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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Real cubic differential systems with a linear center and multiple line at infinity

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Abstract. We classify all cubic differential systems with a linear center and multiple line at infinity up to multiplicity four. For every class with the multiplicity of the line at infinity four the center problem is solved. It is proved that the monodromic points are of the center type if the first three Lyapunov quantities vanish.

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Keywords: cubic differential system, multiple invariant line, the center problem.

Sisteme diferenţiale cubice reale cu centru liniar şi linia de la infinit multiplă

Rezumat. Sunt clasificate sistemele diferenţiale cubice cu centru liniar şi linia de la infinit de multiplicitate cel mult patru. Pentru fiecare clasă ce are linia de la infinit de multiplicitate patru este rezolvată problema centrului. Se arată că punctele monodromice sunt de tip centru, dacă se anulează primele trei mărimi Liapunov.

Cuvinte cheie: sistem diferenţial cubic, linie invariantă multiplă, problema centrului.

1. Introduction

Consider the real cubic system of differential equations

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv p(x, y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv q(x, y), \\ \gcd(p, q) = 1, \ (k, l, m, n, p, q, r, s) \neq 0. \end{cases} \quad (1)$$

The critical point $(0, 0)$ of system (1) is a *linear center*, i.e. for the linearization of (1) the origin is a center, but for system (1) it is either a focus or a center. The problem of distinguishing between a center and a focus is called *the center problem*. It is well known that $(0, 0)$ is a center for system (1) if and only if the Lyapunov quantities $L_1, L_2, ..., L_j, ...$ vanish (see, for example, [2], [7], [8], [9]). Also, the critical point $(0, 0)$ is a center if system (1) has an axis of symmetry ([8]), either an analytical first integral or an integrating factor in some neighborhood of $(0, 0)$.

We suppose that at infinity system (1) has at most four distinct critical point, i.e.

$$sx^4 + (k+q)x^3y + (m+n)x^2y^2 + (l+p)xy^3 + ry^4 \not\equiv 0. \tag{2}$$

The homogeneous system associated to system (1) has the form

$$\begin{cases} \dot{x} = yZ^2 + (ax^2 + cxy + fy^2)Z + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y,Z), \\ \dot{y} = -(xZ^2 + (gx^2 + dxy + by^2)Z + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y,Z). \end{cases}$$

Denote $\mathbb{X} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$, $\mathbb{X}_\infty = P(x,y,Z)\frac{\partial}{\partial x} + Q(x,y,Z)\frac{\partial}{\partial y}$ and $E_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$. The polynomial E_∞ has the form $E_\infty = C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 + \cdots + C_8(x,y)Z^6$, where $C_j(x,y),\ j = 2,...,8$ are polynomials in x and y. For example,

$$\begin{aligned} C_4(x,y) = {} & M_4(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) \\ & + M_4(b,a,d,c,g,f,l,k,n,m,q,p,s,r,y,x), \end{aligned} \tag{3}$$

where

$$\begin{aligned} & M_4(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) = \\ & \quad (agk + dgk + 2k^2 - g^2m + agq + kq - a^2s + ads - 2cgs - 2ms - s^2)x^6 + \\ & \quad (3adk + d^2k + 2bgk - dgm + 3km + 2agn + 2kn - 2g^2p + 2adq - cgq - mq \\ & \quad +2abs - 3acs - cds - 4fgs - 3ks - 4ps - qs)x^5y + (5abk + 3bdk + 2cdk - fgk \\ & \quad +3agl + 3kl + 2adm - cgm + m^2 + a^2n + 3adn - agp - 3dgp + 2kp + 3abq - acq \\ & \quad -3fgq - kq - 3pq - 3g^2r - 2c^2s - 4afs - 3dfs - 4ms - 6rs)x^4y^2 + \\ & \quad (2b^2k + 4bck + dfk + 4abm + bdm + cdm - 2fgm + lm + adp - d^2p - 2bgp \\ & \quad -2cgp + mp - 2np - 2agr - 5dgr + kr - 5qr)x^3y^3. \end{aligned}$$

We say that the line at infinity $Z = 0$ has *multiplicity* ν if $C_2(x,y) \equiv 0, ..., C_\nu(x,y) \equiv 0$, $C_{\nu+1}(x,y) \not\equiv 0$, i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides E_∞.

The algebraic line $f(x,y) = 0$ is called *invariant* for (1) if there exists a polynomial $K \in \mathbb{C}[x,y]$ such that the identity $\mathbb{X}(f) \equiv f \cdot K(x,y)$ holds. Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

Cubic differential systems with multiple invariant straight lines (including the line at infinity) were studied in [6], [13], [16], and the center problem for (1) with invariant straight lines was considered in [2], [3], [4], [5], [10], [12], [14], [15], [17].

In [11], using the complex variables $A, B, C, D, F, G, K, L, M, N, P, Q, R, S$, where

$$\begin{aligned}
&A = g - b - c + i(a + d - f),\ C = 2(b + g + i(a + f)),\\
&F = c + g - b + i(a - d - f),\ K = r + s - m - n + i(k - l - p + q),\\
&M = n - m - 3(r - s) + i(3(k + l) + p + q),\\
&P = m + n + 3(r + s) + i(3(k - l) + p - q),\\
&R = m - n - r + s + i(k + l - p - q),\\
&B = \overline{A},\ D = \overline{C},\ G = \overline{F},\ L = \overline{K},\ N = \overline{M},\ Q = \overline{P},\ S = \overline{R},
\end{aligned} \tag{4}$$

cubic differential systems (1) with multiple line at infinity are classified.

In this paper we obtain the classification of these systems in real coefficients. Moreover, in the classes for which the multiplicity of the line at infinity is four, the center problem is solved.

2. Classification of cubic systems with multiple line at infinity

2.1. Cubic systems with the line at infinity of multiplicity two.

In [11], using the variables $A, B, ..., R, S$, it was shown that the line at infinity in (1) has multiplicity two if and only if one of the following three sets of conditions holds:

$$K = L = R = S = 0,\ Q = MN/P,\ MN \neq 0; \tag{5}$$

$$M = N = P = Q = 0,\ R = KL/S; \tag{6}$$

$$P = KN/S,\ R = KL/S,\ Q = MS/K,\ M \neq 0. \tag{7}$$

Taking into account (4), we obtain the following sets of solutions:

- in Case (5):

$$k = m = p = r = 0,\ q = l,\ s = n; \tag{8}$$

$$k = n = p = s = 0,\ q = l,\ r = m; \tag{9}$$

$$k = p,\ l = q,\ m = pq/s,\ n = s,\ r = pq/s; \tag{10}$$

- in Case (6):

$$k = m = p = r = 0,\ q = -3l,\ s = -n/3; \tag{11}$$

$$k = n = p = s = 0,\ q = -3l,\ r = -m/3; \tag{12}$$

$$k = -p/3,\ l = -q/3,\ m = -pq/(3s),\ n = -3s,\ r = pq/(9s); \tag{13}$$

- in Case (7):

$$k = m = p = r = 0; \tag{14}$$

$$k = m = n = p = q = s = 0; \tag{15}$$

$$k = n = p = s = 0,\ l = qr/m; \tag{16}$$

$$k = s = 0,\ l = nr/p,\ q = mn/p; \tag{17}$$

$$l = rs/k,\ n = ps/k,\ q = ms/k. \tag{18}$$

We change the coefficients $l \to -3l,\ n \to -n/3$ (respectively, $\{l \to -3l,\ m \to -m/3\}$ and $\{m \to m/9,\ p \to -p/3,\ q \to -q/3,\ r \to -r/3,\ s \to -3s\}$) in (1). This implies that the set of conditions (11) (respectively, (12) and (13)) is equivalent with the set of conditions (8) (respectively, (9) and (10)). Conditions (8) are contained in (14), while $\{(9),\ m = 0\}$, $\{(9),\ m \neq 0\}$, $\{(10),\ k = 0\}$, $\{(10),\ k \neq 0\}$ are contained in (14), (16), (14), (18), respectively.

In this way we proved the following Lemma.

Lemma 2.1. *The line at infinity of system* $\{(1),(2)\}$ *has multiplicity at least two if and only if one of conditions* (14) − (18) *holds.*

2.2. Cubic systems with the line at infinity of multiplicity three.

In complex variables $A, B, ..., R, S$ the line at infinity has the multiplicity at least three for cubic system $\{(1),(2)\}$ if and only if the coefficients of $\{(1),(2)\}$ verify one of the following six set of conditions:

$$K = L = R = S = 0,\ F = BM/P,\ G = AP/M,\ N = P^2/M,\ Q = P; \tag{19}$$

$$K = L = R = S = 0,\ D = CN/P,\ F = BP/N,\ G = AN/P,\ Q = MN/P; \tag{20}$$

$$M = N = P = Q = 0,\ C = DR/L,\ F = BR/L,\ G = AL/R,\ S = KL/R; \tag{21}$$

$$\begin{gathered} M = N = P = Q = 0,\ F = -(CL^2 - DLR - BR^2)/(LR), \\ G = (AL^2 + CLR - DR^2)/(LR),\ S = KL/R,\ KL^3 - R^4 = 0; \end{gathered} \tag{22}$$

$$D = CL/R,\ F = BR/L,\ G = AL/R,\ P = NR/L,\ Q = LM/R,\ S = KL/R; \tag{23}$$

$$\begin{gathered} D = L(FKL^2 - BKLR - FLMR + BMR^2 + CR^3)/R^4, \\ G = -L(FKL - BKR - AR^2)/R^3,\ N = -(KL^3 - L^2MR - R^4)/R^3, \\ P = -(KL^3 - L^2MR - R^4)/(LR^2),\ Q = LM/R,\ S = KL/R \end{gathered} \tag{24}$$

(see [11]).

Using (4), we will solve in the real coefficient $a, b, c, d, f, g, k, l, m, n, p, q, r, s$ of (1) the equalities (19)-(24). Firstly, we solve the equalities:

$$K = L = R = S = 0;$$
$$M = N = P = Q = 0.$$

We obtain, respectively

$$k = p,\ l = q,\ m = r,\ n = s; \tag{25}$$

$$k = -p/3,\ l = -q/3,\ m = -3r,\ n = -3s. \tag{26}$$

Suppose (25) is realised. Then equations (19) give us two series of conditions:

$$d = k = l = n = p = q = s = 0,\ b = g,\ m = r,\ r \neq 0; \tag{27}$$

$$a = (gq + fs - bq)/s,\ c = dq/s,\ k = l = p = q,\ m = q^2/s,\ n = s,\ r = q^2/s, \tag{28}$$

and (20) yields the conditions

$$a = c = f = k = m = p = r = 0,\ l = q,\ n = s; \tag{29}$$

$$l = m = q = r = 0,\ b = fs/p,\ d = cs/p,\ g = as/p,\ k = p, n = s; \tag{30}$$

$$b = fq/r,\ d = cq/r,\ g = aq/r,\ k = p,\ l = q,\ m = r,\ n = pq/r,\ s = pq/r. \tag{31}$$

Let equalities (26) hold. Then,

set (21) implies:

$$a = c = f = k = m = p = r = 0,\ l = -q/3,\ n = -3s; \tag{32}$$

$$l = m = q = r = 0,\ b = -3fs/p, d = -3cs/p, g = -3as/p, k = -p/3, n = -3s; \tag{33}$$

$$\begin{aligned} &b = -fq/(3r),\ d = -cq/(3r),\ g = -aq/(3r),\ k = -p/3, l = -q/3, \\ &m = -3r,\ n = -pq/(3r),\ s = pq/(9r); \end{aligned} \tag{34}$$

set (22) yields:

$$c = k = l = m = p = q = r = 0,\ a = -f/3,\ n = -3s,\ s \neq 0; \tag{35}$$

$$d = k = l = n = p = q = s = 0,\ b = -g/3,\ m = -3r,\ r \neq 0; \tag{36}$$

$$\begin{aligned} &l = m = q = r = 0,\ b = -fp/(9s),\ g = (dp + 3cs - ap)/(9s), \\ &k = -p/3,\ n = -3s,\ p^2 - 27s^2 = 0; \end{aligned} \tag{37}$$

$$\begin{aligned} &a = -(cq^3 + 3dq^2r - 27cqr^2 + 72gqr^2 - 81dr^3)/(24q^2r), \\ &f = (cq^3 + 3dq^2r - 24bqr^2 - 3cqr^2 - 9dr^3)/(8q^2r), \\ &p = -3k = -q(q^2 - 27r^2)/(3(q^2 - 3r^2)),\ l = -q/3, \\ &m = -3r,\ n = -3s = q^2(q^2 - 27r^2)/(9r(q^2 - 3r^2)). \end{aligned} \tag{38}$$

set (23) gives:

$$b = d = g = l = n = q = s = 0; \tag{39}$$

$$a = c = f = k = m = p = r = 0; \tag{40}$$

$$k = l = n = p = r = s = 0,\ a = gm/q,\ c = dm/q,\ f = bm/q; \tag{41}$$

$$k = n = p = s = 0,\ a = gr/l,\ c = dr/l,\ f = br/l,\ m = qr/l; \tag{42}$$

$$n = p = 0,\ a = gk/s,\ c = dk/s,\ f = bk/s,\ m = kq/s,\ r = kl/s; \tag{43}$$

$$a = gp/n,\ c = dp/n,\ f = bp/n,\ k = ps/n,\ m = pq/n,\ r = lp/n. \tag{44}$$

The system $\{Q = LM/R, S = KL/R\}$ has the solutions:

(i) $k = l = n = p = r = s = 0,\ q \neq 0$; (ii) $l = n = p = r = 0,\ m = kq/s$;
(iii) $l = r = 0,\ k = ps/n,\ m = pq/n$; (iv) $l = q = 0,\ s = n$;
(v) $k = rs/l,\ m = qr/l,\ p = nr/l.$

Nextly, we solve (24) taking into account the equalities $(i) - (v)$. We obtain conditions

$$a = f = k = l = m = n = p = r = s = 0,\ q \neq 0; \tag{45}$$

in Case (i); conditions

$$f = k = l = m = n = p = r = 0,\ c = aq/s; \tag{46}$$

$$l = n = p = r = 0,\ b = fs/k,\ d = cs/k,\ m = kq/s,\ q = k; \tag{47}$$

in Case (ii); conditions

$$k = l = m = p = r = 0,\ a = fs/n,\ c = fq/n; \tag{48}$$

$$\begin{aligned} &l = r = 0,\ a = (cnpq - cn^3 + dn^2p + gnp^2 - dp^2q)/(n^2p),\ f = bp/n, \\ &k = (pq - n^2)/p,\ m = pq/n,\ s = n(pq - n^2)/p^2; \end{aligned} \tag{49}$$

in Case (iii); conditions

$$b = g = k = l = m = n = q = r = s = 0,\ p \neq 0; \tag{50}$$

$$b = k = l = n = q = r = s = 0,\ d = gp/m; \tag{51}$$

$$k = l = n = q = s = 0,\ d = bp/r,\ g = bm/r; \tag{52}$$

in Case (iv); and conditions

$$\begin{aligned} &a = (fl^4 - bl^3r - fl^2nr + blnr^2 + flqr^2 + glr^3 - bqr^3)/(l^2r^2), \\ &c = (bl^2r - fl^3 + flnr + dlr^2 - bnr^2)/(l^2r),\ k = (l^3 - lnr + qr^2)/r^2, \\ &m = (qr)/l, p = nr/l,\ s = l(l^3 - lnr + qr^2)/r^3. \end{aligned} \tag{53}$$

in Case (v). From the above, the next statement follows.

Lemma 2.2. *The line at infinity of system* $\{(1), (2)\}$ *has multiplicity at least three if and only if one of the series of conditions* $(27) - (53)$ *holds.*

It is easy to show that under each of the conditions (27)−(38), (42), (47), (48) and (51) the polynomial (3), i.e. $C_4(x, y)$, is not equivalently zero. Therefore, the assertion of the following Lemma is true.

Lemma 2.3. *Under each of the conditions* (27)−(38)*,* (42)*,* (47)*,* (48) *and* (51) *the line at infinity of system* $\{(1), (2)\}$ *has multiplicity exactly three.*

2.3. Cubic systems with the line at infinity of multiplicity four.

According to [11], the line at infinity in complex variables $A, B, ..., R, S$ has the multiplicity at least four for cubic system $\{(1), (2)\}$ if and only if the coefficients of $\{(1),(2)\}$ verify one of the following two sets of conditions:

$$\begin{array}{l} D = CS/K,\ F = BK/S,\ G = AS/K,\ L = -S^4/K^3, \\ M = S, N = R = -S^3/K^2,\ Q = -P = S^2/K; \end{array} \tag{54}$$

$$\begin{array}{l} A = 2(K^3L + S^4)/(S^2(BK - FS)) - S(BK - 2FS)/(KL), \\ C = 2K^3L + S^4)/(KS(BK - FS)) - (BK^4L - 2FK^3LS - FS^5)/(K^2LS^2), \\ D = (FK^2L + BS^3)/(K^2L) + 2(K^3L + S^4)/(K^2(BK - FS)), \\ G = FS^3/(K^2L) + 2(K^3L + S^4))/(KS(BK - FS)), \\ M = (K^3L + 2S^4)/S^3,\ N = (2K^3L + S^4)/(K^2S),\ P = (2K^3L + S^4)/(KS^2), \\ Q = (K^3L + 2S^4)/(KS^2),\ R = KL/S. \end{array} \tag{55}$$

The equalities $M = S$, $Q = -P$ occur if $l = -k$, $m = -2r - s n = -r - 2s$. The last equalities together with $D = CS/K$, $F = BK/S$, $G = AS/K$, $L = -S^4/K^3, R = -S^3/K^2$, $P = -S^2/K$ from (54) give us the following three series of conditions:

$$a = c = f = k = l = m = n = p = r = s = 0,\ q \neq 0; \tag{56}$$

$$b = d = g = k = l = m = n = q = r = s = 0,\ p \neq 0; \tag{57}$$

$$\begin{array}{l} b = fs/k,\ d = cs/k,\ g = as/k,\ l = -k,\ m = 2k^2/s - s, \\ n = k^2/s - 2s,\ p = -2k + k^3/s^2,\ q = 2k - s^2/k,\ r = -k^2/s. \end{array} \tag{58}$$

Similarly, in Case (55), solving the system of equalities $M = (K^3L + 2S^4)/S^3$, $N = (2K^3L + S^4)/(K^2S)$, $P = (2K^3L + S^4)/(KS^2), Q = (K^3L + 2S^4)/(KS^2)$, $R = KL/S$, we get:

$$k = s = 0,\ l = p/2,\ m = n/2,\ q = n^2/(2p),\ r = p^2/(2n); \tag{59}$$

$$\begin{array}{l} l = (ps^2 - k^3)/(2s^2),\ m = (3k^3 + ps^2)/(2ks),\ n = ps/k, \\ q = (3k^3 + ps^2)/(2k^2),\ r = -k(k^3 - ps^2)/(2s^3); \end{array} \tag{60}$$

$$k = l = m = n = p = r = 0; \tag{61}$$

$$k = l = m = n = q = s = 0. \tag{62}$$

Then under each of conditions (59)−(62), we solve the system

$A = 2(K^3L + S^4)/(S^2(BK - FS)) - S(BK - 2FS)/(KL)$,

$C = 2K^3L + S^4)/(KS(BK - FS)) - (BK^4L - 2FK^3LS - FS^5)/(K^2LS^2)$,

$D = (FK^2L + BS^3)/(K^2L) + 2(K^3L + S^4)/(K^2(BK - FS))$,

$G = FS^3/(K^2L) + 2(K^3L + S^4))/(KS(BK - FS))$
and obtain, in Case (59):

$$k = s = 0,\ a = -np/(2(cn - dp)),\ b = (2c^2n - 2cdp + p^2)/(2(cn - dp)),\\ f = p(4c^2n^2 - 6cdnp + 2d^2p^2 + np^2)/(2n^2(cn - dp)),\ g = -n^2/(2(cn - dp)),\\ l = p/2,\ m = n/2,\ q = n^2/(2p),\ r = p^2/(2n); \tag{63}$$

in Case (60):

$$b = (-agk^4 + k^5 + a^2k^3s - 2k^3s^2 + agkps^2 - k^2ps^2 - a^2ps^3)/(2k^2s\cdot\\ (gk - as)),\ l = (-k^3 + ps^2)/(2s^2),\ m = (3k^3 + ps^2)/(2ks),\\ c = -(2g^2k^5 - 5agk^4s - k^5s + 3a^2k^3s^2 + 2k^3s^3 - agkps^3 + k^2ps^3\\ +a^2ps^4)/(2k^2s^2(gk - as)),\ q = (3k^3 + ps^2)/(2k^2),\\ d = (-g^2k^4 + 3agk^3s + k^4s - 2a^2k^2s^2 + g^2kps^2 - 2k^2s^3 - agps^3\\ -kps^3)/(2k^2s(gk - as)),\ r = -k(k^3 - ps^2)/(2s^3)\\ f = -(g^2k^5 - 3agk^4s + k^5s + 2a^2k^3s^2 - g^2k^2ps^2 - 2k^3s^3 + 3agkps^3\\ -k^2ps^3 - 2a^2ps^4)/(2ks^3(-gk + as)),\ n = ps/k; \tag{64}$$

in Case (61):

$$b = c = f = k = l = m = n = p = q = r = 0,\ s = a(d - a),\ a(d - a) \neq 0; \tag{65}$$

$$f = k = l = m = n = p = r = 0,\ a = cs/q,\ c = b, g = s(bdq - q^2 - b^2s)/(bq^2); \tag{66}$$

in Case (62):

$$a = d = g = k = l = m = n = p = q = s = 0,\ r = b(c - b),\ b(c - b) \neq 0; \tag{67}$$

$$g = k = l = m = n = q = s = 0, b = ar/p, d = a, f = r(acp - p^2 - a^2r)/(ap^2). \tag{68}$$

In this way we proved the following Lemma.

Lemma 2.4. *The line at infinity of system* $\{(1), (2)\}$ *has multiplicity at least four if and only if one of the series of conditions* (56)−(58), (63)−(68) *holds.*

Remark 2.1. The substitution $\{x \leftrightarrow y,\ t \to -t,\ a \leftrightarrow b,\ c \leftrightarrow d,\ f \leftrightarrow g,\ k \leftrightarrow l,\ m \leftrightarrow n,\ p \leftrightarrow q,\ r \leftrightarrow s\}$ reduces system {(1),(57)} (respectively, {(1),(67)}, {(1),(68)}) to system {(1),(56)} (respectively, {(1),(65)}, {(1),(66)}).

3. Center conditions for cubic system $\{(1),(2)\}$ with the line at infinity of multiplicity at least four

Lemma 3.1. *The following twelve sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for system* (1)*:*

$$a=b=c=f=k=l=m=n=p=r=s=0,\ q=dg,\ dg\neq 0; \tag{69}$$

$$\begin{gathered} b=\pm f, c=\pm(a+f),\ d=a+f, g=\pm a,\\ k=-l=\pm m=-n=-p=q=\mp r=\pm s=\pm(a^2-f^2)/2; \end{gathered} \tag{70}$$

$$\begin{aligned} &b=(fs)/k, c=(k((k^2+s^2)^2-2(a^2-f^2)s^3))/((a+f)s^2(k^2-s^2)),\\ &d=((k^2+s^2)^2-2(a^2-f^2)s^3)/((a+f)s(k^2-s^2)), g=(as)/k, l=-k,\\ &m=(2k^2-s^2)/s, n=(k^2-2s^2)/s, p=(k(k^2-2s^2))/s^2,\\ &q=(2k^2-s^2)/k, r=-k^2/s,\ k^2(k^2+s^2)-(a+f)s(ak^2-fs^2)=0; \end{aligned} \tag{71}$$

$$\begin{aligned} &k=s=0,\ a=(cd-p)/(2c),\ b=c/2,\ f=c^3(cd-3p)/(2(cd-p)^2),\\ &g=(cd-p)^2/(2c^3),\ l=p/2,\ m=p(cd-p)/(2c^2),\ n=p(cd-p)/c^2,\\ &q=p(cd-p)^2/(2c^4),\ r=c^2p/(2(cd-p)); \end{aligned} \tag{72}$$

$$\begin{aligned} &b=(-agk^4+k^5+a^2k^3s-2k^3s^2+agkps^2-k^2ps^2-a^2ps^3)/(2k^2s\cdot\\ &(gk-as)),\ l=(-k^3+ps^2)/(2s^2),\ m=(3k^3+ps^2)/(2ks),\\ &c=-(2g^2k^5-5agk^4s-k^5s+3a^2k^3s^2+2k^3s^3-agkps^3+k^2ps^3\\ &+a^2ps^4)/(2k^2s^2(gk-as)),\ q=(3k^3+ps^2)/(2k^2),\\ &d=(-g^2k^4+3agk^3s+k^4s-2a^2k^2s^2+g^2kps^2-2k^2s^3-agps^3\\ &-kps^3)/(2k^2s(gk-as)),\ r=k(ps^2-k^3)/(2s^3),\\ &f=-(g^2k^5-3agk^4s+k^5s+2a^2k^3s^2-g^2k^2ps^2-2k^3s^3+3agkps^3\\ &-k^2ps^3-2a^2ps^4)/(2ks^3(-gk+as)),\ n=ps/k,\\ &(k^3-2ks^2-ps^2)(s(3k^3-ps^2)(-agk+k^2+a^2s)+2k(k^2+s^2)(g^2k^2\\ &-2agks-k^2s+a^2s^2))+4k^2s^2(gk-as)^2(k^2+s^2)=0; \end{aligned} \tag{73}$$

$$\begin{aligned} &f=k=l=m=n=p=r=0,\ a=bs/q,\ c=b,\\ &d=b(2s^2-q^2)/(qs),\ g=(b^2s^2-b^2q^2-q^2s)/(bq^2); \end{aligned} \tag{74}$$

$$\begin{aligned} &b=k(k^2+a^2s+s^2-agk)/(s(as-gk)),\\ &c=k(g^2k^2-4agks+k^2s+3a^2s^2+s^3)/(s^2(as-gk)),\\ &d=(k^2s+a^2s^2+s^3-g^2k^2)/(s(as-gk)),\\ &f=k^2(g^2k^2-3agks+k^2s+2a^2s^2+s^3)/(s^3(as-gk)),\\ &p=3l=3k^3/s^2,\ m=n=3k^2/s,\ q=3k,\ r=k^4/s^3,\\ &g^2k^3-3agk^2s+k^3s+2a^2ks^2-g^2ks^2+ags^3+ks^3=0; \end{aligned} \tag{75}$$

$$\begin{aligned} &b=-as/k,\ c=(2ak^2s-gk^3-as^3)/(ks^2),\ d=(ak-gs)/k,\\ &f=(gk-2as)/s,\ l=-k,\ m=(2k^2-s^2)/s,\ n=(k^2-2s^2)/s,\\ &p=k(k^2-2s^2)/s^2,\ q=(2k^2-s^2)/k,\ r=-k^2/s,\\ &g^2k^3-3agk^2s+k^3s+2a^2ks^2-g^2ks^2+ags^3+ks^3=0; \end{aligned} \tag{76}$$

$$b=c=f=g=k=l=m=n=p=q=r=0,\ s=a(d-a),\ a(d-a)\neq 0; \tag{77}$$

$$b=c=f=k=l=m=n=p=q=r=0,\ d=2a,\ s=a^2,\ a\neq 0; \tag{78}$$

$$a=f=g=k=l=m=n=p=r=s=0,\ c=b,\ d=q/b,\ q\neq 0. \tag{79}$$

Proof. When one of the series of conditions (69)−(74) holds, system (1) has an affine invariant straight line l_1 and a Darboux integrating factor of the form $\mu(x,y)=1/l_1$.

In Case (69): $l_1=dy+1$;

in Case (70): $l_1=(a-f)(x\mp y)\pm 1$;

in Case (71): $l_1=(k^2+s^2)(kx-sy)+(a+f)s^2$;

in Case (72): $l_1=py+c$;

in Case (73): $l_1=2k^3s(k^3-2ks^2-ps^2)+(gk-as)(3k^3-ps^2)(2k^2sx-k^3y+ps^2y)$;

in Case (74): $l_1=b(sx+qy)-s$.

Under the conditions (75) the equalities $CF-DG=AD^3-BC^3=AF^3-BG^3=A^4L^3-B^4K^3=A^2N^3-B^2M^3=A^2R^3-B^2S^3=C^4L-D^4K=C^2N-D^2M=C^2R-D^2S=F^4K-G^4L=F^2M-G^2N=F^2S-G^2R=KN^2-LM^2=KR^2-LS^2=MR-NS=P-Q=0$ hold. Therefore, system {(1),(2)} has an axis of symmetry and the origin is a center ([8]).

In Case (76), system (1) has the integrating factor of the Darboux form:

$$\mu(x,y)=l_1^{\alpha_1}l_2^{\alpha_2}l_3^{\alpha_3}l_4^{\alpha_4},$$

where

$$\begin{aligned} &l_1=ks+(gk-as)(sx+ky),\qquad l_3=Exp[2s(gk-as)x+k(sx+ky)^2],\\ &l_2=Exp[sx+ky],\ l_4=Exp[(sx+ky)(s(ags+ks-g^2k)x+(g^2k^2-3agks+k^2s\\ &+2a^2s^2)y+2(as-gk)(sx+ky)^2)/3],\quad \alpha_1=(k^2+s^2)^2(g^4k^3-ag^3k^2s-5g^2k^3s\\ &+g^4ks^2+4agk^2s^2+2k^3s^2-ag^3s^3-g^2ks^3)/(8k^3(as-gk)^4),\\ &\alpha_2=(g^2k-ags-ks)(k^2+s^2)^2/(2k^2(gk-as)^3),\\ &\alpha_3=(k^2+s^2)(gk^2-2aks-gs^2)/(2k^2s^3(gk-as)),\\ &\alpha_4=k(k^2+s^2)^2/(2k^2s^3(gk-as)^2). \end{aligned}$$

In Case (77), Ox is an axis of symmetry for (1).

In Case (78), system (1) has a polynomial first integral $F(x,y)=6(x^2+y^2)+4gx^3+12ax^2y+3a^2x^4$.

In Case (79), system (1) has an integrating factor $\mu(x, y) = (1 + bx)Exp[-q(3qx^2 + 2bqx^3 + 6by + 6b^2xy)/(6b^2)]$. □

Remark 3.1. The line at infinity for system {(1),(2),(78)} has multiplicity five.

Theorem 3.1. *Cubic system* {(1), (2)} *with the line at infinity of multiplicity four has at the origin a center if and only if the first three Lyapunov quantities vanish.*

Proof. To prove Theorem 3.1 we compute the first three Lyapunov quantities (see [2], [9]) L_1, L_2, L_3 for each set of conditions (56), (58), (63)–(66) (see Remark 2.1). In the expressions for L_j we will neglect the non-zero factors.

In Case (56) the first Lyapunov quantity is $L_1 = bd + dg - q$ while $L_1 = 0$ gives $q = d(b + g)$. Then $\{L_2 = b(3b + 5g) = 0,\ L_3 = b(2064b^3 + 183bd^2 + 4982b^2g + 305d^2g + 2660bg^2 + 230g^3) = 0\} \Longrightarrow \{b = 0\} \Longrightarrow$ Lemma 3.1, (69).

Let conditions (58) hold. Then $L_1 = (a+f)(s^2-k^2)s^2c+k((k^2+s^2)^2-2(a^2-f^2)s^3)$. If $s = \pm k$, then $\{L_1 = 0, L_2 = 0, L_3 = 0\} \Longleftrightarrow \{k = \pm(a^2 - f^2)/2,\ c = \pm(a + f)\} \Longrightarrow$ Lemma 3.1, (70). If $s^2-k^2 \neq 0$, then $L_1 = 0 \Longrightarrow c = k((k^2+s^2)^2-2(a^2-f^2)s^3))/((a+f)(k^2 - s^2)s^2) \Longrightarrow L_2 = f_1f_2$, $L_3 = f_1f_3$, where

$$\begin{aligned}
f_1 &= k^2(k^2 + s^2) - (a + f)s(ak^2 - fs^2),\\
f_2 &= 2k^4 + k^2s(2s + (a + f)(3a + 5f)) - (a + f)(5a + 3f)s^3,\\
f_3 &= (a + f)k^2(9a^3 + 39a^2f + 55af^2 + 25f^3 + 40as + 56fs) - s^2(15a^4 + 64a^3f\\
&\quad +98a^2f^2 + 64af^3 + 15f^4 - 28k^2 + 40a^2s + 64afs + 24f^2s + 4s^2).
\end{aligned}$$

If $f_1 = 0$, then Lemma 3.1, (71). Assume that $(a + f)(k^2 - s^2)sf_1 \neq 0$, then system $\{f_2 = 0, f_3 = 0\}$ is incompatible.

Under conditions (63) the first Lyapunov quantity is $L_1 = g_1g_2$, where $g_1 = c^2n - cdp + p^2$ and $g_2 = n(n^4 - p^4) - 2p(cn - dp)(dn^2 + 2cnp - dp^2)$.

If $g_1 = 0$, then Lemma 3.1, (72). Let $g_1 \neq 0$. The second Lyapunov quantity reduced by g_2 has the form

$$\begin{aligned}
L_2 &= (2cd + d^2 - c^2)(c^2 + 2cd - d^2)(2d^2 + n)p^6 + 2cd(24c^4d^2 - 80c^2d^4 + 15c^4n\\
&-62c^2d^2n + 7d^4n - 2c^2n^2 + 2d^2n^2)p^5 + (32c^8d^2 - 112c^6d^4 + 16c^8n - 208c^6d^2n\\
&+600c^4d^4n - 8c^6n^2 + 112c^4d^2n^2 - 52c^2d^4n^2 + 4d^6n^2 + c^4n^3 - 6c^2d^2n^3 + d^4n^3)p^4+\\
&(-4cdn(24c^8 - 88c^6d^2 - 32c^6n + 194c^4d^2n - 40c^2d^4n + 12c^4n^2 - 20c^2d^2n^2 + 6d^4n^2\\
&-c^2n^3 + d^2n^3)p^3 + 4c^2n^2(16c^8 - 76c^6d^2 + 28c^4d^4 - 4c^6n + 112c^4d^2n - 70c^2d^4n+\\
&2c^4n^2 - 15c^2d^2n^2 + 15d^4n^2)p^2 + 8c^5dn^3(8c^4-16c^2d^2-16c^2n+21d^2n)p + 16c^8d^2n^4.
\end{aligned}$$

It is easy to show that system $\{g_2 = 0, L_2 = 0, np(cn - dp)g_1 \neq 0\}$ has no solutions.

In Case (64) the first Lyapunov quantity has the form $L_1 = \varphi_0\varphi_1$ and the second quantity reduced by φ_1 looks as $L_2 = (3k^3 - ps^2)(k^3 - 2ks^2 - ps^2)\varphi_0\varphi_2$, where

$$\begin{aligned}\varphi_0 &= (k^3 - 2ks^2 - ps^2)(s(3k^3 - ps^2)(-agk + k^2 + a^2 s) + 2k(k^2 + s^2)(g^2k^2\\ &\quad -2agks - k^2 s + a^2 s^2)) + 4k^2 s^2 (gk - as)^2 (k^2 + s^2),\\ \varphi_1 &= g^2k^3 - 3agk^2 s + k^3 s + 2a^2 ks^2 - g^2 ks^2 + ags^3 + ks^3,\\ \varphi_2 &= (k^2 + s^2)^2 g^5 (kg - as) + (k^2 + s^2) g^3 (8ak^2 s^2 - ksg(9k^2 + s^2))\\ &\quad +6k^3 s^2 g^2 (3k^2 + s^2) - 12agk^4 s^3 - 4k^5 s^3.\end{aligned}$$

If $\varphi_0 = 0$ (respectively, $\{3k^3 - ps^2 = 0,\ \varphi_1 = 0\}$; $\{k^3 - 2ks^2 - ps^2 = 0,\ \varphi_1 = 0\}$), then Lemma 3.1, (73) (respectively, Lemma 3.1, (75); Lemma 3.1, (76)).

System $\{\varphi_1 = 0,\ \varphi_2 = 0\}$ is incompatible because $Resultant[\varphi_1, \varphi_2, a] = 32k^{11}s^8 \neq 0$.

Under conditions (65) $L_1 = g(2a - d)$. If $g = 0$, then Lemma 3.1, (77), and if $d = 2a$, then Lemma 3.1, (78).

Finally, when conditions (65) hold the first Lyapunov quantity is $L_1 = (bq^2 + dqs - 2bs^2)(q^2 + b^2 s - bdq)$. If $bq^2 + dqs - 2bs^2 = 0$, then Lemma 3.1, (74), and if $q^2 + b^2 s - bdq = 0$, $bq^2 + dqs - 2bs^2 \neq 0$, then $d = (q^2 + b^2 s)/(bq) \Rightarrow L_2 = b^6 q^4 s = 0 \Rightarrow s = 0 \Rightarrow$ Lemma 3.1, (79). □

Theorem 3.2. *With exactness of a centro-affine transformation of coordinates, the cubic system* $\{(1), (2)\}$ *with the line at infinity of multiplicity four has at the origin a center if and only if one of the series of conditions* (69)–(79) *holds.*

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