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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Multifrequency system with multipoint and integral conditions

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Abstract. For the multifrequency system of equations with linear delays and multipoint and integral conditions, the existence and uniqueness of the solution in space is proved. The method of averaging over fast variables is substantiated and the error of the method is estimated, which obviously depends on the small parameter. The obtained result is illustrated by a model example.
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Sistem multifrecvenţă cu multipunct şi condiţii integrale

Rezumat. Pentru sistemul multifrecvenţă de ecuaţii cu intârzieri liniare şi multipunct şi condiţii integrale, se demonstrează existenţa şi unicitatea soluţiei în spaţiu. Se fundamenteaza metoda de mediere asupra variabilelor rapide si se estimeaza eroarea metodei, care depinde evident de parametrul mic. Rezultatul obţinut este ilustrat printr-un exemplu.
Cuvinte cheie: sistem multifrecvenţă, condiţie integrală, metoda medierii, parametru mic, rezonanţă, estimarea erorilor, variabile lente si rapide.

1. Introduction

An important problem in nonlinear mechanics is the study of oscillatory systems, which in the process of evolution pass through resonance. In many cases, such systems are described by differential equations of the form

$$\frac{da}{d\tau} = \varepsilon X(\tau, a, \varphi), \quad \frac{d\varphi}{d\tau} = \omega(\tau, a) + \varepsilon Y(\tau, a, \varphi), \tag{1}$$

where $a \in D \subset \mathbb{R}^n$, $\varphi \in \mathbb{R}^m$, $0 \le \varepsilon$ – small parameter, $\tau = \varepsilon t$, vector-functions X and Y 2π-periodic by variables $\varphi_1, \ldots, \varphi_m$.

An effective method of research and construction of approximate solutions of such systems is the method of averaging on the cube of periods over fast variables φ_ν [1, 2, 3]. As a result, we obtain a much simpler system of equations

$$\frac{d\overline{a}}{dt} = \varepsilon X_0(\tau, \overline{a}), \quad \frac{d\overline{\varphi}}{dt} = \omega(\tau, \overline{a}) + \varepsilon Y_0(\tau, \overline{a}), \tag{2}$$

Such a procedure does not always lead to the correct result and the deviation of slow variables $\|a(t,\varepsilon)-\overline{a}(\varepsilon t)\| = O(1)$ on the time intervals of the length $O(\varepsilon^{-1})$. The reason for this is the resonance of frequencies, the condition of which is

$$\big(k,\omega(\tau,a)\big) := k_1\omega_1(\tau,a)+\cdots+k_m\omega_m(\tau,a) \cong 0,\ \ \|k\| \neq 0. \tag{3}$$

Therefore, it is necessary to impose additional conditions to ensure that the error of the averaging method was $O(\varepsilon^{\alpha})$, $\alpha > 0$, on the time interval of the length $[0, L\varepsilon^{-1}]$, $L = const > 0$. Such results have been obtained in many works, for example [1, 2, 3].

To adequately describe the processes in applied problems, it is necessary to take into account the delay factor. Multifrequency systems with delay and initial conditions and various types of integral conditions were studied in [4, 5, 6] and others. For such systems, the effect of delay was found, in particular, on the frequency resonance condition [7].

In addition to substantiating the method of averaging and establishing non-improving estimates for the error of the method of the order ε^{α}, $0 < \alpha \leq 1/m$, sufficient conditions for the existence and uniqueness or existence of the solution of the original problems are obtained.

In this paper, the following results are obtained for a multifrequency system with a finite number of delays and multipoint and integral conditions.

2. Methods and materials used

Consider a multifrequency system of equations of the form

$$\frac{da}{d\tau} = X(\tau, a_{\Lambda}, \varphi_{\Theta}), \tag{4}$$

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_{\Lambda}, \varphi_{\Theta}), \tag{5}$$

the solution of which satisfies the condition

$$\sum_{\nu=1}^{r} \alpha_{\nu} a(x_{\nu}) = \int_{\tau_1}^{\tau_2} f(\tau, a_{\Lambda}, \varphi_{\Theta})d\tau, \tag{6}$$

$$\sum_{\nu=1}^{r} \beta_{\nu} \varphi(x_{\nu}) = \int_{\tau_1}^{\tau_2} g(\tau, a_{\Lambda}, \varphi_{\Theta})d\tau, \tag{7}$$

where $0 \leq x_1 < \cdots < x_r \leq L$, $0 \leq \tau_1 < \tau_2 \leq L$, $\tau = \varepsilon t \in [0, L]$, $a \in D$ – limited closed area in $\mathbb{R}^n$, $\varphi \in \mathbb{R}^m$, parameter $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, $a_{\Lambda} = (a_{\lambda_1}, \ldots, a_{\lambda_p})$, $\varphi_{\Theta} = (\varphi_{\theta_1}, \ldots, \varphi_{\theta_q})$, $0 < \lambda_1 < \cdots < \lambda_p \leq 1$, $0 < \theta_1 < \cdots < \theta_q \leq 1$, α_{ν}, β_{ν} – given numbers, $a_{\lambda_i}(\tau) = a(\lambda_i\tau)$, $\varphi_{\theta_j}(\tau) = \varphi(\theta_j\tau)$. Vector-functions X, Y, f and g defined and smooth enough for all variables in the area $G = [0, L] \times D^p \times R^{mq}$, 2π-periodic by components of the vector φ_Θ.

Multifrequency ODE systems with integrated conditions on $[0, L]$ by the averaging method were first studied in [1]. Problems with integral conditions of various kinds are actively studied and applied in applied problems [8, 9, 10].

Corresponding (4)–(7) averaged over fast variables $d\varphi_{\theta_1} \dots d\varphi_{\theta_q}$ the problem takes the form

$$\frac{d\overline{a}}{d\tau} = X_0(\tau, \overline{a}_\Lambda), \tag{8}$$

$$\frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y_0(\tau, \overline{a}_\Lambda), \tag{9}$$

$$\sum_{\nu=1}^{r} \alpha_\nu \overline{a}(x_\nu) = \int_{\tau_1}^{\tau_2} f_0(\tau, \overline{a}_\Lambda) d\tau, \tag{10}$$

$$\sum_{\nu=1}^{r} \beta_\nu \overline{\varphi}(x_\nu) = \int_{\tau_1}^{\tau_2} g_0(\tau, \overline{a}_\Lambda) d\tau. \tag{11}$$

The averaged problem is much simpler compared to the original problem (4)–(7). In particular, the problem (8), (10) for slow variables $\overline{a}$ integrates independently of fast variables $\overline{\varphi}$. If the solution $\overline{a}(\tau)$ and the initial value $\overline{\varphi}(0) = \overline{\psi}$ at $\tau = 0$ found, then finding fast variables is reduced to the problem of integration. Let

$$b := \sum_{\nu=1}^{r} \beta_\nu \neq 0. \tag{12}$$

Then the initial value $\overline{\psi} = \overline{\varphi}(0; \overline{y}, \psi, \varepsilon)$ of the problem (9)–(11) takes the form

$$\overline{\psi} = b^{-1} \Big[\int_{\tau_1}^{\tau 2} g_0(\tau, \overline{a}_\Lambda(\tau)) d\tau - \sum_{\nu=1}^{r} \beta_\nu \int_{0}^{\tau_\nu} \big(\frac{\omega(\tau)}{\varepsilon} + Y_0(\tau, \overline{a}_\Lambda(\tau)) \big) \Big] d\tau,$$

the solution of the problem (9), (11) is

$$\overline{\varphi}(\tau; \overline{y}, \overline{\psi}, \varepsilon) = \overline{\psi} + \overline{\varphi}(\tau; \overline{y}, 0, \varepsilon).$$

Let us investigate the existence and uniqueness of a continuous differentiated solution of the problem (4)–(7) for a fairly small $\varepsilon \in (0, \varepsilon^*)$, $\varepsilon^* \leq \varepsilon_0$, assuming the existence of the unique solution to the averaged problem. We will also construct an estimate of the method of averaging of the form

$$\|\kappa(\tau; y, \psi, \varepsilon) - \overline{\kappa}(\tau; \overline{y}, \overline{\psi}, \varepsilon)\| \leq c_1 \varepsilon^\alpha, \tag{13}$$

where $\kappa(\tau; y, \psi, \varepsilon) = \big(a(\tau; y, \psi, \varepsilon), \varphi(\tau; y, \psi, \varepsilon)\big), \overline{\kappa}(\tau; \overline{y}, \overline{\psi}, \varepsilon) = \big(\overline{a}(\tau; \overline{y}), \overline{\varphi}(\tau; \overline{y}, \overline{\psi}, \varepsilon)\big)$, $\alpha = (mq)^{-1}$, $c_1 = const > 0$.

The resonance condition at the point τ for the system of equations (4)–(7) is formulated in [7] and has the form

$$\gamma_k(\tau) := \sum_{\nu=1}^{q} \theta_\nu\big(k_\nu, \omega_\nu(\theta_\nu \tau)\big) = 0,$$

$k_\nu \in \mathbb{Z}$, $\|k_1\| + \cdots + \|k_m\| \neq 0$.

3. Obtained results and discussion

Let $\overline{a}(0;\overline{y}) = \overline{y}$, $Q(\overline{y})$ – matrix of the order n of the form

$$Q(\overline{y}) = \sum_{\nu=1}^{r} \alpha_\nu \frac{\partial \overline{a}(x_\nu;\overline{y})}{\partial \overline{y}} - \sum_{i=1}^{p} \int_{\tau_1}^{\tau_2} \frac{\partial f_0(\tau, \overline{a}_\Lambda(\tau;\overline{y})}{\partial a_{\lambda_i}} \frac{\partial \overline{a}(\lambda_i \tau, \overline{y})}{\partial \overline{y}} d\tau.$$

Theorem 3.1. *Let the conditions be met:*

1) *vector-functions X and Y 2π-periodic by components of the vector φ_Θ, for each $\varepsilon \in (0, \varepsilon_0]$, a sufficient number of times differentiated by variables τ, a_{λ_i}, φ_{θ_j}, namely $F \in C^l_\tau(G, \sigma)$, $F \in C^l_{a_{\lambda_i}}(G, \sigma)$, $F \in C^{l-1}_{\varphi_{\theta_j}}(G, \sigma)$, $F := (X, Y)$, where $l \geq qm + 1$, $G = [0, L] \times D^p \times \mathbb{R}^m q$, by constant $\sigma > 0$ limited vector-functions X, Y and their partial derivatives in G;*
2) *$\omega_\nu \in C^{mq-1}[0, L]$, $\nu = \overline{1, m}$ and Wronsky determinant is not zero for $\tau \in [0, L]$;*
3) *there exists the unique solution to the averaged problem* (8),(10)*, there is only one solution to the averaged problem $\overline{a}(\tau;\overline{y})$ of which is the area D with some ρ-circumference;*
4) *the inequality holds* (12) *and $detQ(\overline{y}) \neq 0$.*

Then for rather small $\varepsilon^* \in (0, \varepsilon_0]$ there is the unique solution to the problem (4)–(7) *with initial conditions $\overline{y} + \mu \in D_1 \subset D$ and $\overline{\psi} + \xi \in \mathbb{R}^m$,*

$$\|\mu\| \leq c_2 \varepsilon^\alpha, \quad \|\xi\| \leq c_3 \varepsilon^\alpha, \quad \alpha = (mq)^{-1},$$

for which inequality holds (13) *for each $\varepsilon \in (\varepsilon_0, \varepsilon^*]$ and $\tau \in [0, L]$, $c_1 > 0$ and does not depend on ε.*

The proof of Theorem 3.1 is based on the application of the oscillatory integral estimate

$$I_k(t, \varepsilon) = \int_0^t g(s, \varepsilon)\Big[\exp\Big(\frac{i}{\varepsilon}\int_0^s \gamma_k(z) dz\Big)\Big] ds,$$

as for $\tau \in [0, L]$, $\varepsilon \in (0, \varepsilon_0]$, $g \in C^1[0, L]$ $\forall \varepsilon \in (0, \varepsilon_0]$ and the fulfillment of condition 4 of the theorem takes the form

$$\|I_k(t,\varepsilon)\| \leq \sigma_1 \varepsilon^{\alpha}\Big(\sup\|g(s,\varepsilon)\| + \frac{1}{\|k\|_\Theta}\sup\Big\|\frac{dg(s,\varepsilon)}{ds}\Big\|\Big), \tag{14}$$

where $\sigma_1 > 0$ and does not depend on ε, $\|k\|_\Theta = \sum\limits_{\nu=1}^{q} \theta_\nu \|k_\nu\|$.

Oscillatory integrals for combinational frequencies $\gamma_k := k_1\omega_1(\tau) + \cdots + k_m\omega_m(\tau)$, $k \neq 0$, were built and used for multifrequency ODE systems in the works of A.M. Samoilenko and R.I. Petryshyn [1].

When proving the theorem, a simpler scheme of proving the existence and uniqueness of the solution of the problem is proposed (4)–(7) and weaker restrictions on the right side (4), (5).

Proof. If conditions 1-3 are met, it is proved [4], that the existence of the unique solution $\kappa(\tau; y, \psi, \varepsilon)$, $y \in D_1 \subset D$, every point $y \in D_1$ enters D together with ρ-circumference. Also for $y = \overline{y} + \mu \in D_1$

$$\|\kappa(\tau; y, \psi, \varepsilon) - \overline{\kappa}(\tau; y, \psi, \varepsilon)\| \leq \overline{c}_1 \varepsilon^{\alpha}, \tag{15}$$

$(\tau, \varepsilon) \in [0, L] \times (0, \varepsilon_0]$, $\overline{c}_1 > 0$ and does not depend on ε.

Let $\mu \in \mathbb{R}^n$ and

$$\|\mu\| \leq c_4 \varepsilon^{\alpha} \leq \rho/2. \tag{16}$$

The value of c_4 will be defined below. It follows that when $\varepsilon \leq \varepsilon_1 = \min(\varepsilon_0, (\rho/2c_4)^{mq})$ the solution $a(\tau; \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon)$ lies in D with $(\rho/2)$-circumference.

From the equations (4), (8) and smoothness of vector-functions X and f for the vector μ in the initial condition for the solution $a(\tau; \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon)$ the equation is received of the form

$$\mu = \Phi_1(\mu, \xi, \varepsilon),$$

where vector-function

$$\Phi_1(\mu, \xi, \varepsilon) = -Q^{-1}(\overline{y})\Big[\sum_{\nu=1}^{r} \alpha_\nu \big(R_{1,\nu}(\mu) + (a(x_\nu; \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon) - \overline{a}(x_\nu; \overline{y} + \mu)) +$$
$$+ R_2(\mu, \xi, \varepsilon)\big) + R_3(\mu)\Big],$$

$$R_{1,\nu}(\mu) = \overline{a}(x_\nu; \overline{y} + \mu) - \overline{a}(x_\nu; \overline{y}) - \frac{\partial \overline{a}(x_\nu; \overline{y})}{\partial \overline{y}}\mu,$$

$$R_2(\mu, \xi, \varepsilon) =$$

$$= \int_{\tau_1}^{\tau_2} \big[f(\tau, a_\Lambda(\tau, \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon), \varphi_\Theta(\tau, \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon)) - f_0(\tau, \overline{a}_\Lambda(\tau; \overline{y} + \mu))\big] d\tau,$$

$$R_3(\mu)=\int\limits_{\tau_1}^{\tau_2}\big[f_0(\tau,\overline{a}_\Lambda(\tau;\overline{y}+\mu))-f_0(\tau,\overline{a}_\Lambda(\tau;\overline{y}))-\sum_{j=1}^{p}\frac{\partial f_0(\tau,\overline{a}_\Lambda(\tau;\overline{y}))}{\partial\overline{a}_{\lambda_j}}R_{1,\nu}(\mu)\big]d\tau.$$

For esimation $R_{1,\nu}(\mu)$ let us apply the estimate [2]

$$\|h(b)-h(a)-\sum_{\nu=1}^{n}\frac{\partial h(a)}{\partial x_\nu}(b_\nu-a_\nu)\|\le C\|b-a\|^2, \tag{17}$$

where h – vecor-function of the variable $x\in D\subset\mathbb{R}$, $h\in C^2(D)$. Then $\|R_{1,\nu}(\mu)\|\le c_{3,\nu}\|\mu\|^2$.

Let us write down R_2 in the form

$$R_2(\tau;\mu,\xi,\varepsilon)=\sum_{\|k\|>0}\int\limits_{\tau_1}^{\tau_2}f_k(\tau,a_\Lambda)exp\big(i\sum_{j=1}^{q}(k_j,\varphi_{\theta_j})\big)+$$

$$+\int\limits_{\tau_1}^{\tau_2}\big[f_0(\tau,a_\Lambda(\tau,\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon))-f_0(\tau,\overline{a}_\Lambda(\tau;\overline{y}+\mu))\big]d\tau=$$

$$=R_{2,1}(\tau;\mu,\xi,\varepsilon)+R_{2,2}(\tau;\mu,\xi,\varepsilon).$$

The estimate $R_{2,2}(\tau;\mu,\xi,\varepsilon)$ is obtained on the basis of the estimation of the deviation of the solutions of the systems (4), (5) and (8), (9) with the same initial conditions (15), and the estimate $R_{2,1}$ from the estimation of the integral (14). We get

$$\|R_{2,2}(\tau;\mu,\xi,\varepsilon)\le\int\limits_{\tau_1}^{\tau_2}\sum_{\nu=1}^{p}\|\frac{\partial f_0}{\partial a_{\lambda_\nu}}\|\cdot\|a_{\lambda_\nu}(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)-\overline{a}_{\lambda_\nu}(\tau;\overline{y}+\mu)\|d\tau\le$$

$$\le\sum_{\nu=1}^{p}\sigma_{1,\nu}\int\limits_{\tau_1}^{\tau_2}\overline{c}_1\varepsilon^\alpha d\tau=c_5\varepsilon^\nu,\quad \|R_{2,1}(\tau;\mu,\xi,\varepsilon)\|\le c_6\varepsilon^\alpha,$$

where $c_6=c_6(\sigma_1)>0$, $0<\varepsilon\le\varepsilon_2$.

Let us build an estimate $R_3(\tau;\mu)$, applying inequalities (14) and (17). We will get the estimate

$$\|R_3(\tau;\mu)\|\le\int\limits_{\tau_1}^{\tau_2}\big(\sum_{\nu=1}^{p}\sigma_{1,\nu}\|\overline{a}_{\lambda_\nu}(\tau;\overline{y}+\mu)-\overline{a}_{\lambda_\nu}(\tau;\overline{y})\|+$$

$$+\sum_{\nu=1}^{p}\sigma_{1,\nu}\|R_{1,\nu}\|\big)d\tau\le(\tau_2-\tau_1)\sum_{\nu=1}^{p}\sigma_{1,\nu}(\overline{c}(\tau_2)+c_1\|\mu\|)\|\mu\|=\overline{c}_7\|\mu\|\varepsilon^\alpha+\overline{\overline{c}}_7\|\mu\|^2,$$

where $\overline{c}_7=(\tau_2-\tau_1)\overline{c}\sum\limits_{\nu=1}^{p}\sigma_{1,\nu}$, $\overline{\overline{c}}_7=(\tau_2-\tau_1)\overline{c}_1\sum\limits_{\nu=1}^{p}\sigma_{1,\nu}$.

So, for $\Phi_1(\mu,\xi,\varepsilon)$ at $\varepsilon \le \varepsilon_3 = \min(\varepsilon_1,\varepsilon_2)$ the estimate received is

$$\|\Phi_1(\mu,\xi,\varepsilon)\| \le \|Q^{-1}(\overline{y},\overline{\psi})\|\big(\sum_{\nu=1}^{r}|\alpha_\nu|\|R_{1,\nu}\| + (c_4+c_6)\varepsilon^\alpha + \overline{c}_7\|\mu\| + \overline{\overline{c}}_7\|\mu\|^2\big) \le$$
$$\le c_8\big(\overline{c}_9 + \overline{\overline{c}}_9\|\mu\|\big)\varepsilon^\alpha + c_8\|\mu\|^2,$$

where $c_8 = (\overline{\overline{c}}_7 + c_1\sum_{\nu=1}^{r}|\alpha_\nu|)\|Q^{-1}(\overline{y},\overline{\psi})\|$, $\overline{c}_9 = (c_4+c_6)/c_8$, $\overline{\overline{c}}_9 = \overline{c}_7/c_8$.

Let in the inequality (15) $c_2 = 2c_8\overline{c}_9$, $\varepsilon \le \varepsilon_4 = \min(\varepsilon_3, (2c_8)^{-mq}, (8c_8^2\overline{c}_9)^{-mq})$. The estimate received is

$$\|\Phi_1(\mu,\xi,\varepsilon)\| \le \|\mu\| \le c_2\varepsilon^\alpha.$$

Reflection $\Phi_1 : S_1 \to S_1$, where $S_1 = \{\mu \in \mathbb{R}^n : \|\mu\| \le c_2\varepsilon^\alpha\}$.

Consider the matrix derivative

$$\frac{\partial\Phi_1(\mu,\xi,\varepsilon)}{\partial\xi} = -Q^{-1}(\overline{y})\Big[\frac{\partial}{\partial\xi}\sum_{\|k\|>0}\int_{\tau_1}^{\tau_2} f_k(\tau,a_\Lambda)exp(i\sum_{j=1}^{q}(k_j,\varphi_{\theta_j}))d\tau+$$
$$+\sum_{\nu=1}^{p}\int_{\tau_1}^{\tau_2}\frac{\partial f_0(\tau,\tilde{a}_\Lambda)}{\partial a_{\lambda_\nu}}\frac{\partial}{\partial\xi}(a_{\lambda_\nu}-\tilde{a}_\lambda)d\tau + \sum_{\nu=1}^{p}\int_{\tau_1}^{\tau_2}\frac{\partial R_{4,\nu}(\mu,\xi,\varepsilon)}{\partial\xi}d\tau\Big] =$$
$$= R_4(\mu,\xi,\varepsilon) + R_5(\mu,\xi,\varepsilon)$$

Applying the estimation of the oscillatory integral (14) and estimates similar to (15) for derivatives deviations of solutions

$$\max_j\big(\|\frac{\partial}{\partial\xi}(\varphi_{\theta_j}-\tilde{\varphi}_{\theta_j})\|, \|\frac{\partial}{\partial\xi}(a_{\lambda_\nu}-\tilde{a}_{\lambda_\nu})\|\big) \le c_{10}\varepsilon^\alpha, \tag{18}$$

for $R_4(\mu,\xi,\varepsilon)$ we will receive

$$\|R_4(\mu,\xi,\varepsilon)\| \le \|Q^{-1}(\overline{y})\|\Big[\sum_{\|k\|>0}\big(\sum_{\nu=1}^{p}\sup_{G_1}\|\frac{\partial f_k(\tau,a_\Lambda)}{\partial a_{\lambda_\nu}}\|\big)c_{10}\lambda_\nu(\tau_2-\tau_1)\varepsilon^\alpha+$$
$$+\sum_{\|k\|>0}\sup_{G_1}\|f_k(\tau,a_\Lambda)\|\sum_{j=1}^{q}|k_j|c_{10}\theta_j\varepsilon^\alpha+$$
$$+m\sum_{\|k\|>0}\big(\|k\|_\Theta\sup_{G_1}\|f_k(\tau,a_\Lambda)\| + \sup_{G_1}\|\frac{df_k(\tau,a_\Lambda)}{d\tau}\|\big)\varepsilon^\alpha\Big] = c_{11}\varepsilon^\alpha,$$

Now consider the estimate $R_5(\mu,\xi,\varepsilon)$. Since

$$\|\frac{\partial R_{5,\nu}(\mu,\xi,\varepsilon)}{\partial\xi}\| \le c_{12,\nu}\varepsilon^\alpha + c_{13,\nu}\|\xi\|, \quad \nu = \overline{1,p},$$

then, using the estimate (18), we will get

$$\|R_5(\mu,\xi,\varepsilon)\| \le \|Q^{-1}\|\Big(\sum_{\nu=1}^{p}\int_{\tau_1}^{\tau_2}\sigma_{1,\nu}c_{10}\varepsilon^{\alpha}\lambda_\nu\tau d\tau + \sum_{\nu=1}^{p}\int_{\tau_1}^{\tau_2}(c_{12,\nu}\varepsilon^{\alpha}+c_{13,\nu}\|\xi\|)d\tau\Big) =$$

$$= 0.5\|Q^{-1}\|\big((\tau_2-\tau_1)^2 c_{10}\varepsilon^{\nu}\sum_{\nu=1}^{p}\sigma_{1,\nu}\lambda_\nu +$$

$$+\varepsilon^{\nu}(\tau_2-\tau_1)\sum_{\nu=1}^{p}c_{12,\nu} + \|\xi\|(\tau_2-\tau_1)\sum_{\nu=1}^{p}c_{13,\nu}\big) = \overline{c}_{14}\varepsilon^{\alpha} + c_{15}\|\xi\|.$$

Thus, in the end we have

$$\|\frac{\partial\Phi_1(\mu,\xi,\varepsilon)}{\partial\xi}\| \le c_{14}\varepsilon^{\alpha} + c_{15}\|\xi\|, \quad c_{14} = c_{11} + \overline{c}_{14}.$$

Choose $\xi \in \mathbb{R}^m$ and ε so, that

$$\|\xi\| \le c_{16}\varepsilon^{\alpha}, \quad \varepsilon \le \varepsilon_5 = \min(\varepsilon_4, (8c_{14})^{-mq}),$$

where $c_{16} = \min(c_{14}/c_{15}, c_{18})$, where c_{18} – will be indicated below.

Then we get

$$\|\frac{\partial\Phi_1(\mu,\xi,\varepsilon)}{\partial\xi}\| \le \frac{1}{4} \tag{19}$$

for each $\varepsilon \le \varepsilon_5$, $\|\mu\| \le c_2\varepsilon$, and $\|\xi\| \le c_{16}\varepsilon^{\alpha}$.

Consider the derivative of the vector function

$$\frac{\partial\Phi_1(\mu,\xi,\varepsilon)}{\partial\mu} = -Q^{-1}\Big[\sum_{\nu=1}^{r}\alpha_\nu\Big(\frac{\partial}{\partial\mu}R_{1,\nu}(\mu)+$$

$$+\frac{\partial}{\partial\mu}(a(x_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(x_\nu;\overline{y}+\mu))\Big) + \frac{\partial}{\partial\mu}R_2(\mu,\xi,\varepsilon) + \frac{\partial}{\partial\mu}R_3(\mu)\Big].$$

Since $\|R_{1,\nu}(\mu)\| = O(\|\mu\|^2)$, then $\|\frac{\partial R_{1,\nu}(\mu)}{\partial\mu}\| = O(\mu)$, therefore

$$\|\frac{\partial R_{1,\nu}(\mu)}{\partial\mu}\| \le c_{17,\nu}\|\mu\|. \tag{20}$$

Let us construct an estimate for the matrix derivative

$$\frac{\partial R_2(\mu,\xi,\varepsilon)}{\partial\mu} = \int_{\tau_1}^{\tau_2} \frac{\partial}{\partial\mu}\big(f_0(\tau,a_\Lambda(\tau)) - f_0(\tau,\tilde a_\Lambda(\tau))\big)d\tau+$$

$$+\sum_{\|k\|>0}\int_{\tau_1}^{\tau_2}\frac{\partial}{\partial\mu}\big[\big(f_k(\tau,a_\Lambda(\tau)) - f_k(\tau,\tilde a_\Lambda(\tau))\big)exp\big(i\sum_{j=1}^{q}(k_j,\varphi_{\theta_j}(\tau))\big)\big]d\tau+$$

$$+\sum_{\|k\|>0}\int_{\tau_1}^{\tau_2}\frac{\partial}{\partial\mu}\big[f_k(\tau,\tilde a_\Lambda(\tau))exp\big(i\sum_{j=1}^{q}(k_j,\varphi_{\theta_j}(\tau))\big)\big]d\tau =$$

$$= R_{21}(\mu,\xi,\varepsilon) + R_{22}(\mu,\xi,\varepsilon) + R_{23}(\mu,\xi,\varepsilon).$$

From the smoothness of the vector function $f_0(\tau,a_\Lambda)$ by a_Λ we obtain

$$\|R_{21}(\mu,\xi,\varepsilon)\| \le \sum_{\nu=1}^{p}\sigma_{3,\nu}c_{18}\lambda_\nu(\tau_2-\tau_1)\varepsilon^\alpha + c_{19}\varepsilon^\alpha = c_{20}\varepsilon^\alpha.$$

The estimate for $R_{22}(\mu,\xi,\varepsilon)$ is build as for $R_{21}(\mu,\xi,\varepsilon)$:

$$\|R_{22}(\mu,\xi,\varepsilon)\| \le \sum_{\|k\|>0}\int_{\tau_1}^{\tau_2}\sum_{\nu=1}^{p}\big(\|\frac{\partial f_k(\tau,\tilde a_\Lambda)}{\partial a_{\lambda_\nu}}\|\cdot\|a_\Lambda(\tau)-\tilde a_\Lambda(\tau)\|+$$

$$+\|\frac{\partial}{\partial\mu}R_{2\nu k}(\mu,\xi,\varepsilon)\|\big)d\tau \le c_{21}\varepsilon^\alpha.$$

Further we have

$$\frac{\partial R_{23}(\mu,\xi,\varepsilon)}{\partial\mu} = \sum_{\|k\|>0}\int_{\tau_1}^{\tau_2}\big\{\sum_{\nu=1}^{p}\frac{\partial f_k(\tau,\tilde a_\Lambda)}{\partial\tilde a_{\lambda_\nu}}\frac{\partial\tilde a_{\lambda_\nu}(\tau)}{\partial\mu}exp\big(i\sum_{j=1}^{q}(k_j,\varphi_{\theta_j}(\tau))\big)+$$

$$+if_k(\tau,\tilde a_\Lambda(\tau))\big[\sum_{j=1}^{q}(k_j,\frac{\partial}{\partial\mu}\big(\varphi_{\theta_j}(\tau)-\tilde\varphi_{\theta_j}(\tau)\big)\big]exp\big(i\sum_{j=1}^{q}(k_j,\varphi_j(\tau))\big)+$$

$$+i\sum_{j=1}^{q}\big(k_j,\frac{\partial}{\partial\mu}\tilde\varphi_{\theta_j}(\tau)\big)exp\big(i\sum_{j=1}^{q}(k_j,\varphi_j(\tau))\big)\big\}d\tau = R_{23}^{(1)}(\mu,\xi,\varepsilon) + R_{23}^{(2)}(\mu,\xi,\varepsilon) + R_{23}^{(3)}(\mu,\xi,\varepsilon).$$

Applying the estimates (17) and (18), we will obtain

$$\|R_{23}^{(2)}(\mu,\xi,\varepsilon)\| \le \sum_{\|k\|>0}\int_{\tau_1}^{\tau_2}\sup_{G_1}\|f_k(\tau,a_\Lambda)\|\sum_{j=1}^{q}\|k_j\|c_{18}\theta_j\varepsilon_\alpha \le$$

$$\le \big(0.5(\tau_2^2-\tau_1^2)c_{18}\sum_{\|k\|>0}\|k\|_\Theta\sup_{G_1}\|f_k(\tau,a_\Lambda)\|\big)\varepsilon^\alpha = c_{22}\varepsilon^\alpha.$$

Let

$$F_k(\tau)=\sum_{\nu=1}^{p}\frac{\partial f_k(\tau,\tilde a_\Lambda(\tau;\overline{y}+\mu))}{\partial \tilde a_{\lambda_\nu}}\frac{\partial \tilde a_{\lambda_\nu}(\tau;\overline{y}+\mu)}{\partial\mu}.$$

There exist constants $\overline{c}_{23}$ and $\overline{\overline{c}}_{23}$ such as

$$\|F_k(\tau)\|=\sum_{\nu=1}^{p}\sup_{G_1}\|\frac{\partial f_k(\tau,a_\Lambda)}{\partial a_{\lambda_\nu}}\|\|\frac{\partial a_{\lambda_\nu}(\tau)}{\partial\mu}\|\le\overline{c}_{23},\quad \|\frac{dF_k(\tau)}{d\tau}\|=\le\overline{\overline{c}}_{23}.$$

Then $\|R_{23}^{(1)}(\mu,\xi,\varepsilon\|+\|R_{23}^{(3)}(\mu,\xi,\varepsilon\|\le c_{24}\varepsilon^\alpha$, where $c_{24}=c_{24}(\overline{c}_{23},\overline{\overline{c}}_{23})$.

Summarizing the estimates obtained, we receive

$$\|\frac{\partial R_{23}(\mu,\xi,\varepsilon)}{\partial\mu}\|\le(c_{22}+c_{24})\varepsilon^\alpha,$$

and for the norm of the derivative Φ_1 by μ

$$\|\frac{\partial\Phi_1(\mu,\xi,\varepsilon)}{\partial\mu}\|\le\|Q^{-1}(\overline{y})\|\big[\big(\sum_{\nu=1}^{r}|\alpha_\nu|c_{12,\nu}\big)\|\mu\|+$$

$$+\big(c_{18}(\sum_{\nu=1}^{r}|\alpha_\nu|x_\nu)+c_{19}+c_{20}+c_{21}+c_{22}+c_{24}\big)\varepsilon^\alpha\big]=\overline{c}_{25}\|\mu\|+\overline{\overline{c}}_{25}\varepsilon^\alpha.$$

Since $\|\mu\|\le c_2\varepsilon^\alpha$, then

$$\|\frac{\partial\Phi_1(\mu,\xi,\varepsilon)}{\partial\mu}\|\le(c_2\overline{c}_{25}+\overline{\overline{c}}_{25})\varepsilon^\alpha\le\frac{1}{4},\tag{21}$$

if $\varepsilon\le\varepsilon_6=min(\varepsilon_5,(4(c_2\overline{c}_{25}+\overline{\overline{c}}_{25}))^{-1}$.

Consider the question of finding the value of a vector ξ.

Before constructing the equation for the ξ of the form

$$\xi=\Phi_2(\mu,\xi,\varepsilon)\tag{22}$$

and estimating the norm of the vector-function Φ_2 and its derivatives by vector variables μ,ξ, let us introduce the following notation:

$$P(\overline{y},\overline{\psi})=\sum_{\nu=1}^{r}\beta_\nu\sum_{i=1}^{p}\int_0^{x_\nu}\frac{\partial Y_0(\tau,\overline{a}_\Lambda)}{\partial a_{\lambda_i}}\frac{\partial\overline{a}_{\lambda_i}}{\partial\overline{y}}d\tau-\sum_{i=1}^{p}\int_{\tau_1}^{\tau_2}\frac{\partial g_0(\tau,\overline{a}_\Lambda)}{\partial a_{\lambda_i}}\frac{\partial\overline{a}_{\lambda_i}}{\partial\overline{y}}d\tau,$$

$$\Psi_k=\sum_{\nu=1}^{q}\big(k_\nu,\varphi_{\theta_\nu}\big).$$

Subtract from (7) relevant averaging conditions (11) and separate the linear parts in them by μ. After the transformation, we obtain

$$b\xi+P(\overline{y},\overline{\psi})\mu=\Phi(\mu,\xi,\varepsilon),\tag{23}$$

where

$$\Phi(\mu,\xi,\varepsilon)=\int\limits_{\tau_1}^{\tau_2}\Big[\big(g_0(\tau,a_\Lambda)-g_0(\tau,\tilde a_\Lambda)\big)+P_1(\tau,\mu)+\sum_{k\neq 0}g_k(\tau,a_\Lambda)\exp\big(i\Psi_k\big)\Big]d\tau-$$

$$-\sum_{\nu=1}^{r}\beta_\nu\Big[\int\limits_0^{x_\nu}\big(Y_0(\tau,a_\Lambda)-Y_0(\tau,\tilde a_\Lambda)\big)+P_{2,\nu}(\tau,\mu)+\sum_{k\neq 0}Y_k(\tau,a_\Lambda)\exp\big(i\Psi_k\big)\Big]d\tau=$$

$$=R_6(\mu,\xi,\varepsilon)+R_7(\mu,\xi,\varepsilon),$$

$P_1(\tau,\mu)$ and $P_{2,\nu}(\tau,\mu)$ – remnants after separating the linear part of μ, their norms are in order $\|\mu\|^2$.

From the equation (23) and the equation $\mu=\Phi_1(\mu,\xi,\varepsilon)$ we obtain

$$\xi=b^{-1}\Big(\Phi(\mu,\xi,\varepsilon)-P(\overline{y},\overline{\psi})\Phi_1(\mu,\xi,\varepsilon)\Big)=:\Phi_2(\mu,\xi,\varepsilon).$$

Let us construct an estimate of the norm of the vector-function $\Phi(\mu,\xi,\varepsilon)$ and its derivatives by μ and ξ according to the scheme of estimates for $\Phi_1(\mu,\xi,\varepsilon)$. We obtain

$$\|R_6(\mu,\xi,\varepsilon)\|\le c_{27}\varepsilon^\alpha,\quad \|R_7(\mu,\xi,\varepsilon)\|\le\Big(\sum_{\nu=1}^{p}|\beta_\nu|c_{26,\nu}\Big)\varepsilon^\alpha=c_{28}\varepsilon^\alpha.$$

Therefore,

$$\|\Phi\|\le(c_{27}+c_{28})\varepsilon^\alpha,$$

once $\varepsilon\in(0,\varepsilon_5]$, $\|\mu\|\le c_2\varepsilon^\alpha$, $\|\xi\|\le c_{16}\varepsilon^\alpha$.

Returning to the equation (22), we obtain Φ_2 such an estimate:

$$\|\Phi_2(\mu,\xi,\varepsilon)\|\le\big(\|\Phi\|+\|P\|\cdot\|\Phi_1\|\big)/b\le c_{29}\varepsilon^\alpha,$$

where

$$c_{29}=(c_{27}+c_{28}+c_2c_{30})/b,\quad \sup_{G_1}\Big|\frac{\partial g_0(\tau,a_\Lambda)}{\partial a_{\lambda_\nu}}\Big|\le\sigma_{3,\nu},$$

$$c_{30}=\sigma_2^{-1}\sum_{\nu=1}^{p}|\beta_\nu|\lambda_\nu^{-1}\Big[\sigma_{2,\nu}\big(e^{\sigma_2\lambda_\nu x_\nu}-1\big)+\sigma_{3,\nu}\big(e^{\sigma_2\lambda_\nu\tau_2}-e^{\sigma_2\lambda_\nu\tau_1}\big)\Big].$$

Let $c_3=\min(c_{16},c_{29})$. Then $\Phi_2:S_2\to S_2=\big\{\xi\in\mathbb{R}^m:\|\xi\|\le c_3\varepsilon^\alpha\big\}$.

Similarly, as for Φ_1 we obtain estimates

$$\|\frac{\partial\Phi_2}{\partial\mu}\|\le c_{31}\varepsilon^\alpha,\quad \|\frac{\partial\Phi_2}{\partial\xi}\|\le\overline{c}_{32}\varepsilon^\alpha. \tag{24}$$

If

$$\varepsilon\le\varepsilon_6=\min(\varepsilon_5,(4c_{31})^{-mq},(4\overline{c}_{32})^{-mq}),$$

then the norms of derivatives are limited to 0.25.

From the estimates (19), (21) and (24), it follows that

$$\|\Phi_3(\mu,\xi,\varepsilon)\| \le 1/2,$$

once $0<\varepsilon\le\varepsilon_6$, $\|\mu\|\le c_2\varepsilon^\alpha$, $\|\xi\|\le c_3\varepsilon^\alpha$, where Φ_3 – matrix of the order $m+n$ is composed of derivatives Φ_1 and Φ_2 by μ and ξ. Hence the reflection Φ – compressive, and on the basis of the theorem on compressive mappings there is the unique solution (μ,ξ) for each $\varepsilon\in(0,\varepsilon_6]$. For the problem (4)–(7) this means the existence of the unique solution in the class of continuously differentiated functions.

Estimation of the error of the method of averaging with a constant $c_{33}=c_1+c_1(L)c_2+c_{18}(c_3)$ follows from inequality (15) at $y=\overline{y}+\mu$ and $\psi=\overline{\psi}+\xi$ and inequality

$$\|\kappa(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)-\overline{\kappa}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \le \overline{c}_1(L)\|\mu\|+c_{18}\varepsilon^\alpha \le \big(c_2\overline{c}_1(L)+c_{18}(c_3)\big)\varepsilon^\alpha = c_1\varepsilon^\alpha.$$

□

Remark 3.1. Under conditions (6) and (7) instead of scalar coefficients α_ν and β_ν can be matrix. In this case, instead of the condition (12) it is needed to impose a condition

$$det\sum_{\nu=1}^{r}\beta_\nu\ne 0.$$

4. Example

Consider a single-frequency system of equations

$$\frac{da}{d\tau}=1+cos\varphi_\theta,\quad \frac{d\varphi}{d\tau}=\frac{1+2\tau}{\varepsilon},\quad \tau\in[0,1]; \tag{25}$$

with integral conditions

$$-3a(0)+2a(\frac{3}{4})=\int_{\frac14}^{\frac34}(1+cos\varphi_\theta(\tau))d\tau,\quad 4\varphi(0)+\varphi(\frac{3}{4})=\int_{\frac14}^{\frac34}(1+cos\varphi_\theta(\tau))d\tau, \tag{26}$$

where $\varphi_\theta(\tau)=k_1\varphi(\tau)+k_2\varphi(\theta\tau)$, $k_1,k_2\in\mathbb{Z}$, $0<|k_1|<|k_2|$, $k_1+k_2\theta=0$.

In the system (25) the resonance is achieved at $\tau=0$, since

$$k_1\omega(\tau)+k_2\theta\omega(\theta\tau)=2\tau(k_1+\theta^2k_2)=0.$$

Solution for the average problem

$$\frac{d\overline{a}}{d\tau}=1,\quad \frac{d\overline{\varphi}}{d\tau}=\frac{1+2\tau}{\varepsilon},\quad -3\overline{a}(0)+2\overline{a}(\frac34)=\frac12, 4\overline{\varphi}(0)+\overline{\varphi}(\frac34)=\frac12$$

takes the form of

$$\overline{a}(\tau)=\tau+\overline{y},\quad \overline{\varphi}(\tau)=(1+\tau)\tau/\varepsilon+\overline{\psi},\quad \overline{y}=1,\quad \overline{\psi}=(\varepsilon-10)/(10\varepsilon).$$

The solution of the problem (25), (26) satisfies the initial condition $a(0) := y = \overline{y} + \mu$, $\varphi(0) := \psi = \overline{\psi} + \xi$ and takes the form

$$a(\tau; 0, y, \xi, \varepsilon) = y + \tau + \int_0^\tau cos(\frac{s^2}{c\varepsilon} - \psi)ds,$$

$$\varphi(\tau; 0, y, \xi, \varepsilon) = \psi + (1+\tau)\tau/\varepsilon, \quad c^{-1} = k_1 + k_2\theta^2.$$

The values μ and ξ in the initial conditions are solutions of the equations

$$\xi = \frac{1}{5}\int_{\frac{1}{4}}^{\frac{3}{4}} cos\Psi(s, \xi, \varepsilon)ds, \quad \Psi(s, \xi, \varepsilon) = \frac{s^2}{c\varepsilon} - \overline{\psi} - \xi,$$

$$\mu = 2\int_0^{\frac{3}{4}} cos\Psi(s, \xi, \varepsilon)ds - \int_{\frac{1}{4}}^{\frac{3}{4}} cos\Psi(s, \xi, \varepsilon)ds.$$

From the estimates of the Fresnel integrals [11], it follows that

$$\xi = \frac{1}{5}\int_0^{\frac{3}{4}} \big(\cos\frac{s^2}{c\varepsilon}\cos(\psi + \xi) + \sin\frac{s^2}{c\varepsilon}\sin(\psi + \xi)\big)ds =$$

$$= \frac{\sqrt{c\varepsilon}}{5}\Big(\int_0^{\frac{3}{4\sqrt{c\varepsilon}}} \cos t^2 dt + \int_0^{\frac{3}{4\sqrt{c\varepsilon}}} \sin t^2 dt\Big) = O(\sqrt{\varepsilon})$$

at $\varepsilon -> 0$. Similarly, we obtain that $\mu = O(\sqrt{\varepsilon})$.

For errors of the averaging method at $\tau = 1$ we obtain

$$a(1, 0, y, \psi, \varepsilon) - \overline{a}(1, 0, \overline{y}) = \mu + 2\int_0^{\frac{3}{4}} cos\Psi(s, \xi, \varepsilon)ds = O(\sqrt{\varepsilon}),$$

$$\varphi(1, 0, y, \psi, \varepsilon) - \overline{\varphi}(1, 0, \overline{\psi}) = \xi + \frac{1}{5}\int_{\frac{1}{4}}^{\frac{3}{4}} cos\Psi(s, \xi, \varepsilon)ds = O(\sqrt{\varepsilon}).$$

5. Conclusion

The paper gives an unimproved estimate of the averaging method for a multifrequency system with a finite number of linear delays and integral conditions on $[\tau_1, \tau_2] \subset [0, L]$. The result according to the scheme presented in the paper is transferred to the case when the integral conditions are given on the union of intervals $[\tau_i, \tau_{i+1}]$ which do not intersect. The condition of the system out of resonance can be weakened, as was done in [1], assuming an equality of zero multiplicity of κ, which does not exceed Wronsky determinant on $[0, L]$. The estimation will be of the order ε^{β}, $\beta = 1/(mq + \kappa)$. The obtained results can be used in the study of complex oscillatory systems in the case of the resonance, as well as in networks of coupled phase oscillators.

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