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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Construction of solutions of integral equations with Stieltjes functionals and bifurcation parameters

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Abstract. The nonlinear Volterra integral equations with loads on the desired solution are studied. Loads are given using the Stieltjes integrals. The equations contain a parameter, for any value of which the equation has a trivial solution. The necessary and sufficient conditions on the values of the parameter are derived in the neighborhood where the equation has nontrivial real solutions.
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Construirea soluţiilor ecuaţiilor integrale cu funcţionale Stieltjes şi parametri de bifurcaţie

Rezumat. În lucrare sunt studiate ecuațiile integrale neliniare Volterra cu sarcini pe soluția dorită. Sarcinile sunt date folosind integralele Stieltjes. Ecuațiile conțin un parametru, pentru oricare valoare a căruia ecuația are o soluție banală. Condițiile necesare și suficiente asupra valorilor parametrului sunt obţinute în vecinătatea în care ecuaţia are soluţii reale nebanale.
Cuvinte cheie: ecuaţii Volterra neliniare, decompoziţia Newton–Puiseux, puncte de bifurcaţie, asimptotic, integrală Stieltjes, sarcini.

1. Problem statement

Let us address the following equation

$$x(t) = \mathcal{L}(x(t), x_\alpha, t, \lambda), \tag{1}$$

where $t \in [0, T]$, the desired function $x(t) \in C_{[0,T]}$, $\lambda \in \mathbb{R}^1$ is bifurcation parameter, x_α is given using the linear Stieltjes functional $\int_{t_1}^{t_2} x(t)\, d\alpha(t)$, where $\alpha(t)$ is limited variation function, $[t_1, t_2] \in (0, T)$. Then x_α can be local as follows $x_\alpha = x(\alpha)$, $\alpha \in (0, T)$ and nonlocal. Nonlinear integral operator $\mathcal{L}(x(t), x_\alpha, t, \lambda)$ is defined as formula (2).

In this manuscript we continue our studies [3].

I. Let us introduce the nonlinear mapping as follows

$$\mathcal{L}(x(t), x_\alpha, t, \lambda) = a(t, \lambda)x_\alpha + \sum_{i+k=l}^{N} f_{ik}(t, \lambda)x^i(t)x_\alpha^k +$$

$$+\int_0^t \sum_{i+k=l}^{N} K_{ik}(t, s\lambda)x^i(s)x_\alpha^k\, ds + R(x(t), x_\alpha, t, \lambda), \tag{2}$$

where $l \geq 2$, $\lambda \in \mathbb{R}^1$, $t \in [0, T]$,

$$||R(x(t), x_\alpha, t, \lambda)|| = O((||x(t)|| + |x_\alpha|)^{N+1}).$$

Functions $a(t, \lambda)$, $f_{ik}(t, \lambda)$, $K_{ik}(t, s, \lambda)$ are continuous and smooth with respect to parameter λ.

For arbitrary λ equation (1) has trivial solution $x(t) = 0$, $x_\alpha = 0$.

Definition 1.1. Point λ_0 is called the bifurcation point of the equation (1), if for any $\varepsilon > 0$, $\delta > 0$ there exist $x(t)$ and λ which satisfy (1) such as $0 < ||x|| < \varepsilon$, $|\lambda - \lambda_0| < \delta$.

In this work we present the conditions in which point $\lambda_0 \in \mathbb{R}^1$ is bifurcation point of equation (1) and asymptotic of the nontrivial branches of small solutions of equation (1) can be constructed.

We obtain the solution of the problem by constructing an equation with respect to the load with bifurcation parameter λ and investigating it using the method of successive approximations, methods of power geometry and rotation of finite-dimensional vector fields. A combination of such methods go back to the classical approaches outlined in [1, 2, 6] and has found application in solving a number of complex problems in mechanics, mathematical physics and energy [4, 5]

2. Construction of the equation with respect to the load and its analysis

For transparency, we first give a single-load statement. Using the method of successive approximations and the implicit singularity theorem in the analytic case, we solve $x(t)$ of the Volterra integral equation (1) as the following series

$$x(t) = \sum_{n=1}^{\infty} a_n(t, \lambda)x_\alpha^n. \tag{3}$$

We successively calculate functions $a_n(t, \lambda)$ as follows

$$a_1(t, \lambda) = a(t, \lambda),$$

$$a_n(t,\lambda) = \frac{1}{n!}\frac{\partial^n}{\partial x_\alpha^n}\mathcal{L}\left(\sum_{k=1}^{n-1} a_k(t,\lambda)x_\alpha^k, x_\alpha, t, \lambda\right)\Bigg|_{x_\alpha=0}, \; n = 2, 3, \dots. \tag{4}$$

Based on the implicit mapping theorem, series (3) will converge in a sufficiently small neighborhood $x_\alpha = 0$. It follows from the above

Lemma 2.1. *Let condition I be fulfilled. Then load x_α satisfies the equation*

$$A_1(\lambda)x_\alpha + \sum_{i=l}^{\infty} A_i(\lambda)x_\alpha^i = 0, \tag{5}$$

where

$$A_1(\lambda) = \int_{t_1}^{t_2} a_1(t,\lambda)\, d\alpha(t) - 1,$$

$$A_i(\lambda) = \int_{t_1}^{t_2} a_i(t,\lambda)\, d\alpha(t), \; i = l, l+1, \dots$$

Proof follows from the possibility of representing the solution $x(t)$ in the form of a series (3) and specifying the load using a linear functional x_α.

Corollary 2.1. *(The necessary conditions of bifurcation) In order that λ_0 be bifurcation point it is necessary $A_1(\lambda_0) = 0$.*

Proof. Equation (5) for all λ has a trivial solution $x_\alpha \equiv 0$. If $A_1(\lambda_0) \neq 0$, then $|\lambda - \lambda_0| \leq \rho_1$ in the neighborhood of $|x_\alpha| < \rho_2$. Based on the implicit function theorem, the small solution of equation (5) is singular. Hence, in this case $x_\alpha \equiv 0$ and λ_0 by Definition 1.1 is not a bifurcation point of the equation (1). □

Corollary 2.2. *Let all coefficients of $A_i(\lambda)$ at λ_0 be zero in equation (5). Then λ_0 is the bifurcation point. Moreover, equation (1) at $\lambda = \lambda_0$ has c–parametric non-trivial $x(t,c)$, depending on a sufficiently small enough parameter c. At $0 < |\lambda - \lambda_0| < \rho_1$ there are no other small solutions to equation (1).*

Proof is obvious. Since by virtue of the conditions of Corollary 2.2, the load x_α at $\lambda = \lambda_0$ in expansion (3) of the solution of equation (1) remains an arbitrary parameter c from the interval $|c| \leq \rho_2$, in which series (3) converges.

Constructive sufficient conditions for the existence of bifurcation points are obtained. By defining real solutions, $x_\alpha \to 0$ at $\lambda \to \lambda_0 + 0$ ($\lambda \to \lambda_0 - 0$) in equation (5) and substitute them into the right-hand side of formula (3).

By Lemma 2.1 to construct the asymptotic function x_α in equation (5), put $\lambda = \lambda_0 + \mu$, where μ is a small real parameter. Let us introduce the conditions:

II. λ_0 is the root of the equation $A_1(\lambda) = 0$ of multiplicity p;

III. $A_l^{(p)}(\lambda_0) \neq 0$.

As a result, equation (5) at $\lambda = \lambda_0 + \mu$ converts to

$$\left(\frac{1}{p!}A_1^{(p)}(\lambda_0)\mu^p + O(|\mu|^{p+1})\right)x_\alpha + (A_l(\lambda_0) + O(|\mu|))x_\alpha^l + O(|x_\alpha|^{l+1}) = 0 \qquad (6)$$

in the vicinity of points $x_\alpha = 0, \mu = 0$. The solution to equation (6) can be found using the Newton-Puso expansion (Newton's diagram defines the exponent $p/(l-1)$):

$$x_\alpha = (c_0 + O(|\mu|))\mu^{\frac{p}{l-1}}, \ c_0 \neq 0.$$

To determine c_0 we get the equation

$$\operatorname{sign}\mu^p \frac{1}{p!}\frac{d^p A_1(\lambda)}{d\lambda^p}\bigg|_{\lambda=\lambda_0} c + A_l(\lambda_0)c^l = 0. \qquad (7)$$

It is, therefore, obvious that when p is odd, these are two equations, namely

$$\frac{1}{p!}A_1^{(p)}(\lambda_0)c + A_l(\lambda_0)c^l = 0, \ \text{in } \mu > 0, \qquad (8)$$

$$-\frac{1}{p!}A_1^{(p)}(\lambda_0)c + A_l(\lambda_0)c^l = 0, \ \text{p } \mu < 0. \qquad (9)$$

For odd p and any l, equation (7) in at least one half-cross point $\mu = 0$ has a simple real solution $c_0 \neq 0$.

Theorem 2.1. *Let conditions I, II be fulfilled. Let λ_0 be the root of the equation $A_1(\lambda) = 0$ of odd multiplicity. Then λ_0 is the bifurcation point of equation (1). Moreover, if conditions III are satisfied, then for even l one can construct asymptotic solutions in the neighbourhood of point λ_0.*

Theorem 2.1 makes possible the representing of other manifolds of solutions of equation (5) in the form of Newton–Puiseux decompositions and admits generalizations.

The described approach allows one to construct solutions of the equation (1) directly as a series of integer or fractional powers of the parameter $\lambda - \lambda_0$, where λ_0 is the bifurcation point. The coefficients of the series, as in the well-known Nekrasov-Nazarov method [12] will be determined from a recurrence sequence of linear equations.

3. Integral equations with vector load and vector bifurcation parameter

Let $x_\alpha = (x_{\alpha_1}, \dots, x_{\alpha_n})$, $\lambda = (\lambda_1, \dots, \lambda_m)$ in equation (1). The values of the load and the bifurcation parameter lie in the vicinity of the zeros of vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Let us consider equation (1) where

$$\mathcal{L}(x(t), x_\alpha, t, \lambda) = \sum_{k=1}^{n} b_k(t, \lambda_1, \dots, \lambda_m) x_{\alpha_k} +$$

$$+ \sum_{s=l}^{\infty} \sum_{j+k_1+\dots+k_n=s} \Big[f_{jk_1\dots k_n}(t, \lambda_1, \dots, \lambda_m) x^j(t) x_{\alpha_1}^{k_1} \dots x_{\alpha_n}^{k_n} +$$

$$\int_0^t K_{jk_1,\dots,k_n}(t, s, \lambda_1, \dots, \lambda_m) x(s)^j x_{\alpha_1}^{k_1} \dots x_{\alpha_n}^{k_n} \, ds \Big],$$

b_k, $f_{jk_1,\dots,k_n}$, $K_{jk_1,\dots,k_n}$ be continuous functions, sufficiently smooth over all λ_i in the neighborhood of $||\lambda - \lambda_0|| \le \delta$. We find sufficient conditions when $\lambda_0 \in \mathbb{R}^m$ is the point of bifurcation and we can construct the asymptotics of real solutions at $\lambda_i = \lambda_{0i} + \mu$, $i = 1, \dots, m$, where $\mu \in [0, \delta]$ (or at $\mu \in [-\delta, 0]$).

We start by constructing a system to determine the vector load. To do this, plot the solution $x(t)$ using the series (10) on the homogeneous forms components of the vector load x_α :

$$\sum_{j=1}^{\infty} a_j(t, \lambda, x_\alpha). \tag{10}$$

Here

$$a_j(t, \lambda, \mu x_{\alpha_1}, \dots, \mu x_{\alpha_n}) = \mu^j a_j(t, \lambda, x_{\alpha_1}, \dots, x_{\alpha_n}).$$

Note that the j-single forms can be computed uniquely following the work of [3].

Lemma 3.1. *The required load x_α in solution (10) must satisfy n non-linear equations with parameters λ_i*

$$A_1(\lambda) x_\alpha + \sum_{j=l}^{\infty} F_j(\lambda, x_\alpha) = 0. \tag{11}$$

Here $F_j = (F_{j1}, \dots, F_{jn})^T$, $\lambda = (\lambda_1, \dots, \lambda_m)$, F_{ji}, $i = 1, \dots, n$ are j-uniform forms of vector components x_α, $A_1(\lambda)$.

Examining system (11) leads to the following result.

Theorem 3.1. *(Necessary bifurcation condition in the vector case) In order for the point $\lambda_0 \in \mathbb{R}^m$ to be a bifurcation point it is necessary that* $\det A_1(\lambda_0) = 0$.

Proof. A point λ_0 can be a bifurcation point of the integral equation if and only if it is a bifurcation point of system (11). But if $\det A_1(\lambda_0) \neq 0$, then for system (11) besides the trivial solution there are no other small solutions based on the implicit function theorem. Hence, due to formula (10) establishing an one-to-one correspondence between a small solution of the integral equation and a small solution of the system (11), the integral equation will also have only a trivial small solution $x(t) \equiv 0$. □

We derive the necessary conditions of bifurcation solutions of equation (1) in vector case. Let matrix $A_1(\lambda)$ be the null matrix in the point λ_0 and, therefore, conditions of Theorem 3.1 be satisfied. Let $\lambda = (\lambda_1^0 + \mu, \cdots, \lambda_m^0 + \mu)$ in system (11), where μ is a small real parameter. Then, system (11) becomes

$$L(x_\alpha, \mu) := B(\mu)x_\alpha + O(||x_\alpha||^2) = 0.$$

Here square matrix $B(\mu) = [b_{ik}(\mu)]_{i,k=1}^n$ is the null matrix in the point $\mu = 0$ due to selection of λ_0. Let the following condition be satisfied:

V. $b_{ik}(\mu) = c_{ik}\mu^{p_k} + r_{ik}(\mu)$, $i, k = 1, \dots n$, $\det[c_{ik}]_{i,k=1}^n \neq 0$, $|r_{ik}(\mu)| = o(|\mu|^{p_k})$, where p_k is natural number.

Then the following theorem is valid.

Theorem 3.2. *Let the load x_α and paramter λ be vectors in equation (1). Let condition V be satisfied and matrix A_1 be the null matrix in the point λ_0 of system (11). If $p_1 + \cdots + p_n$ is odd number, then λ_0 is a bifurcation point.*

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