha_k$. Let N_k denotes the set of all functions from $L_\infty(\Gamma)$, that are continuous in some neighborhoods $u(t_k^{(j)})$ of points $t_k^{(j)}$ $(j = 1, 2, 3, 4)$. Let $a \in N_k$ and $u_k^{(j)}$ $(j = 1, 2, 3, 4)$ be some neighborhoods of points $t_k^{(j)}$, in which the function $a(t)$ is continuous. Put $\gamma_k = \cup_{j=1}^{4} u_k^{(j)}$.

Theorem 3.1. *Let $a \in N_k$, then there exists a function $a_k \in N_k$ such that $a_k(t) = 1$ for $t \in \Gamma \setminus \gamma_k$ and the operator $N = M_k a I - a_k M_k$ is completely continuous in $L_p(\Gamma)$.*

We denote by $l_k^{(1)}$ $(\tilde{l}_k^{(1)})$ the part of the contour Γ, lying inside the region bounded by the contour Γ_k $(= \{\xi : \xi = t - \alpha_k, t \in \Gamma\}$ $\left(\tilde{\Gamma}_k (= \{\zeta : \zeta = t + \alpha_k, t \in \Gamma\})\right)$, and let $(l_k^{(2)} = \Gamma \setminus l_k^{(1)})$ $(\tilde{l}_k^{(2)} = \Gamma \setminus \tilde{l}_k^{(1)})$.

Theorem 3.2. *The following equalities hold*

$$M_k S = h_k M_k, \qquad S M_k = M_k \tilde{h}_k I,$$

where

$$h_k(t) = \begin{cases} 1, & \text{for } t \in l_k^{(1)} \\ -1, & \text{for } t \in l_k^{(2)} \end{cases}, \qquad \tilde{h}_k(t) = \begin{cases} 1, & \text{for } t \in \tilde{l}_k^{(1)} \\ -1, & \text{for } t \in \tilde{l}_k^{(2)} \end{cases}.$$

Corollary 3.1. *Theorem 2 implies the following equalities*

$$M_k P = \tilde{\delta}_k M_k, M_k Q = \left(1 - \tilde{\delta}_k\right) M_k, P M_k = M_k \tilde{\delta}_k I, Q M_k = M_k \left(1 - \tilde{\delta}_k\right) I, \tag{8}$$

where $\delta_k = \frac{1+h_k}{2}$ and $\tilde{\delta}_k = \frac{1+\tilde{h}_k}{2}$.

In what follows, we will assume that the numbers α_k $(k = 1, \ldots, m)$ are such that $\Gamma_j \cup \Gamma \cup \tilde{\Gamma}_k = \emptyset$ $(j, k = 1, \ldots, m)$.

Theorem 3.3. *Let $a \in L_\infty(\Gamma)$, then the operators $M_j a M_k$ $(j, k = 1, \ldots, m)$ are completely continuous in the space $L_p(\Gamma)$. If $a, b \in N_k$, then the following operators are also completely continuous:*

$$PaQbM_k, \quad QaPbM_k, \quad M_k aPbQ, \quad M_k aQbP, \quad PaM_k bQ, \quad QaM_k bP.$$

Theorem 3.4 (Main)**.** *Let $a, b \in \cup_{1 \le k \le m} N_k$ and $c_k \in L_\infty(\Gamma)$. In order for the operator*

$$A = aP_\Gamma + bQ_\Gamma + \sum_{k=1}^{m} c_k M_k \tag{9}$$

to admit regularization in the space $L_p(\Gamma)$ it is necessary and sufficient that regularization was allowed by the operator

$$A_0 = aP + bQ. \tag{10}$$

If the operator A_0 admits regularization, then

$$\operatorname{Ind} A = \operatorname{Ind} A_0. \tag{11}$$

The proof of this theorem uses the following lemma.

Lemma 3.1. *The operator $H = I + \sum_{k=1}^{m} c_k M_k$ admits regularization in the space $L_p(\Gamma)$ and its index is zero.*

Example 3.1. *Let $\alpha_1 = 2$ and $\alpha_2 = -2$. In this case $\tilde{\Gamma}_1 = \Gamma_2$, $\tilde{\Gamma}_2 = \Gamma_1$, $\Gamma_1 \cup \Gamma \cup \tilde{\Gamma}_2 = \{-1\}$ and*

$$(M_1\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t - 2} d\tau, \quad (M_2\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t + 2} d\tau.$$

We denote by K the operator $K = M_1 + M_2$ and by N the set of piecewise continuous functions on Γ and continuous at the points $\tau = \pm 1$.

Theorem 3.5. *Let $a, b \in N$. For the operator $A = aI + bS + K$ to be Noetherian in the space $L_p(\Gamma)$, it is necessary and sufficient that the operator $A_0 = aI + bS$ to have the same property. If operator A_0 is Noetherian, then* $\operatorname{Ind} A = \operatorname{Ind} A_0$.

The proof of this theorem is based on a number of properties of the operator K, which we establish in the following lemmas.

Lemma 3.2. *For every function $h \in N$ there is a function $\tilde{h} \in N$ such that the operator $Kh - \tilde{h}K$ is compact in $L_p(\Gamma)$. Moreover, if $h(t \pm 0) \neq 0$ ($t \in \Gamma$), then $\tilde{h}(t \pm 0) \neq 0$ too.*

This statement is easily deduced from [5].

Lemma 3.3. *The following relations are valid:*

$$SK = K, \qquad KS = -K, \qquad K^2 = 0. \tag{12}$$

Proof. Let $\varphi(t) = \sum_{k=-n}^{n} a_k t^k$ be a trigonometric polynomial, $\varphi_+(t) = \sum_{k=0}^{n} a_k t^k$ and $\varphi_-(t) = \sum_{k=-n}^{-1} a_k t^k$. Then $(S\varphi)(t) = \varphi_+(t) - \varphi_-(t)$ and for each point $t \in \Gamma \setminus \{-1, 1\}$ the equality

$$(K\varphi)(t) = -2\sum_{k=-n}^{-1} a_k \left[(t+2)^k + (t-2)^k\right]$$

is true.

It is easy to show that $SK\varphi = K\varphi$ and since the set of trigonometric polynomials is dense in the space $L_p(\Gamma)$, then $SK = K$. Further we have

$$KS\varphi = K(\varphi_+ - \varphi_-) = 2\sum_{k=-n}^{-1} a_k \left[(t+2)^k + (t-2)^k\right] = -K\varphi.$$

So $KS = -K$. The last relation from (12) easily follows from the first two lemmas. ■

Note also that the statement of Lemma 1 holds for operators of the form $F = I + fK$ $(f \in N)$.

Lemma 3.4. *The operator $F = I + fK$ is Noetherian and* $\operatorname{Ind} F = 0$.

Proof of the theorem 3.5. If the operator $A_0 = aI + bS$ is Noetherian in $L_p(\Gamma)$, then (see [2]) the conditions $a(t \pm 0) + b(t \pm 0) \neq 0$ and $a(t \pm 0) - b(t \pm 0) \neq 0$ $(t \in \Gamma)$ are true.

Let f denote the function $f = 1/(a + b)$ $(\in N)$. Based on Lemmas 3.2 and 3.3, we see that the operator A can be represented in the form

$$A = A_0(I + fK) + T,$$

where T is a compact operator. By Lemma 3.4, the operator $F = I + fK$ is Noetherian and $\operatorname{Ind} F = 0$. Therefore, operator A is also Noetherian and $\operatorname{Ind} A = \operatorname{Ind} A_0$. The sufficiency has been proven.

Let us prove the necessity of the conditions of the theorem. Suppose that the operator $A = aI + bS + K$ is Noetherian, and the operator $A_0 = aI + bS$ is not Noetherian.

Let ε be a positive number such that all operators A', satisfying the condition $\|A - A'\| < \varepsilon$, are Noetherian and $\operatorname{Ind} A' = \operatorname{Ind} A$. Just as in [3], we can construct two Noetherian operators $B_j = a_j I + b_j S$ $(j = 1, 2)$, such that $\|A_0 - B_j\| < \varepsilon$ and $\operatorname{Ind} B_1 \neq \operatorname{Ind} B_2$. By virtue of what was proved above, the operators $A_j = a_j I + b_j S + K$ $(j = 1, 2)$ are Noetherian and $\operatorname{Ind} A_j = \operatorname{Ind} B_j$. Therefore, $\operatorname{Ind} A_1 \neq \operatorname{Ind} A_2$. And since $\|A - A_j\| < \varepsilon$ $(j = 1, 2)$, then $\operatorname{Ind} A_1 = \operatorname{Ind} A_2$. The resulting contradiction proves that the operator A_0 is Noetherian. ■

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